The Long-Time Behaviour of a Stochastic Plankton Food Chain Model

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Abstract: This paper deals with problems of a tri-trophic plankton food chain model under environmental noise. When considering plankton models, people are interested in when the plankton will be persist and extinct in a long time. Sufficient conditions for the existence of an ergodic stationary distribution and extinction are established. This work can predict the variation trend of the population of each species, so as to better manage the population and provide theoretical basis for the safety and protection of the environment.

1 INTRODUCTION

In recent years, with the increasing population in coastal areas and the rapid development of industry and mariculture, coastal ecological environment has been seriously damaged. The frequent occurrence of red tide, sudden outbreak of fish diseases and devastating death of large areas of cultured fish are all related to the destruction of ecological environment (Wang 2018). People have gradually realized the importance of protecting marine environment after paying a huge economic cost, and they are eager to know and use the ocean from a scientific perspective. Therefore, it has become a scientific research strategy for coastal countries to establish a mathematical model that can be applied to marine ecological environment and predict the balance and evolution of marine ecological system. Phytoplankton play an important role in aquatic ecosystem (Guillard 1975). Therefore it is necessary to study the dynamic mechanism of the population growth of phytoplankton.

At present, studying the phytoplankton population growth by ecological model has become a hot research topic. Scholars all over the world have established and analysed a large number of different types of ecological model (such as ordinary differential equations, delayed differential equations, differential equation of diffusion.) to describe the phytoplankton population growth and diffusion process (Abdallah 2003, Fang 2017, Ghosh 2016, He 2017, Samanta 2013, Smith 2015). As far as we know, the mathematical model on the tri-trophic food chain has not been theoretically explored. However, we believe that this study may open many windows for population dynamics and require in-depth research in this area.

In this work, we use the model of Hastings and Powell (Hastings 1991) as a starting point to construct a model that includes intraspecies competition and white noise. The deterministic tri-trophic food-chain model is as follows:

$$\begin{cases} \dfrac{dx}{dt} = x(1-x) - \dfrac{a_1xy}{1+b_1x}, \\ \dfrac{dy}{dt} = \dfrac{a_1xy}{1+b_1x} - \dfrac{a_2yz}{1+b_2y} - d_1y - c_1y^2, \\ \dfrac{dz}{dt} = \dfrac{a_2yz}{1+b_2y} - d_2z - c_2z^2. \end{cases} \quad (1)$$

The model describes the rates of change in densities of a basal species (x), its predator (y) and a top predator (z). It includes logistic growth of the basal species and Holling Type II functional responses. In a typical example, x , y and z might

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represent the population densities of phytoplankton, herbivorous zooplankton and carnivorous zooplankton, respectively. The parameters are as follows: $\frac{a_1xy}{1+b_1x}$ and $\frac{a_2yz}{1+b_2y}$ are predation rates of the two predators, respectively; d_1y and d_2z are mortality rates of the two predators; c_1y^2 and c_2z^2 denote the density regulation of the two predators.

There are four nonnegative equilibria for system (1): $E_0(0,0,0)$, $E_1(1,0,0)$, $E_2(x_1,y_1,0)$ and $E^*(x^*,y^*,z^*)$, where $(x_1,y_1,0)$ and (x^*,y^*,z^*) satisfy

$$\begin{cases} x_1+\dfrac{a_1y_1}{1+b_1x_1}=1, \\ \dfrac{a_1x_1}{1+b_1x_1}-c_1y_1=d_1, \end{cases} \tag{2}$$

$$\begin{cases} x^*+\dfrac{a_1y^*}{1+b_1x^*}=1, \\ \dfrac{a_1x^*}{1+b_1x^*}-\dfrac{a_2z^*}{1+b_2y^*}-c_1y^*=d_1, \\ \dfrac{a_2y^*}{1+b_2y^*}-c_2z^*=d_2, \end{cases} \tag{3}$$

respectively. E_2 is nonnegative equilibria if there is positive solution of Equations (2) and E^* is a positive equilibrium if there is a positive solution of Equation (3). In this paper, we assume E_2 and E^* always exist as nonnegative equilibria.

Environmental stochastic perturbations can affect population dynamics inevitably. Real population systems are always exposed to uncertain environmental factors (Wen 2015, Zhao 2016). Motivated by above facts, in this paper, taking into account the effect of randomly fluctuating environment, we assume that the parameters involved in the model (1) fluctuate around some average value. Thus we study a stochastic plankton food chain model:

$$\begin{cases} dx=x\left[(1-x)-\dfrac{a_1y}{1+b_1x}\right]dt+\sigma_1xdB_1(t), \\ dy=y\left[\dfrac{a_1x}{1+b_1x}-\dfrac{a_2z}{1+b_2y}-d_1-c_1y\right]dt+\sigma_2ydB_2(t), \\ dz=z\left[\dfrac{a_2y}{1+b_2y}-d_2-c_2z\right]dt+\sigma_3zdB_3(t), \end{cases} \tag{4}$$

where $B_1(t)$, $B_2(t)$ and $B_3(t)$ are standard one-dimensional independent Brownian motion, and $\sigma_i>0$ are the intensity of the white noise, $i=1,2,3$.

2 GLOBAL DYNAMICS

When considering plankton models, we are interested in when the plankton will be persist and extinct in a long time. Here, we will talk about the persistence and extinction of system (4).

2.1 Ergodicity

Ergodicity is one of the most significant characteristics, meaning the stochastic plankton model has a stationary distribution which shows the survival of the plankton in the future. Now we state sufficient conditions for the existence and uniqueness of an ergodic stationary distribution of the positive solutions to system (4) in the following theorem.

Theorem 2.1. Assume that the following conditions hold

$$1>\frac{a_1b_1y^*}{1+b_1x^*}, \tag{5}$$

$$c_1>\frac{a_2b_2z^*}{1+b_2y^*} \tag{6}$$

and

$$\frac{\sigma_1^2x^*}{2}+\frac{\sigma_2^2(1+b_1x^*)y^*}{2}+\frac{\sigma_3^2(1+b_1x^*)(1+b_2y^*)z^*}{2} < \min\left\{\begin{array}{l}\left(1-\dfrac{a_1b_1y^*}{1+b_1x^*}\right)(x^*)^2, \\ \left(c_1-\dfrac{a_2b_2z^*}{1+b_2y^*}\right)(y^*)^2, \\ (1+b_1x^*)(1+b_2y^*)c_2(z^*)^2\end{array}\right\}, \tag{7}$$

where $E^*(x^*,y^*,z^*)$ is the positive equilibrium of system (1), then system (4) admits a stationary distribution which is ergodic.

Proof. According to the analysis in (Liu 2019, Zhu 2007), we only need to show there exists a nonnegative C^2 -function V and a neighbourhood U such that LV is negative for any $(x(t),y(t),z(t))\in R_+^3\setminus U$. Construct a C^2 -function $V: R_+^3 \to R$ in the following form

$$V(x,y,z)=\left(x-x^*-x^*\log\frac{x}{x^*}\right) +A_1\left(y-y^*-y^*\log\frac{y}{y^*}\right) +A_2\left(z-z^*-z^*\log\frac{z}{z^*}\right), \quad (8)$$

where $A_1=1+b_1x^*$ and $A_2=(1+b_1x^*)(1+b_2y^*)$.

Making use of $It\hat{o}'s$ formula and combing with (3) lead to

$$\begin{aligned} LV &= -(x-x^*)^2+\frac{a_1b_1y^*(x-x^*)^2}{(1+b_1x^*)(1+b_1x)} +\frac{x^*\sigma_1^2}{2}-A_1c_1(y-y^*)^2 \\ &+\frac{A_1a_2b_2z^*(y-y^*)^2}{(1+b_2y^*)(1+b_2y)}+\frac{A_1y^*\sigma_2^2}{2} -A_2c_2(z-z^*)^2+\frac{A_2z^*\sigma_3^2}{2} \\ &\le -\left[1-\frac{a_1b_1y^*}{1+b_1x^*}\right](x-x^*)^2- A_1\left[c_1-\frac{a_2b_2z^*}{1+b_2y^*}\right](y-y^*)^2-A_2c_2 \cdot(z-z^*)^2+\frac{x^*\sigma_1^2}{2}+\frac{A_1y^*\sigma_2^2}{2}+\frac{A_2z^*\sigma_3^2}{2} \\ &:= -m_1(x-x^*)^2-m_2(y-y^*)^2-m_3 \cdot(z-z^*)^2+\frac{x^*\sigma_1^2}{2}+\frac{A_1y^*\sigma_2^2}{2}+\frac{A_2z^*\sigma_3^2}{2}, \end{aligned} \quad (9)$$

where $m_1,m_2,m_3>0$ according to the conditions (5) and (6).

Define a bounded closed set

$$U=\left\{(x,y,z)\in R_+^3: \varepsilon\le x\le\frac{1}{\varepsilon},\varepsilon\le y\le\frac{1}{\varepsilon},\varepsilon\le z\le\frac{1}{\varepsilon}\right\}, \quad (10)$$

where $0<\varepsilon<1$ is a sufficiently small number. In the set $R_+^3\setminus U$, we can choose ε sufficiently small such that the following conditions hold

$$\varepsilon<\frac{1}{4x^*m_1}\left[m_1(x^*)^2-\left(\frac{x^*\sigma_1^2}{2}+\frac{A_1y^*\sigma_2^2}{2}+\frac{A_2z^*\sigma_3^2}{2}\right)\right], \quad (11)$$

$$\varepsilon<\frac{1}{4y^*m_2}\left[m_2(y^*)^2-\left(\frac{x^*\sigma_1^2}{2}+\frac{A_1y^*\sigma_2^2}{2}+\frac{A_2z^*\sigma_3^2}{2}\right)\right], \quad (12)$$

$$\varepsilon<\frac{1}{4z^*m_3}\left[m_3(z^*)^2-\left(\frac{x^*\sigma_1^2}{2}+\frac{A_1y^*\sigma_2^2}{2}+\frac{A_2z^*\sigma_3^2}{2}\right)\right]. \quad (13)$$

For convenience, we can divide $R_+^3\setminus U$ into six domains,

$$U_1=\{(x,y,z)\in R_+^3: x<\varepsilon\},$$
$$U_2=\{(x,y,z)\in R_+^3: y<\varepsilon\},$$
$$U_3=\{(x,y,z)\in R_+^3: z<\varepsilon\},$$
$$U_4=\left\{(x,y,z)\in R_+^3: x>\frac{1}{\varepsilon}\right\},$$
$$U_5=\left\{(x,y,z)\in R_+^3: y>\frac{1}{\varepsilon}\right\},$$
$$U_6=\left\{(x,y,z)\in R_+^3: z>\frac{1}{\varepsilon}\right\}.$$

Obviously, $R_+^3\setminus U=U_1\cup U_2\cup U_3\cup U_4\cup U_5\cup U_6$. Next, we will prove that $LV(x,y,z)<-1$ for any

$(x,y,z)\in(R_+^3\setminus U)$, which is equivalent to proving it on the above six domains.

If $(x,y,z)\in U_1$, by (9), we have

$$LV(x,y,z)\le -m_1(x-x^*)^2 + \frac{x^*\sigma_1^2}{2}+\frac{A_1y^*\sigma_2^2}{2}+\frac{A_2z^*\sigma_3^2}{2} \le 2m_1x^*\varepsilon - m_1(x^*)^2 + \frac{x^*\sigma_1^2}{2}+\frac{A_1y^*\sigma_2^2}{2}+\frac{A_2z^*\sigma_3^2}{2}\le -\frac{1}{2}\left[m_1(x^*)^2-\left(\frac{x^*\sigma_1^2}{2}+\frac{A_1y^*\sigma_2^2}{2}+\frac{A_2z^*\sigma_3^2}{2}\right)\right]\le 0, \tag{14}$$

which follows from (7) and (11). Similarly, we obtain $LV(x,y,z)<0$ on U_2 using the inequalities (7), (12) and $LV(x,y,z)<0$ on U_3 according to (7), (13).

If $(x,y,z)\in U_4$, $(x,y,z)\in U_5$ or $(x,y,z)\in U_6$, in view of (9), we get $LV(x,y,z)<0$.

Therefore, we have a conclusion that $LV(x,y,z)<0, \forall(x,y,z)\in \mathrm{R}_+^3\setminus U$.

So we obtain that system (4) has an ergodic stationary distribution. This completes the proof.

2.2 Extinction

We establish sufficient criteria for extinction of the plankton in three cases. Before giving the main result, we first give a lemma.

Lemma 2.1. Let $X(t)$ be the solution of the stochastic differential equation

$$dX = X(1-X)dt+\sigma_1 X dB_1(t), \tag{15}$$

then $X(t)$ converges weakly to distribution ν and ν is a probability measure in R_+ such that $\int_0^\infty u\nu(du)=1-\frac{\sigma_1^2}{2}$ and its density is $\left(Q\sigma_1^2u^2p(u)\right)^{-1}$, where

$$Q=\left[\sigma_1^{-2}\left(\frac{\sigma_1^2}{2}\right)^{\frac{2}{\sigma_1^2}-1}\Gamma\left(\frac{2}{\sigma_1^2}-1\right)\right]^{-1}$$

is a normal constant and $p(u)=u^{-\frac{2}{\sigma_1^2}}e^{\frac{2}{\sigma_1^2}u}$, $u>0$.

Since the proof is similar to (Liu 2012), Theorem 4.1 and we omit it here.

According to the Lemma, we get the following result. For simplicity, we introduce the notations $\langle f(t)\rangle_t=\frac{1}{t}\int_0^t f(s)ds$ and $\to^w$ means the convergence in distribution.

Theorem 2.2. Let $(x(t),y(t),z(t))$ be the solution of system (1.4) with any initial value $(x(0),y(0),z(0))\in R_+^3$.

(i) If $1<\frac{\sigma_1^2}{2}$, then all the plankton tend to zero exponentially with probability one, i.e.,

$$\lim_{t\to\infty}x(t)=\lim_{t\to\infty}y(t)=\lim_{t\to\infty}z(t)=0 \text{ a.s.} \tag{16}$$

(ii) If $1>\frac{\sigma_1^2}{2}$ and $a_1\int_0^\infty\frac{u\nu(du)}{a_1+u}<d_1+\frac{\sigma_2^2}{2}$, then $\lim_{t\to\infty}\langle x(t)\rangle_t=1-\frac{\sigma_1^2}{2}$ a.s., $x(t)\to^w\nu$ as $t\to\infty$,

$$\limsup_{t\to\infty}\frac{\log y(t)}{t}\le a_1\int_0^\infty\frac{u\nu(du)}{1+b_1u}-\left(d_1+\frac{\sigma_2^2}{2}\right)<0, \text{ a.s.}$$
$$\limsup_{t\to\infty}\frac{\log z(t)}{t}<0 \text{ a.s.} \tag{17}$$

where ν is a probability measure in R_+ which is defined in Lemma 2.1. That is to say, the phytoplankton $x(t)$ is persistent, while two kinds of zooplankton $y(t),z(t)$ go to extinction with probability one.

(iii) If $1>\frac{\sigma_1^2}{2}$, $a_1\int_0^\infty\frac{u\nu(du)}{a_1+u}>d_1+\frac{\sigma_2^2}{2}$, $1>\frac{a_1b_1y_1}{1+b_1x_1}$, $d_2>\frac{a_2y_1}{1+b_2y_1}$ and

$$\frac{a_2 y_1}{1+b_2 y_1}+\frac{a_2}{1+b_2 y_1}\left[\frac{x_1\sigma_1^2+(1+b_1x_1)y_1\sigma_2^2}{2c_1(1+b_1x_1)}\right]^{\frac{1}{2}}$$
$$< d_2+\frac{\sigma_3^2}{2},$$

then

$$\lim_{t\to\infty}\langle x(t)\rangle_t = 1-\frac{\sigma_1^2}{2} \text{ a.s.}, x(t)\to^{w} \nu \text{ as } t\to\infty,$$
$$\liminf_{t\to\infty}\langle y(t)\rangle_t \geq$$
$$\frac{1}{a_1^2+c_1}\left[a_1\int_0^{\infty}\frac{u\nu(du)}{a_1+u}-\left(d_1+\frac{\sigma_2^2}{2}\right)\right]>0,$$
$$\limsup_{t\to\infty}\frac{\log z(t)}{t}\leq\frac{a_2 y_1}{1+b_2 y_1}-\left(d_2+\frac{\sigma_3^2}{2}\right)$$
$$+\frac{a_2}{1+b_2 y_1}\left[\frac{x_1\sigma_1^2+(1+b_1x_1)y_1\sigma_2^2}{2c_1(1+b_1x_1)}\right]^{\frac{1}{2}}<0, \tag{18}$$

where (x_1, y_1, z_1) is the boundary equilibrium of system (2) and ν is defined in Lemma 2.1. That is to say, the phytoplankton $x(t)$ and herbivorous zooplankton $y(t)$ are persistent, while carnivorous zooplankton $z(t)$ go to extinction with probability one.

The proof is quite similar to Theorem 4.1 in (Liu 2019), so we omit it here.

3 CONCLUSIONS

In this work, we use the model of Hastings and Powell as a starting point to construct a more realistic tri-trophic plankton food chain model (4) that includes environmental perturbation. The persistence and extinction of the plankton in a long time are discussed.

Theorem 2.1 implies that the plankton will be persistent, that is to say, all the phytoplankton and zooplankton will survive in a long time under certain conditions. Theorem 2.2 shows that the plankton will extinct in three cases.

We have explored dynamic behaviors of the food chain model. We believe that this work can predict the variation trend of the population of each species, so as to better manage the population and provide theoretical basis for the safety and protection of the environment.

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