Measures of Joint Default Dependence Risk based on Copulas

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Keywords: Default, Copula, Conditional Dependence Probability, Financing Indexes.

Abstract: This paper studies the problem of forecasting joint default. The default is the result that the credit rating of an obligor, determined by obligor's operating situation and financing state, decreases to some certain degree. The dependence relationship of financing indexes is investigated to judge the credit rating of an obligor and the conditional dependence probability and probability density functions are proposed. A member of conditional dependence risk relationships is completely characterized by the marginal distribution and the copulas of random variables. These results can be applied to investigate the conditional dependence structure and the conditional dependence measure of obligor's assets and of the defaults among obligors.

1 INTRODUCTION

In the economic and financial market environment, a default would have a chain effect on. The default is the result that the credit rating of an obligor, determined by obligor's operating situation and financing state, decreases to some certain degree. A number of studies have investigated credit risk about financial market and default correlation of obligors. The KMV (Kealhofer, 1998) and Credit Metrics (Gupton, et al., 1997) Models are the most important and widely used industry models. A core assumption of the KMV and Credit Metrics Models is the multivariate normality of the latent variables, where the latent variables often interpreted as the value of the obligor's assets. In these models default of an obligor occurs if the latent variables fall below some threshold which often interpreted as the value of the obligor's liabilities. Defaults are predictable since the values of assets are continuous process. Indeed, at any time investors know the nearness of the assets to the default threshold, so that they are warned in advance when a default is imminent. However, for bond prices and credit spreads, prices converge continuously to their default-contingent values can not appear at all. This means that they fail to be consistent in particular with the observed contagion phenomena, although the existing structural approaches provide important insights into the relation between firms' fundamentals and correlated default events as well as practically most valuable tools (Kay, 2004). A benchmark study was provided on the basis of time to default in credit scoring using survival analysis and identifying hidden patterns in credit risk survival data using Generalised Additive Models (Dirick, 2017, Claeskens, 2017, Baesens, 2017, Djeundje, 2019, Crook, 2019).

The default of an obligor is an asymptotical accumulating process of firm's assets decreasing. The default will occur when the operating situation and financing are distressed to some certain degree. The indexes characterizing the credit rating of an obligor are dependence on each other. It is useful of judging the probability of default of an obligor and running the credit risk to investigate the dependence structure of indexes.

In this paper we provide a dependence model of multivariate indexes based on copula functions for forecasting the obligor's default and the conditional dependence relationship of some indexes. Based on the properties of probability, we present the conditional dependence probability and density functions.

2 COPULAS AND THE LATENT VARIABLE MODEL

Copulas are simply the joint distribution function of random vectors with standard uniform marginal distributions. The most important result in the copula

Huang, A. and Yi, W.
Measures of Joint Default Dependence Risk based on Copulas.
DOI: 10.5220/0011203700003440
In *Proceedings of the International Conference on Big Data Economy and Digital Management (BDEDM 2022)*, pages 654-657
ISBN: 978-989-758-593-7

framework is due to (Sklar, 1959). That is, the copula connects a multivariate distribution to its margins in such a way that it captures the entire dependence structure in the multivariate distribution. Their value in statistics is that they provide a way of understanding how marginal distributions of single risks are coupled together to form joint distributions of groups of risks.

Let F be a joint distribution function with continuous margins $F_1,\cdots,F_m$. Then there exists a unique copula $C:[0,1]^m \to [0,1]$ such that

$$F(x_1,\cdots,x_m)=C(F_1(x_1),\cdots,F_m(x_m)) \quad (1)$$

holds. Conversely, if C is a copula and $F_1,\cdots,F_m$ are distribution functions, then the function F by (1) is a joint distribution function with margins $F_1,\cdots,F_m$.

For example, in the credit application, if the latent variables $X=(X_1,\cdots,X_m)$ have a multivariate Gaussian distribution with correlation matrix ρ, then the copula of X may be represented by $C_\rho^{Ga}(u_1,\cdots,u_m)=\Phi_\rho(\Phi^{-1}(u_1),\cdots,\Phi^{-1}(u_m))$, where Φ_ρ denotes the joint distribution function of a standard $m-$ dimensional normal random vector with correlation matrix ρ, and Φ is the distribution function of univariate standard normal. C_ρ^{Ga} is called as the Gaussian copula which characterizes the dependence structure of the latent variables.

(David 2000) studied the default correlation via the copula function approach. In his model the random vector $X=(X_1,\cdots,X_m)^T$ are interpreted as time-until-default which implicate the survival time of each defaultable entity or financial instrument, and the thresholds $D_1,\cdots,D_m$ are all set to take the value T, the time horizon. (Frey 2001, McNeil 2001) proposed a latent variable model combine to copula functions to investigate the credit. For random vector X with continuous marginal distributions, deterministic cut-off levels vector $D_1,\cdots,D_m$ and the binary random vector $(Y_1,\cdots,Y_m)^T$, such that the following relationship holds:

$$Y_i=1 \Leftrightarrow X_i \le D_i.$$

We define the $Y_i=1$ as default of obligor i at time t and $Y_i=0$ as non-default.

3 DEFAULT FORECASTING MODELS OF OBLIGOR

In this section, we take into account the defaultable probability of an obligor. Suppose that the credit quality of the obligor at time t is completely determined by its financing index $X=(X_{1t},\cdots,X_{mt})^T$ which is a $m-$ dimensional observable random vector with continuous marginal distributions F_i and density functions f_i, $i=1,\cdots,m$, respectively. The financing indexes correlations are calibrated by assuming that they follow a factor model, where the underlying factors are interpreted as a set of macroeconomic variables. Let $D_1,\cdots,D_m$ be a vector of deterministic cut-off levels of financing indexes, respectively, for determining the thresholds D_i, $i=1,\cdots,m$, an option pricing technique based on historical firm value data is used. Under normal operation, commonly $X_{it}>D_i, i=1,\cdots,m$, when the credit rating of the obligor decreases, some financing indexes become $X_{i_k t}\le D_{i_k}$, $\{i_1,\cdots,i_k\}\subset\{1,\cdots,m\}$. Moreover, the procedure of the credit rating decreasing is an asymptotic process. The lower the credit rating is, the higher the probability of obligor's default is. Let $X=(X_{1t},\cdots,X_{mt})^T$ have joint distribution H such that

$$H(x_1,x_2,\cdots,x_m)=P\{X_{1t}\le x_1,X_{2t}\le x_2,\cdots,X_{mt}\le x_m\},$$

By Sklar's theorem, there exists a copula function such that

$$H(x_1,x_2,\cdots,x_m)=C(F_1(x_1),F_2(x_2),\cdots,F_m(x_m)).$$

So that copula C represents the probability of event $X_{1t}\le x_1,X_{2t}\le x_2,\cdots,X_{mt}\le x_m$ and simultaneously describes the dependence structure of financing indexes. The joint probability of all financing indexes being less than the cut-off levels can be calculated by

$$P\{X_{1t}\le D_1,X_{2t}\le D_2,\cdots,X_{mt}\le D_m\}$$
$$=C(F_1(D_1),F_2(D_2),\cdots,F_m(D_m)).$$

Now we consider that j financing indexes $X_{i_1 t},\cdots,X_{i_j t}$ among $X_{1t},\cdots,X_{mt}$ are more than the cut-off levels and others $m-j$ financing indexes are less than the cut-off levels. For simplify representing, we take into account the probability $P\{X_{1t}>D_1,\cdots X_{jt}>D_j,X_{j+1t}\le D_{j+1}\cdots,X_{mt}\le D_m\}$,

$$P\{X_{1t}>D_1,\cdots X_{jt}>D_j,X_{j+1t}\le D_{j+1}\cdots,X_{mt}\le D_m\}$$
$$=C(1,\cdots,1,F_{j+1}(D_{j+1}),\cdots,F_m(D_m))-\sum \mathrm{sgn}(c)C(c,F_{j+1}(D_{j+1}),\cdots,F_m(D_m))$$
(2)

Where $c=(F_1(D_1),F_2(D_2),\cdots,F_j(D_j))$, $F_i(D_i)=1$ or $F_i(D_i)$, $i=1,\cdots,j$,

$$\mathrm{sgn}(c)=\begin{cases}1, & if\ \ F_i(D_i)\neq 1 for\ \ an odd\ \ number\ \ of\ \ i's,\\ -1, & if\ \ F_i(D_i)\neq 1 for\ \ an even number\ \ of\ \ i's.\end{cases}$$

$i=1,2,\cdots,j$.

By the properties of copula functions, we know the $C(1,\cdots,1,F_{j+1}(D_{j+1}),\cdots,F_m(D_m))$ is the marginal copula of the copula C of X_t (and then is the copula of variables $(X_{j+1t},\cdots,X_{mt})$). Others are similar.

One is often interested in estimating or forecasting certain conditional probability, such as under the condition of some financing indexes lower than the cut-off level values, calculating the probability of which the remains are more than the cut-off levels.

$$P\{X_{1t}>D_1,\cdots X_{jt}>D_j\,|\,X_{j+1t}\le D_{j+1}\cdots,X_{mt}\le D_m\}$$
$$=\frac{P\{X_{1t}>D_1,\cdots X_{jt}>D_j,X_{j+1t}\le D_{j+1}\cdots,X_{mt}\le D_m\}}{P\{X_{j+1t}\le D_{j+1}\cdots,X_{mt}\le D_m\}}$$
$$=\frac{C(1,\cdots,1,F_{j+1}(D_{j+1}),\cdots,F_m(D_m))-\sum \mathrm{sgn}(c)C(c,F_{j+1}(D_{j+1}),\cdots,F_m(D_m))}{C(1,\cdots,1,F_{j+1}(D_{j+1}),\cdots,F_m(D_m))} \quad (3)$$

In similarly, we can obtain the conditional probability

$P\{X_{1t}\le D_1,\cdots,X_{jt}\le D_j\,|\,X_{j+1t}\le D_{j+1},\cdots,X_{mt}\le D_m\}$.

Furthermore, the conditional probability of $(X_{j+1t}\le D_{j+1},\cdots,X_{mt}\le D_m)$ under the condition $(X_{1t}>D_1,\cdots,X_{jt}>D_j)$.

$$P\{X_{j+1t}\le D_{j+1}\cdots,X_{mt}\le D_m\,|\,X_{1t}>D_1,\cdots X_{jt}>D_j\}$$
$$=\frac{P\{X_{1t}>D_1,\cdots X_{jt}>D_j,X_{j+1t}\le D_{j+1}\cdots,X_{mt}\le D_m\}}{P\{X_{1t}>D_1,\cdots X_{jt}>D_j\}}$$
$$=\frac{C(1,\cdots,1,F_{j+1}(D_{j+1}),\cdots,F_m(D_m))-\sum \mathrm{sgn}(c)C(c,F_{j+1}(D_{j+1}),\cdots,F_m(D_m))}{1-\sum \mathrm{sgn}(c)C(c,1,\cdots,1)} \quad (4)$$

According to the definition and properties of probability, we can obtain the conditional probability density function

$$f_{X_{j+1}\cdots X_m|X_1\cdots X_j}(x_{j+1},\cdots,x_m)$$
$$=\lim_{\substack{\Delta x_{j+1}\to 0^+\\ \vdots \\ \Delta x_m\to 0^+}}\frac{P\{x_{j+1}<X_{j+1t}\le x_{j+1}+\Delta x_{j+1}\cdots,x_m<X_{mt}\le x_m+\Delta x_m\,|\,X_{1t}>D_1,\cdots X_{jt}>D_j\}}{\Delta x_{j+1}\cdots\Delta x_m}$$
$$=\frac{C_{j+1,\cdots,m}(1,\cdots,1,F_{j+1}(x_{j+1}),\cdots,F_m(x_m))-\sum \mathrm{sgn}(c)C_{j+1,\cdots,m}(c,F_{j+1}(x_{j+1}),\cdots,F_m(x_m))}{1-\sum \mathrm{sgn}(c)C(c,1,\cdots,1)}$$
$$\cdot f_{j+1}(x_{j+1})\cdots f_m(x_m). \quad (5)$$

where

$$C_{j+1,\cdots,m}(\cdots,u_{j+1},\cdots,u_m)=\frac{\partial^{m-j}C(\cdots,u_{j+1},\cdots,u_m)}{\partial u_{j+1}\cdots\partial u_m}.$$

In this section, we achieve some dependence conditional probability functions and its density functions under the conditions of some financing indexes lower or more than the cut-off levels. These results are very useful in credit risk management, since the risk analysts need to analyse the conditional dependence structure and conditional dependence measure of financing indexes according to given some conditions. Especially, the conditional dependence risk probability and density functions have particular meaning when the threshold values equal some certain values such as $D_i=0$ and $D_i=VaR_\alpha(X_i)$.

In realistic application, it is very difficult that the default of an obligor is exactly forecasted by the obligor's credit rating or its operating situation. The default is a result which is affected by many factors. Obligors have the same credit rating or similar operating situation, but they have different credit results. The default is capable of contagion among obligors. Whenever an obligor suddenly defaults, investors learn about the default threshold of closely associated business partner obligors. This updating leads to "contagious" jumps in credit spreads of business partners.

4 CONCLUSIONS

In this paper, we have studied the dependence structure of financing indexes of obligor, and the conditional dependence risk probability and the conditional dependence risk density functions. We mainly focus on the scenarios under the conditions such as $\left(X_{j+1t} \leq D_{j+1}, \cdots; X_{mt} \leq D_m\right)$ and $\left(X_{1t} > D_1, \cdots; X_{jt} > D_j\right)$. A member of conditional dependence risk relationships is completely characterized by the marginal distribution and the copulas of random variables. These results can be applied to investigate the conditional dependence structure and the conditional dependence measure of obligor's assets and of the defaults among obligors.

FUNDING

This work was sponsored by Project Supported by the Program for Innovation Team Building at Institutions of Higher Education in Chongqing (Grant No.KJTD201321) and the National Social Science Fund of China (Grant No. 15XJY023).

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