

# Statistical Model Checking for Probabilistic Temporal Epistemic Logics

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**Abstract:** Interpreted Systems and epistemic temporal logics have been employed extensively to study the notion of knowledge in Multi-Agent Systems. New model checking algorithms, as well as adaptations of existing algorithms to this setting have been reported. For the most part, these algorithms have focused on exhaustive state space exploration based approaches. While these approaches yield accurate results to model checking queries, they are often expensive for realistic scenarios. So much so that, many of the applications studied in academic literature deal with small state spaces. In order to scale to real life multi-agent systems with large state spaces, an alternative to exhaustive exploration based techniques is needed. Statistical Model Checking was proposed to alleviate this problem when model checking stochastic systems against temporal logic queries. In this paper, we extend this technique to epistemic temporal logics. The first version of the approach, which we call the vanilla approach, would be to simply generate Monte Carlo samples of the runs of the system and evaluate the query on them. The advantage that SMC is expected to bring is greatly diminished due to the knowledge operator in such systems of logic. For large systems, this would entail an exhaustive exploration of epistemically accessible global states. Our major contribution is to introduce a sampling based approach for the knowledge operator as well. We show that this results in significant performance gains at the expense of a marginal loss in accuracy (1-2% in experimental results) for most epistemic operators. Specifically, we show evidence of a dramatic improvement in time complexity for large Multi-Agent Systems. We substantiate the effectiveness of the approach through case studies that involve a large number of agents.

## 1 INTRODUCTION

Multi-Agent Systems (MASs) have been used extensively in the recent past to model Artificial Intelligence, emergent behavior and cooperative and adversarial systems. Several complex systems and decision support systems have been modeled and studied as MAS. Such systems have been subjected to study and analysis through techniques like formal methods in Computer Science. Of particular interest is analysis through model checking.

Traditionally, model checking has been used extensively to study reactive systems (Baier and Katoen, 2008). Model Checking involves algorithmically deciding whether or not a system satisfies a desired property. Naturally, both the system and the desired property have to be described formally to be fed to the algorithm. The system is usually described through some variant of a Kripke Structure and the property, as a formula in an appropriate temporal logic (Ben-

Ari et al., 1981; Pnueli, 1977; Vardi, 1996). For example, modeling formalisms like Labeled Transition Systems have successfully captured evolution dynamics of reactive systems. However, they fall short of being able to model features of MAS that are pertinent to artificial intelligence and decision making.

Fagin et al. (Fagin et al., 2003) proposed the use of epistemic modal logic to capture the notion of *knowledge* possessed by an agent. The knowledge operator that they defined for computationally grounded semantics (“Agent  $i$  knows that  $\phi$ ”) can also be extended to other epistemic modalities like *group knowledge*, *distributed knowledge* and *common knowledge*. The introduction of the *Interpreted Systems* formalism allowed MAS to be modeled by Kripke-like structures enhanced by the *epistemic* or the *knowledge* component, giving a computationally grounded semantics to these modalities. Thus, it became possible to model both dynamics and knowledge possessed by agents naturally. Coupled with various systems of Epistemic Temporal Logic (ETL), this provided a potent tool for analysis of MAS. In this work, we use a probabilistic extension of Interpreted Systems. We mention

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that other versions of modal logic with, for example, deontic and doxastic modalities have also been proposed, which we will not consider in this work (Kaufmann et al., 2008).

For stochastic reactive systems, an alternative approach to exact model checking algorithms (that involve exhaustive state space exploration) was proposed (Nimal, 2010; Agha and Palmkog, 2018; Younes et al., 2010; Legay et al., 2010). This approach, called Statistical Model Checking (SMC) is essentially based on a Monte-Carlo sampling of the runs of the system. Intuitively, the fraction of runs that evaluate the formula to *TRUE* decides the outcome of the model checking query. This variant has found particular traction in situations where a trade-off between accuracy and time/space complexity is acceptable (Younes et al., 2004). This approach has been extended to different variants of Stochastic Multi Agent Systems (SMASs) as well (Herd et al., 2015b; Herd et al., 2015a).

In this work, we propose an SMC algorithm to check Probabilistic Epistemic Temporal logic K-PCTL(B) against a stochastic version of Interpreted Systems. A straightforward adaptation of the SMC approach to this problem would be to simulate several runs of the system and evaluate the epistemic temporal logic formulas on them. Indeed, this does give a first cut SMC algorithm. However, a bottleneck remains for efficient evaluation of the knowledge operator at a time step.

In the worst case, a (Stochastic) Multi Agent System with  $n$  agents having  $k$  local states each, has a state space of size  $k^n$ . This results in a running time that is exponential in the number of agents. For systems that have a large number of agents, such time complexity is expensive—most of the case studies reported in literature involve small sized models. As example of a system whose analysis can be expensive, consider the spread of a pandemic among  $n$  agents. Suppose  $\phi$  is a propositional formula that evaluates to *TRUE* iff at least 60% agents are infected. A pertinent query that would involve both temporal and epistemic operators is “What is the probability that agent  $i$  knows that  $\phi$  before  $\tau$  time steps?”. Another example, which has been constructed by Wan et al (Wan et al., 2013), is that of online supermarkets. This system has two agents, a customer agent and a server agent. A useful query in this context is “what is the probability that when a customer places an order online, she knows that at least 90% of the items will be shipped successfully?”.

Given this, we propose a sampling approach to evaluate the knowledge operator as well. This needs some innovation in the way the sampling is done, in

accordance with the knowledge operator. We analyze the impact of this sampling approach on the accuracy, and the resulting speed-up. We show empirical evidence for a substantial improvement over the brute force SMC for queries of nesting level of one. We remark that we do not report evaluation of our sampling approach against numerical approaches, as it is well known that they do not scale well for systems with a large state space. While the advantage of the sampling algorithm over the brute force SMC is to be expected, what is surprising is the degree of accuracy that is achieved for a running time that is several orders of magnitude lesser.

The paper is structured as follows. We begin, in the next section, with a brief introduction to some preliminary ideas, terminology and notation that will be used in the rest of the paper. In section 3, we discuss the SMC algorithms for various epistemic operators, both brute force (which we call the “Vanilla” SMC algorithm) and the Epistemic Sampling (ES) algorithm. In section 4, we report case studies that substantiate the effectiveness of the ES algorithm. To put our work in context with some of the existing state of the art, we discuss related work in section 5. We conclude the paper in section 6, with a brief discussion of future directions.

## 2 PRELIMINARIES

In this section, we discuss some basic definitions and concepts that will be useful subsequently.

A Multi Agent System consists of several agents, each of which can be in one of several *local* states. It is also customary to designate a special *environment* agent that is distinct from all other agents. Formally, let  $\mathcal{A} = \{1, \dots, n\}$  denote a set of  $n$  agents. Corresponding to each agent  $i \in \mathcal{A}$  is a set  $L_i$  of *local states*, and a set of actions  $Act_i$ . We denote the state of agent  $i$  at time  $t$  by  $l_i(t) \in L_i$ . Similarly,  $l_e(t)$  for the environment. Evolution of the system is essentially change of states of these agents. The state space of the entire system can therefore be described set of *global* states  $G \subseteq L_1 \times L_2 \times \dots \times L_n$ . One can then talk of a global state  $g(t) \in G$  at time  $t$  that is a tuple aggregated from the local states:  $g(t) = (l_1(t) \dots l_n(t) l_e(t))$ .

Associated with each agent  $i$  is a protocol  $A_i$ , and with the environment, a protocol  $A_e$  that changes its local state at each time step, in accordance with the corresponding the action:

$$l_i(t) \xrightarrow{A_i : \alpha \in Act_i} l_i(t+1)$$

for  $l_i(t), l_i(t+1) \in L_i$ . Similarly, for the environment

agent:

$$l_e(t) \xrightarrow{A_e : \alpha \in Act_e} l_e(t+1)$$

Consequently, the global state evolves from  $g(t) = (l_1(t) \dots l_n(t) l_e(t))$  to  $g(t+1) = (l_1(t+1) \dots l_n(t+1) l_e(t+1))$ . We can also define the *initial* global state:  $g_0 = (l_1^m, \dots, l_n^m)$ , where  $l_i^m$  are the local initial states. We denote the local state of an agent  $i$  within a global state  $g(t)$  by  $g(t)_i$ .

We can also define a labeling function  $\mathcal{L} : G \rightarrow 2^{AP}$ , where  $AP$  is a set of atomic propositions that hold true on a global state. An *Interpreted System* then, is the tuple  $\langle (L_i, A_i)_{i \in \mathcal{A}}, g_0, AP, \mathcal{L} \rangle$ , where  $i$  runs over all agents and  $g_0$  is a starting global state.

The evolution protocol is specific to the application. Since we are interested in stochastic multi-agent systems in this paper, we now discuss evolution for such systems.

## 2.1 Stochastic Multi Agent Systems

A simple example of stochastic evolution is the case when the agents are modeled as a Discrete Time Markov Chain (DTMC). The evolution protocol  $A_i$  for an agent  $i$  is then simply a stochastic matrix  $P_i$  with elements  $p_{l_i, l'_i}$ :

$$l_i(t) \xrightarrow{p_{l_i, l'_i}} l'_i(t+1),$$

where  $p_{l_i, l'_i}$  is the probability of transition from the state  $l_i$  to  $l'_i$  and  $\sum_{l'_i} p_{l_i, l'_i} = 1$ . In the simplest setting, action symbols may be omitted. The Stochastic Multi Agent System is then merely an aggregation of DTMCs—one DTMC for each agent. For incorporating synchronization and interaction, they can be augmented with action symbols (Delgado and Benevides, 2009). For the purposes of demonstrating the sampling approach over epistemic operators, the former formulation suffices. Indeed, this is what we employ in this paper.

## 2.2 Epistemic Accessibility

For each agent  $i$ , we define the *epistemic accessibility* relation  $\sim_i$  between the global states in  $G$  at time snapshot  $t$  as follows:  $g(t) \sim_i g'(t)$  if and only if  $g(t)_i = g'(t)_i$ . In other words, two global states belong to the same equivalence class with respect to  $\sim_i$  if and only if the local state of agent  $i$  is the same in both the global states. Epistemically, these global states are indistinguishable for agent  $i$ . Alternatively, these global states are said to be epistemically accessible to each other. Informally speaking, if a fact “holds” in all such global states, the agent is said to *know* the fact. We will formalize this notion later in subsection

2.4, while defining the semantics of epistemic formulas.

We are now in a position to define *global model* for an SMAS, which is a tuple  $\langle G, g_0, \{P_i, \sim_i\}_{i \in \mathcal{A}}, AP, \mathcal{L} \rangle$  where

- $G$ , the set of global states,  $g_0$  an initial global state,  $P_i$  the stochastic transition matrices for each agent  $i$  and  $\sim_i$ , the epistemic accessibility relations are as defined earlier.
- $AP$  is a set of *atomic propositions*
- $\mathcal{L} : G \rightarrow 2^{AP}$  is a labeling function that assigns atomic propositions to the states in  $G$ .

Such a structure can be subjected to analysis against desired properties through model checking. In this paper, we will use K-PCTL(B) as the temporal epistemic logic for formulating queries. As a running example, we model simple epidemic dynamics in this formalism. At any given point in time, every individual (agent) in the population is in one of the following health states (Kermack and McKendrick, 1927): Susceptible (S), Infected (I) or Recovered (R). The transition from S to I to R is modeled as a simple three-state DTMC. We use one atomic proposition  $a$  on the set of global states that evaluates to *TRUE* for all and only those global states with at least 60% of the agents in the I state. Then an example query of interest is: “what is the probability that agent  $i$  knows that at least 60% of the population is infected?”

## 2.3 Measurability

An important criterion for statistical model checking, and indeed probabilistic model checking, is that the set of runs or trajectories should form a measurable set (Agha and Palmkog, 2018). It turns out that this is indeed the case for a large variety of practical stochastic systems—ranging from DTMCs to stochastic discrete event systems. In particular, in our setting, the definitions of the measurable space—the sample space, the sigma algebra and the probability measure—that is valid for Probabilistic Computation Tree Logic (PCTL) model checking for DTMCs remain unchanged (Baier and Katoen, 2008). As we will see in the next section, the only change will be in the statistical model checking algorithm’s ability to estimate the probability accurately. Specifically, the algorithm for assigning truth values to the (state) knowledge operators will be a (one-sided error) randomized algorithm, prone to false positives.

## 2.4 K-PCTL(B)

Probabilistic extensions to Epistemic Temporal Logics have been explored extensively in the past. For

example, Delgado and Benevides (Delgado and Benevides, 2009) introduced the Probabilistic Epistemic Temporal Logic (PETL), which was revisited by Fu et al (Fu et al., 2018) as K-PCTL. K-PCTL is PCTL augmented with epistemic modalities. Since PCTL(B), (PCTL with bounded *Until*) is a fragment of K-PCTL(B), we will review PCTL syntax and semantics when discussing K-PCTL(B). For statistical model checking, the most conducive approach is to replace the *Until* fragment of K-PCTL with the *bounded Until* fragment. This is because, as will be seen shortly, traditional SMC algorithms work by sampling finite-length runs of the system. We call this version the Bounded Probabilistic Computation Tree Logic of Knowledge (K-PCTL(B)).

Syntactically, K-PCTL(B) has the following grammar<sup>1</sup>:

$$\begin{aligned} \phi & := \text{TRUE} \mid a \mid \phi \wedge \phi \mid \neg \phi \mid K_i \phi \mid E_{\mathcal{G}} \phi \mid D_{\mathcal{G}} \phi \mid C_{\mathcal{G}} \phi \\ & \quad \mid Pr_{=?} \psi \\ \psi & = X \phi \mid \phi U^{\leq \tau} \phi \end{aligned}$$

where  $a \in AP$  is an atomic proposition.

We need the following terminology before discussing the semantics of the above grammar. A path fragment  $\sigma = g_0 g_1 \dots g_T$  is a sequence of global states in the execution of the system  $\mathcal{M}$  over  $T$  discrete time units and where each state  $g_i$  corresponding to state of the system at  $i^{\text{th}}$  time instant i.e.  $\sigma[i] = g_i$ . The fact that a state  $g$  (or a path  $\sigma$ ) satisfies a K-PCTL(B) state formula  $\phi$  (or a path formula  $\psi$ ) is denoted by  $g \models \phi$  (or  $\sigma \models \psi$ ). Building upon the  $\sim_i$  relation, we can define relations that involve multiple agents. These relations are  $\sim_{\mathcal{G}}^E = \bigcup_{i \in \mathcal{G}} \sim_i$ ,  $\sim_{\mathcal{G}}^D = \bigcap_{i \in \mathcal{G}} \sim_i$  and  $\sim_{\mathcal{G}}^C$ , the transitive closure of  $\sim_i$ , where  $\mathcal{G}$  is a group of agents. These relations in turn are useful in defining epistemic operators that involve a group of agents. Important examples are the *group*, *distributed* and *common* knowledge operators.

Semantics of the K-PCTL(B) is as follows. We first list semantics for the state formulas, which are common in probabilistic temporal logic, along with the epistemic fragment.

State formulas:

$$\begin{aligned} g \models \text{TRUE} & \\ g \models a & \quad \text{iff } a \text{ is true in the state } s_0 \\ g \models \Phi_1 \wedge \Phi_2 & \quad \text{iff } g \models \Phi_1 \text{ and } g \models \Phi_2 \\ g \models \neg \Phi & \quad \text{iff } g \not\models \Phi \end{aligned}$$

Epistemic formulas:

$$\begin{aligned} g \models K_i \phi & \quad \text{iff } \forall g' \in G, \\ & \quad g \sim_i g' \implies g' \models \phi \\ g \models E_{\mathcal{G}} \phi & \quad \text{iff } g \sim_{\mathcal{G}}^E g' \implies g' \models \phi \\ g \models D_{\mathcal{G}} \phi & \quad \text{iff } g \sim_{\mathcal{G}}^D g' \implies g' \models \phi \\ g \models C_{\mathcal{G}} \phi & \quad \text{iff } g \sim_{\mathcal{G}}^C g' \implies g' \models \phi \end{aligned}$$

The  $K_i \phi$  operator says that ‘‘agent  $i$  knows that  $\phi$  (is *TRUE*)’’ if and only if  $\phi$  is *TRUE* in all epistemically accessible global states—namely the global states that are epistemically indistinguishable for agent  $i$ .

The *group* knowledge operator  $E_{\mathcal{G}}$  evaluates to *TRUE* if and only if every agent in the group  $\mathcal{G}$  knows that  $\phi$ .

The *distributed* knowledge operator  $D_{\mathcal{G}}$  evaluates to *TRUE* if and only if the agents in the group  $\mathcal{G}$  together know that  $\phi$ . In other words, while individual agents may not know that  $\phi$  is *TRUE*, it can be inferred from the knowledge possessed by the individual agents.

Finally, the *common* knowledge operator  $C_{\mathcal{G}} \phi$  evaluates to *TRUE* if all the agents in the group  $\mathcal{G}$  know that  $\phi$ , they all know that they know  $\phi$ , *ad infinitum*. We use a syntactic characterization on the model for designing the algorithm.

Now, we discuss the semantics of the path formulas, noting that  $\sigma[t]$  denotes the  $t^{\text{th}}$  state in the path:

Path formulas:

$$\begin{aligned} \sigma \models X \phi & \quad \text{iff } \sigma[1] \models \phi \\ \sigma \models \phi_1 U^{\leq \tau} \phi_2 & \quad \text{iff } \exists j \leq \tau \mid \sigma[j] \models \phi_2, \\ & \quad \text{and } \forall t < j \mid \sigma[t] \models \phi_1 \end{aligned}$$

Finally,  $Pr_{=?} \psi$  estimates the probability measure of the paths  $\sigma$  that start in  $g$ , and satisfy  $\psi$ .

We remark that PCTL(B) is K-PCTL(B) without the epistemic fragment, namely,  $K_i, E_{\mathcal{G}}, D_{\mathcal{G}}$  and  $C_{\mathcal{G}}$ . PCTL is PCTL(B) augmented with the *unbounded until* operator  $U$  with the semantics:

$$\sigma \models \phi_1 U \phi_2 \text{ iff } \exists j \mid \sigma[j] \models \phi_2, \text{ and } \forall t < j \mid \sigma[t] \models \phi_1.$$

## 2.5 Statistical Model Checking

Statistical Model Checking (SMC) works by Monte Carlo sampling. Runs of the system are generated

<sup>1</sup>We note that there exists another popular variant of the probabilistic operator:  $Pr_{\bowtie \theta}$ , where  $\bowtie \in \{<, \leq, \geq, >\}$  is a comparison operator and  $\theta \in [0, 1]$ .

through simulation and the temporal logic formula is evaluated. SMC comes in two flavors. The first one, generally referred to as *qualitative* SMC, involves techniques like hypothesis testing to decide whether a formula is *TRUE* or not. The second one, called the *estimation* technique, tries to estimate the probability of a formula being true. In any case, the approach works by generating several runs (say,  $N$ ) of the system and evaluating the query on each run. In the estimation problem, a Bernoulli random variable  $b_r$  is set to 1 iff the formula evaluates to *TRUE* the  $r^{\text{th}}$  run. An estimate of the probability that the formula is true is  $\frac{\sum b_r}{N}$ . Clearly, as the number of runs grows, the accuracy of the estimate improves. Tail inequalities like the Chernoff-Hoeffding bounds can be used to estimate the minimum number of runs needed for a desired accuracy and confidence.

### 3 SMC ALGORITHMS FOR EPISTEMIC OPERATORS K-PCTL(B)

In this section, we discuss our main algorithms. Starting with the basic knowledge operator in combination with temporal operators, we move on to *group*, *distributed* and *common* knowledge operators.

#### 3.1 SMC for $K_i\phi$ (with Temporal Operators)

We first discuss the model checking subroutine for the epistemic operator  $K_i\phi$ . We have to decide if a global state  $g$  satisfies  $K_i\phi$ . An exhaustive algorithm would work by looking at all the global states that are epistemically accessible from  $g$  via the relation  $\sim_i$  ((Delgado and Benevides, 2009)). The global state  $g \models K_i\phi$ , if and only if  $\phi$  is *TRUE* at all such states. Once the truth value of  $K_i\phi$  is ascertained, it is straightforward to evaluate larger formula like  $Pr_\gamma[K_i\phi_1 U^{\leq c} K_j\phi_2]$  through standard algorithms (Baier and Katoen, 2008).

The computational bottleneck in this approach lies in the following scenario. If there are  $n$  agents each having  $k_i$  local states, the number of epistemically accessible states can potentially be  $\prod_i k_i$ . For example, if all agents have the same number  $k$  of local states, the global state space is of size  $k^n$ . Exploring the entire state space in the worst case is computationally very expensive.

However, this gives a first-cut SMC algorithm. The algorithm proceeds by generating sample runs of the system, as in the case of the SMC algorithm for

probabilistic temporal logics. For evaluation of the epistemic operators, the algorithm resorts to exhaustively visiting all epistemically accessible valid states. We will call this the “Vanilla” SMC algorithm.

Can we extend the sampling procedure for evaluating the truth value of epistemic operators as well? The approach that we propose is to sample a “small” number  $M$  of epistemically accessible states and check if  $\phi$  is *TRUE* or not, for only these states. We call this the “Epistemic Sampling” (ES) Statistical Model Checking approach. The approach for  $K_i\phi$  is outlined in Algorithm 1. Let the global state of the system at time  $t$  be  $g(t)$ . The procedure  $K_{op}$  for evaluating the knowledge operator takes in as input the current global state  $g(t)$ , the agent  $i$  for whom the knowledge operator is to be evaluated, a user decided sample size  $M$ , along with the SMAS. In the current global state, suppose agent  $i$  has the local state  $l_i$ . Then in line 3, the algorithm calls a sampling routine to construct a sample  $S$  of the global states that are epistemically accessible to  $g$ . The sampling routine is detailed in Algorithm 2. The procedure returns *TRUE* if and only if all states in  $S$  satisfy  $\phi$ . The sampling routine works as follows. A global state is generated uniformly at random. One way of doing this is as follows. For each agent, generate its local state (uniformly) at random. Other flavors could do this without replacement. Line 6 is crucial, and depends on the application at hand. A global state generated in this way may be “illegal” due to two reasons: (a) the state is semantically not possible in the system and (b) it is epistemically inaccessible from  $g$ . Otherwise, such a state would be called *legal*. This decision is made by the *isLegal* function in line 6. A total of  $M$  legal states generated this way are populated into the set  $S$  and is returned. If *isLegal* is efficiently computable (say in  $O(n^c)$ , for some constant  $c$ ), then the routine takes  $O(Mn^c)$  steps. This is a significant saving in comparison to the  $k^n$  steps needed to explore the state space exhaustively.

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Algorithm 1.

```

1: procedure  $K_{op}(g, i, \Phi, M, SMAS)$ 
2:   Let the current (local) state of agent  $i$  be  $l_i$ .
3:    $S \leftarrow$  A  $M$ -sized set of epistemically accessible
      global states
4:   if  $\phi$  is TRUE in all states in  $S$  then
5:     Return “yes”
6:   else
7:     Return “No”

```

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However, this sampling has consequences on accuracy. The following lemma is easy to see.

**Lemma 1.** *If  $\phi$  is TRUE on all epistemically accessible states, then Algorithm 1 returns TRUE. In other words, the probability of false negatives is zero.*

Algorithm 2.

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```

1: procedure SMC( $M, SMAS$ )
2:    $Count \leftarrow 0, S \leftarrow \Phi$ 
3:   while  $Count \leq M$  do
4:     Sample a global state uniformly at random
5:     //For all agents  $j \neq i$ , generate a local state
6:      $g$ 
7:     if  $isLegal(g)$  then
8:        $S \leftarrow S \cup g$ 
9:        $Count \leftarrow Count + 1$ 
10:    if no new global states to sample then
11:      break
12:  Return  $S$ 

```

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On the other hand, the formula  $\phi$  may not be *TRUE* on all the epistemically accessible global states. Then, the algorithm could report a false positive if the sampling routine does not generate at least one global state where  $\phi$  is not *TRUE*. This leads to the following lemma.

**Lemma 2.** *If  $\phi$  is FALSE on a fraction  $f > 0$  of all epistemically accessible global states, then the Algorithm 1 will output FALSE with probability at least  $1 - (1 - f)^M$ .*

*Proof.* Suppose  $g \not\models K_i\phi$ . Let the fraction of epistemically accessible global states that do not satisfy  $\phi$  be at least  $f$ . Then, in  $M$  samples, the probability that a satisfying global state is always picked is at most  $(1 - f)^M$ . This is the probability of false positive at any given time step. Therefore, the probability that the false positive does not occur is at least  $1 - (1 - f)^M$ .  $\square$

Therefore, the larger the fraction of global states where  $\phi$  is *FALSE*, the higher the chance that such a state will be picked, and therefore, the lower the chance of a false positive.

Hence, for example,

**Corollary 1.** *For the formula  $Pr_{=?}(T \ U^{\leq \tau} \ K_i\phi)$ , the probability that a false positive does not occur is at least  $(1 - (1 - f)^M)^\tau$ .*

We can also bound the false positive probability for the bounded until fragment when both the operands are epistemic operators.

**Corollary 2.** *Let  $K_i\phi$  and  $K_j\psi$  be FALSE at every step. Let the probability of false positive evaluation by the ES algorithm at any point in time be, for  $K_i\phi$  be  $p_1$  and that for  $K_j\psi$  be  $p_2$ . Then, for the formula  $Pr_{=?}[K_j\psi \ U^{\leq \tau} \ K_i\phi]$ ,  $Pr[\text{False positive}] \leq (1 - p_1)^M(1 - p_2)^M$ .*

*Proof.* Suppose a run is (falsely) declared *TRUE* for the formula, at time  $t$ . This will happen if  $K_i\phi_1$

is (falsely) evaluated to *TRUE* for  $t - 1$  steps, after which  $K_j\psi_2$  is (falsely) evaluated to *TRUE*. For the corresponding false positive probabilities, the probability of false positive is  $(1 - p_1)^{M(t-1)}(1 - p_2)^M$ . This is bounded from above by  $(1 - p_1)^M(1 - p_2)^M$ .  $\square$

We remark that  $Pr_{=?}[K_j\psi_1 \ U^{\leq \tau} \ K_i\phi_2]$  can evaluate to *FALSE* in other ways as well. For example, the  $K_i\phi$  can be *TRUE* at all steps before  $\tau$ , but  $K_j\psi$  never evaluates to *TRUE* for all  $t \leq \tau$ . Analogous analysis can be performed for these cases individually, which we omit in the interest of space.

As a consequence of the above discussion, it is easy to see that a given run of the system has the following properties. A run that evaluates to *TRUE* in the Vanilla SMC algorithm will also be evaluated to *TRUE* by the ES algorithm. However, some runs in the Vanilla SMC that evaluate to *FALSE* might be evaluated to *TRUE* by the ES algorithm. Hence, some of the Bernoulli random variables would take the value 1, when in fact they should be 0.

Recall that the probability of false positives can be controlled by the number  $M$  of epistemically accessible states sampled. The presence of false positives shifts the average of the distribution. It is easy to show that if the false positive rate is small, so is the error in probability estimation.

**Theorem 3.** *Let the probability of false positive evaluation of the formula in the ES algorithm be  $p_{fp} \approx \frac{1}{\delta N}$  for  $\delta > 0$ . Then, the probability that the estimate given by the ES algorithm deviates from the actual mean  $p$  is the same as that for the Vanilla algorithm.*

*Proof.* Consider the vanilla algorithm that has  $N$  runs. Let the outcomes of the  $N$  runs be defined by independent Bernoulli random variable  $X_i$ ,  $1 \leq N$ , and  $\sum_{i=1}^N X_i = X$ . Let  $p$  be the actual probability of the formula being satisfied. Then, a simple application of the Chernoff bound shows that

$$Pr(X \geq (1 + \delta)Np) \leq \left( \frac{e^\delta}{(1 + \delta)^{(1 + \delta)}} \right)^{Np},$$

for any  $\delta > 0$ .

Consider now the Epistemic Sampling algorithm. As we say previously, in this algorithm, some runs may yield false positives. Let the number of runs now be  $N'$ . Let the sum of the  $N'$  Bernoulli random variables be  $X'$ . The mean shifts to  $p + p_{fp}$  where  $p_{fp}$  is the term due to false positives. We now bound from above the probability that this random variable  $X'$  is larger than the  $(1 + \delta)Np$ . Then, an easy derivation from first principles follows. Recall the folklore Chernoff bound:

$$Pr(X' \geq c) \leq \frac{E[e^{rX'}]}{e^{cr}} \quad (1)$$

Using  $c = (1 + \delta)Np$  and  $r = \ln(1 + \delta)$  for  $\delta > 0$ , we get

$$Pr(X' \geq (1 + \delta)Np) \leq \frac{e^{N'p\delta}}{(1 + \delta)^{(1 + \delta)Np}} e^{N'pfp\delta} \quad (2)$$

If the same number of runs  $N$  is used, the bound becomes:

$$Pr(X' \geq (1 + \delta)Np) \leq \left( \frac{e^\delta}{(1 + \delta)^{(1 + \delta)}} \right)^{Np} e^{Npfp\delta} \quad (3)$$

Thus, for a sufficiently small false positive rate, the bound does not loosen much.  $\square$

### 3.2 Group, Distributed and Common Knowledge

The sampling algorithms for the group and the distributed knowledge operators are straightforward. We first discuss them. The group operator  $E_G$  is evaluated as follows. It is easy to see that  $g \models E_G(\phi)$  evaluates to *TRUE* if and only if  $g \models K_i\phi$  for all agents  $i \in G$ . This suggests an easy extension to the Epistemic Sampling algorithm: For all the agents in the group  $G$ , evaluate  $K_i\phi$  through sampling. Return *TRUE* if and only if it is *TRUE* for all such evaluations.

For the distributed knowledge operator, the following sampling based algorithm, listed in Algorithm 3 is easy to conceive. In a sample of epistemically accessible states for the agents in  $G$ , check if the formulas imply  $K_i\phi$ . Return *TRUE* if and only if it is *TRUE* for all such evaluations.

Algorithm 3: Algorithm for evaluating Distributed knowledge Operator.

---

```

1: procedure DISTRIBUTED KNOWL-
   EDGE( $gState, G, \phi$ )
2:    $M \leftarrow$  number of agents in group  $G$ 
3:   for  $i \leftarrow 1$  to  $M$  do
4:      $S_i \leftarrow$  a sample of epistemically related
       states of  $gState$  w.r.t. agent  $i$ 
5:    $S \leftarrow S_1 \cap S_2 \cap \dots \cap S_M$ 
6:   Check whether the formulas satisfied in  $S$  imply  $\phi$ 
7:   if all the states in  $S$  satisfies  $\phi$  then
8:     return TRUE
9:   else
10:    return FALSE

```

---

Semantically, the common knowledge operator is the reflexive and transitive closure of the *group* knowledge operator. This leads to a simple algorithm. The common knowledge operator can be evaluated through a recursive (depth-first) search of the epistemically accessible global states in  $G$ . This is done

by generating a target number of samples of epistemically accessible states. For each state, if the state does not satisfy  $\phi$ , the algorithm returns *FALSE*. Otherwise, the sampling continues recursively from that state. The algorithm is detailed in Algorithm 4.

In the next section, we provide empirical evidence that this approach yields comparable accuracy in significantly lesser running time.

Algorithm 4: Evaluating Common Knowledge Operator By Finding Transitive Closure.

---

```

1: procedure COMMON KNOWL-
   EDGE( $gState, A_i, \phi, List, currIndex$ )
2:    $List.add(gState)$   $\triangleright$  Initially List is empty
3:    $i \leftarrow 0$ 
4:   while  $i < sampleSize$  do
5:      $newState \leftarrow$  Generate random epistemically
       accessible state from agent  $A_i$ 
6:     if List does not contain  $newState$  then
7:        $List.add(newState)$ 
8:      $i \leftarrow i + 1$ 
9:   while  $currIndex < listSize$  do  $\triangleright$  Initially
       currIndex is zero
10:     $currState \leftarrow List.get(currIndex)$ 
11:     $currIndex \leftarrow currIndex + 1$ 
12:    if  $\phi$  is not satisfied in  $currState$  then
13:      return FALSE
14:    else
15:       $nextState \leftarrow List.get(currIndex)$ 
16:       $CommonKnowledge(nextState, A_i, \phi,$ 
17:         $List, currIndex)$ 
18:    return TRUE

```

---

## 4 CASE STUDIES

We discuss two case studies. All experiments for the two studies were conducted on the system with Ubuntu 18.04.2 LTS (64-bit) operating system, Intel(R)Xeon(R) CPU E5-2690 v4 @ 2.60GHz Processor, 62 GiB RAM, and a 2TB hard disk. One way of estimating the number of runs needed, that is commonly used for SMC, is to appeal to the Chernoff Hoeffding bound. For the estimated probability  $p'$  to be within  $\epsilon$  of  $p$  with a high probability  $1 - \delta$ , one needs at least  $\ln \frac{2/\delta}{2\epsilon^2}$  samples. Therefore, we evaluate the queries over 2335 runs which corresponds to  $\epsilon = 0.03$  and  $\delta = 0.03$ .

### 4.1 Epidemic Spreading

Consider a disease which is spreading across a population of  $n$  agents. An agent progresses from being susceptible to being infected. To focus the exposition on the model checking aspect, we use a simple

Table 1: Results of Query 1. Number of runs: 2335. The query evaluation time is measured in seconds.

$n$	Sample Size ( $M = n \log n$ )	Prob Est (ES)	Prob Est (Vanilla)	Time (ES)	Time (Vanilla)
10	23	0.788	0.760	0.043	0.4
15	40	0.784	0.781	0.079	5.12
20	60	0.751	0.765	0.111	119
25	80	0.773	0.776	0.151	2734
30	102	0.792	0.778	0.190	56942

Table 2: Results of Query 2. Number of runs: 2335. The query evaluation time is measured in seconds.

$n$	Sample Size ( $M = n \log n$ )	Prob Est (ES)	Prob Est (Vanilla)	Time (ES)	Time (Vanilla)
10	23	0.60	0.60	0.506	3.338
15	40	0.58	0.52	0.818	128
20	60	0.56	0.48	1.281	4693
25	80	0.52	0.42	1.905	169448
30	102	0.50	0.40	3.519	$\geq 500$ hrs

epidemic “spread” dynamic—a susceptible agent gets infected with probability 0.1 at each time step and an infected agent recovers at each time step with probability 0.1. Initially, all the agents are susceptible.<sup>2</sup>

The first query asks “what is the probability that agent  $i$  knows that 60% people are infected within 10 time steps”:

$$Pr_{=?}(TRUE \ U^{\leq 10} K_i(\text{Atleast } 60\% \text{ Agents Are Infected}))$$

For all the queries, the number of global states sampled,  $M$  in the algorithm listing, is  $n \log n$ , where  $n$  is the number of agents. We remark that  $n \log n$  is a modest sample size, which may be increased for greater accuracy. We evaluate this query for different values of  $n$  in  $\{10, 15, 20, 25, 30\}$ . We compare performance of the Vanilla SMC with the proposed Epistemic Sampling approach. The results are detailed in Table 1. The sample size, that is, the number of epistemically accessible global states is  $n \log n$  (column 2). It can be seen from columns 3 and 4 that the estimates are close. What is striking is the difference between the times taken (columns 5 and 6). The Vanilla SMC that visits every epistemically accessible global state takes significantly higher time.

The second, somewhat artificial query, asks “what is the probability that agent  $i$  knows that number of recovered people are at most 60% Until agent  $j$  knows that number of infected people are at least 80%, within 15 time steps”:

$$Pr_{=?}(K_i(\phi_1) \ U^{\leq 15} K_j(\phi_2)), \text{ where } \phi_1 \text{ stands for}$$

<sup>2</sup>In general, the spread of an epidemic is modeled elaborately using, for example, differential equations (Kermack and McKendrick, 1927).

“atmost 60% agents are recovered” and  $\phi_2$  stands for “at least 80% agents are infected.

We evaluate this query for  $n$  in  $\{10, 15, 20, 25, 30\}$ . Table 2 shows the results for the query. Notice that the mismatch increases for a more complex query. This is to be expected, since we keep the number of runs same, in spite of higher chances of false positives. Nevertheless, the difference between times taken continues to be significant.

## 4.2 Group, Distributed and Common Knowledge

We now discuss results for the epidemic case study in the context of the group, distributed and common knowledge operators. We report results of the cases when the numbers of agents are  $n$  in  $\{10, 15, 20, 25, 30\}$ . Further, we randomly select four agents to form the group  $\mathcal{G}$ .

### 4.2.1 Group Knowledge ( $E_{\mathcal{G}}$ )

In the epidemic setting, we use the group knowledge operator to ask the query “what is the probability that every agent in group  $\mathcal{G}$ , knows that 60% people are infected with in 10 time steps”:

$$Pr_{=?}(TRUE \ U^{\leq 10} E_{\mathcal{G}}(\phi_1)), \text{ where } \phi \text{ evaluates to } TRUE \text{ iff at least } 60\% \text{ agents are infected.}$$

Table 3 shows the results. We observe that the probability estimates are close; however the ES approach yields a significant gain in terms of running time over the Vanilla SMC.

Table 3: Results for  $Pr_{=?}(TRUE \ U^{\leq 10} \ E_G(\phi_1))$ . Number of runs: 2335. The query evaluation time is measured in seconds.

$n$	Sample Size ( $M = n \log n$ )	Prob Est (ES)	Prob Est (Vanilla)	Time (ES)	Time (Vanilla)
10	23	0.769	0.752	0.048	3.78
15	40	0.764	0.745	0.072	83
20	60	0.767	0.751	0.108	1461
25	80	0.765	0.745	0.141	11211
30	102	0.774	0.755	0.619	256850

Table 4: Results for  $Pr_{=?}(TRUE \ U^{\leq 10} \ D_G(\phi_1))$ . Number of runs: 2335. The query evaluation time is measured in seconds.

$n$	Sample Size ( $M = n \log n$ )	Prob Est (ES)	Prob Est (Vanilla)	Time (ES)	Time (Vanilla)
10	23	0.739	0.734	0.049	0.178
15	40	0.756	0.748	0.075	3.814
20	60	0.773	0.768	0.098	105
25	80	0.795	0.782	0.145	2526
30	102	0.802	0.791	0.206	58588

Table 5: Results for  $Pr_{=?}(TRUE \ U^{\leq 10} \ C_G(\phi_1))$ . Number of runs: 2335. The query evaluation time is measured in seconds.

$n$	Sample Size ( $M = n \log n$ )	Prob Est (ES)	Prob Est (Vanilla)	Time (ES)	Time (Vanilla)
10	23	0.768	0.760	0.306	4.886
15	40	0.789	0.773	0.588	180
20	60	0.799	0.772	1.084	3256
25	80	0.795	0.780	1.728	31234
30	102	0.801	0.785	2.860	>150 hrs

#### 4.2.2 Distributed Knowledge ( $D_G$ )

As an example of the distributed knowledge operator, we ask “what is the probability that combined knowledge among agents in group  $G$ , implies that 60% people are infected with in 10 time steps”:  $Pr_{=?}(TRUE \ U^{\leq 10} \ D_G(\phi_1))$ , where  $\phi$  is  $TRUE$  iff at least 60% agents are infected. Table 4 shows the results of the query.

#### 4.2.3 Common Knowledge ( $C_G$ )

Finally, as an example of the common knowledge operator, we discuss the query “what is the probability that it is common knowledge among the  $G$ , is 60% people are infected, within 10 time steps”:  $Pr_{=?}(TRUE \ U^{\leq 10} \ C_G(\phi_1))$ , where  $\phi_1$  evaluates to  $TRUE$  iff at least 60% agents are infected. Table 5 shows the results. When needing to perform the closure, it can be seen that the ES algorithm fares much better than the Vanilla algorithm in terms of

running time. There is, as expected, some loss in accuracy. Unsurprisingly, the running time for the common knowledge is greater than that of group and distributed knowledge queries. The number of states to be explored is the least for the distributed knowledge operator, since it involves intersection. On the other hand, sampling closure takes the most time. Intermediate is the group knowledge operator, since it involves taking union of epistemically accessible states for various agents in the group  $G$ .

### 4.3 Online Shopping

The second case study that we report was used by Wan et al (Wan et al., 2013) while discussing their adaptation of the PCTL algorithm for the epistemic operators. The setting is as follows. There is an online shopping system, with three agents: a customer, a server and the environment. When the customer requests a delivery, the system will deliver the goods successfully 95% of the time, and fail 5% of the time.

Table 6: Results for the Online Shopping problem. Number of runs: 2335. The query evaluation time is measured in seconds.

Number of Agents	Sample Size	$Pr_{=?}(K_c(X(Success)))$	$K_c(Pr_{\geq 0.9}(X(Success)))$	Time (in seconds)
3	1	0.943	true	0.328
3	2	0.917	true	0.378

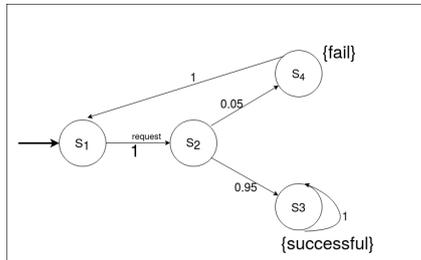


Figure 1: DTMC model of online shopping.

Figure 1 illustrates a DTMC model of this system. The customer’s local state is  $S_1$ , and the server’s state is  $S_2$ . The environment agent’s states are  $\{S_3, S_4\}$ , resulting in global states  $\{S_1, S_2, S_3, S_4\}$ . We ask the following query: “when a customer places an order, she knows that at least 90% of the items will be shipped successfully”:  $K_c(Pr_{\geq 0.9}(X(Successful)))$ . Table 6 shows the result of this query. These results are consistent with those reported by Wan et al (Wan et al., 2013).

## 5 RELATED WORK

Several variants of Interpreted Systems and ETLs have been proposed, along with the necessary model checking algorithms. Much of early model checking efforts for MAS was in the context of non-deterministic systems (Lomuscio et al., 2003; Lomuscio and Raimondi, 2006; Kong and Lomuscio, 2017). Indeed, excellent tools like MCMAS (Model Checker for Multi-Agent Systems) have been developed that implement these model checking algorithms for MAS (Lomuscio et al., 2017).

However, as in the case of reactive systems, it is desirable to do a quantitative analysis in the case of Stochastic MAS (SMAS). A rich theory of probabilistic model checking exists for analyzing stochastic reactive systems. Discrete (and continuous) Time Markov Chains (DTMCs) are popular examples of modeling formalisms for such systems. Temporal logics enhanced with the probability operator like PCTL are good examples of query languages (Hansson and Jonsson, 1994). To fill the gap for stochastic MAS, stochastic versions of Interpreted Systems and also epistemic temporal logics were introduced (Wan et al., 2013; Delgado and Benevides, 2009; Huang

and Luo, 2013; Fu et al., 2018). Often, the Interpreted System is modeled by some variant or a composition of Markov chains (Wan et al., 2013). While composition do introduce an element of non-determinism, as in the case of Markov Decision Processes, it was still amenable to quantitative model checking (Delgado and Benevides, 2009). The model checking algorithm, for say K-PCTL (PCTL augmented with the knowledge operator), has a time complexity that is polynomial in the size of the model and the epistemic temporal logic formula (Wan et al., 2013; Fu et al., 2018).

## 6 CONCLUSIONS AND FUTURE DIRECTIONS

In this work, we discussed a statistical model checking approach for answering queries in the probabilistic epistemic temporal logic K-PCTL(B). We focused on the estimation problem, whereas qualitative questions that make use of statistical methods like hypothesis testing have also been discussed extensively in the context of probabilistic temporal logics. Adapting these techniques to answer K-PCTK(B) queries would be a natural next step.

Further, we propose to construct a user friendly tool for nested queries involving epistemic and temporal operators, on the lines of MCMAS (Lomuscio et al., 2017) and Uppaal-SMC (David et al., 2015). Empirical evaluation of the tool against large systems and complex queries that involve several knowledge and temporal operators would yield interesting insights to model checking stochastic MASs. Extension of this technique to other modalities that have practical applications would also be an interesting future direction.

The approach discussed in this paper is particularly conducive for large systems with many agents, but a small loss in accuracy in answering the queries is tolerable. Many MASs display this characteristic. A practical direction is to explore such applications. We used a simple epidemic propagation model to demonstrate the ES algorithm, but extending it to realistic spread models is an example of such a practical direction. Other applications could arise from the domain of robotics (motion planning), IoT etc.

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