Eigenvalue and Eigenvector Expansions for Image Reconstruction

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Abstract: In medical imaging modality, such as X-ray computerized tomography, image reconstruction from projection is to produce the density distribution within the human body from estimates of its line integrals along a finite number of lines of known locations. Generalized Analytic Reconstruction from Discrete Samples (GARDS) can be derived by the Singular Value Decomposition analysis. In this paper, by discretizing the image reconstruction problem, we applied GARDS to the problem and evaluated the image quality. We have computed the condition number in the case of changing the views and the normalized mean square error in the case of changing the views and the number of the eigenvectors. We have showed that the error decreases with increasing the number of eigenvectors and the number of views.

1 INTRODUCTION

In medical imaging modality, such as X-ray computerized tomography (CT) and positron emission tomography (PET), image reconstruction from projection is to produce the density distribution within the human body from estimates of its line integrals along a finite number of lines of known locations (Herman, 2009; Kak et al., 1998; Imimya, 1985). In mathematically the problem of image reconstruction can be formulated by the Fredholm integral equation of the first kind. Because of the ill-posed nature, it is difficult to solve strictly this integral equation. Up to now many image reconstruction methods have been proposed by the research development regardless of imaging modality (Stark, 1987; Natterer and Wubbeling, 2001).

It is necessary to seek the solution of linear inverse problems with discrete data. In general, to solve the problems, we have to deal with the normal solutions, least-squares solution, generalized inverses, pseudo inverse and Moore-Penrose generalized invers (Bertero et al., 1985; Bertero et al., 1988; Andrews and Hunt, 1977). These methods depend on a general formulation by defining a mapping from an infinite dimensional function space into a finite dimensional vector space.

Although observed data can be discretized experimentally, original object which we want to seek are modeled continuous object. This continuous-discrete relation means that the object space is defined as continuous, while the observation space is discrete. So, this relation can be called a C-D mapping. In generalized model based on the C-D mapping, An analytical expression of object space by continuous base functions can be derived by the Singular Value Decomposition (SVD) analysis. This method is named a Generalized Analytic Reconstruction from Discrete Samples (GARDS) (Ohyama and Barrett, 1992). In reconstruction algorithm with GARDS, there is a paper which it could be analyzed with conjugate gradient algorithm by preconditioning the coefficient matrix using a polynomial function (Yamaya et al., 2000). But it is not to compute all eigen values and eigen vectors of the GARDS matrix directly. It is necessary to reveal the property of the GARDS matrix. It is more important mathematically to reveal the spectrum and the properties of bounded self-adjoint operator in Hilbert space (Reed and Simon, 1972; Kuroda, 1980).

In this paper, by discretizing the image reconstruction problem, we applied GARDS to the problem and evaluated the image quality. To implement GARDS, it is necessary to compute all eigenvalues and eigenvectors of symmetric matrix. We computed these by the Jacobi method. Moreover, we computed the condition number of the matrix and the normalized mean square error (NMSE) in reconstructed image. We have showed that the error

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decreases with increasing the number of eigenvectors and the number of views.

2 REVIEW OF GARDS

The observed data g_i can be viewed as the components of a vector which will be called the data vector $\boldsymbol{g}$ in the data space Y or the finite dimensional Hilbert space. The unknown characteristics of the sample, denoted by $\boldsymbol{f}$, will be called the object in continuous object space X or the element in infinite dimensional Hilbert space. Then, the C-D mapping model can be defined by

$$g_i = \int K_i(x) f(x)\, dx, \qquad i = 1, \dots, N, \tag{1}$$

where x denotes a space variable and the kernel $K_i(x)$ describes the interaction between the incident radiation and the object. If K is the bounded linear operator, the model can be defined by

$$\boldsymbol{g} = K\boldsymbol{f}, \tag{2}$$

that is, $K: X \ni \boldsymbol{f} \mapsto \boldsymbol{g} \in Y$. Let us consider a back projection operator K^+ from the discrete observation space to the continuous object space, that is, $K^+: Y \to X$. Because KK^+ be of a matrix of $n \times n$ elements, where n is the number of observed data, we can obtain the set $\{\lambda_i^2, \boldsymbol{v}_i\}$ by singular value decomposition. Here, λ_i^2 are eigenvalues and $\{\boldsymbol{v}_i\}_{i=1}^n$ are the eigenvectors, which satisfy the equation,

$$KK^+\boldsymbol{v}_i = \lambda_i^2 \boldsymbol{v}_i. \tag{3}$$

The element of KK^+, k_{ij}, is given by

$$k_{ij} = \int K_i(x)\, K_j(x) dx. \tag{4}$$

Therefore, KK^+ can be of a self-adjoint operator on a Hilbert space. Eigenvectors corresponding to distinct eigenvalues of KK^+ are orthogonal. The eigenvalues are real number. If K has a bounded inverse, K^{-1}, from eq. (3) we can derive

$$u_i = \frac{1}{\lambda_i} K^+ v_i, \tag{5}$$

where $\{u_i\}$ is orthogonal base, which satisfies

$$K^+Ku_i = \lambda_i^2 u_i. \tag{6}$$

If eigenvalues are ordered in a descending order and the last number of non-zero eigenvalue is R, we can reconstruct the object f_{re} as

$$f_{re} = K^+\left(\sum_{i=1}^{R} \frac{\langle v_k, g\rangle}{\lambda_i^2} v_k\right), \tag{7}$$

where $\langle\cdot,\cdot\rangle$ is inner product in Hilbert space. This eq. (7) gives the Moore-Penrose type reconstruction by GARDS.

3 COMPUTER SIMULATIONS

To confirm the effectiveness of the method, computer simulations were carried out. First, the continuous object space and the data space are discretized in a reconstruction problem. A Cartesian grid of the square observation plane, called pixels, is introduced into the region of interest (ROI) so that it covers the whole observation plane that has to be reconstructed in infinite-dimensional Hilbert space. The pixels are numbered in some manner. We set the top left corner pixel 1and bottom right corner pixel M with Raster scanning. The object to be reconstructed is approximated by a constant uniform value f_j throughout the j-th pixel, for $j = 1,2,\cdots,\mathrm{M}$. Thus, the vector $\boldsymbol{f} = \{f_j\}_{j=1}^M$ in $\mathbb{R}^M$ is the discretized version of the object (Censor et al., 2008). For our simulations we assumed the parallel beam scanning model for data correction in CT. In this mode an array of sources is arranged in a line. In opposite side over the ROI an array of detectors is arranged in a line. The set of one detector-source pair for which line integrals are estimated is divided into D elements at equidistant. We assumed projection angle $\theta = [0, \pi[$, and the number of views is the number of the angle discretized at evenly space. The total number of all discretized line is $N = \mathrm{View}\times\mathrm{D}$.

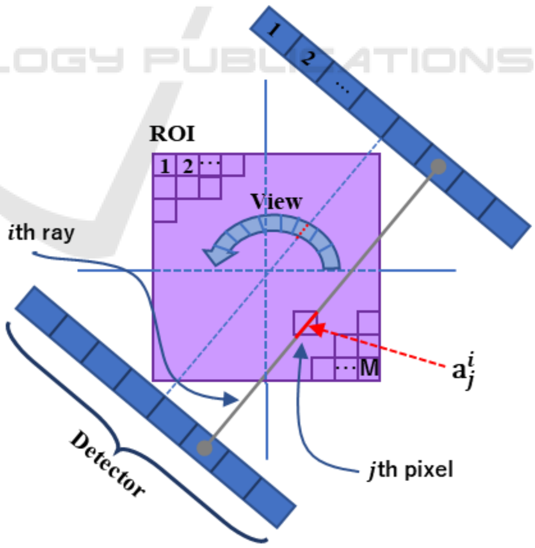

Figure 1: The discretized model of the image reconstruction problem.

We set the left detector element to 1 at $\theta = 0$ and the right detector element to N at last View. In this setting i indicates any detector-source elements and $i = 1,2,\cdots,N$. Thus, the vector $\boldsymbol{g} = \{g_i\}_{i=1}^N$ in $\mathbb{R}^N$ is the data vector. We denote the length of intersection of the i-th line with the j-th pixel by a_j^i, for all $i = 1,2,\cdots,N$, $j = 1,2,\cdots,M$, that is,

$$A = \{a_{ij}\}. \tag{8}$$

A can be of $N \times M$ matrix (Aoyagi et al., 2020). Figure 1 shows the discretized model of the image reconstruction problem.

Our algorithm is following scheme.

Step 1: Compute AA^t.

Step 2: Compute all eigenvalues and eigenvectors of AA^t by the Jacobi method.

Step 3: For all k, compute the inner product of eigenvector $\boldsymbol{v}_k$ and data vector $\boldsymbol{g}$, and divide it by corresponding eigenvalue λ_k.

Step 4: Compute the linear combination of eigenvectors $\boldsymbol{v}_k$ and corresponding coefficients which are computed by step 3.

Step 5: Operate A^t on the vector which is computed by step 4.

The transpose of A is denoted by A^t.

Figure 2 shows 6 original test images, discretized 32×32 pixels and text based phantoms. Figure 3 shows the projection data by setting 32 views and 32 detectors per view in parallel beam. Figure 4 shows the reconstructed images with our algorithm by using projection data which are shown in Fig. 3. In this case we set 32 views with 32 line per view. The matrix size is 1024×1024.

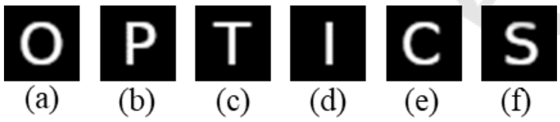

Figure 2: The original test images (32×32pixel, 8bit/pixel).

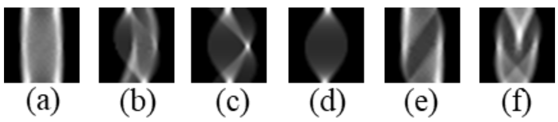

Figure 3: The projection data (Sinogram: 32 Detectors, 32 Views and 8bit/pixel).

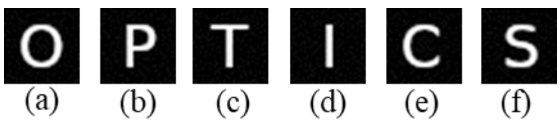

Figure 4: The reconstructed images. 32 detectors and 32 views.

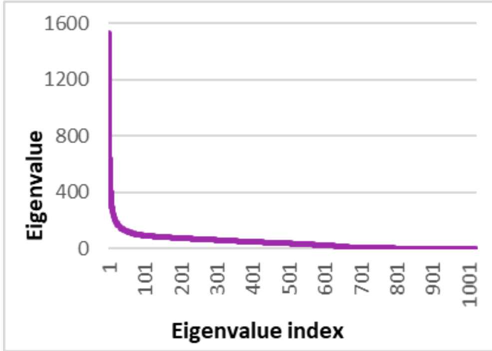

Figure 5: Plot of the Eigenvalues with decreasing order.

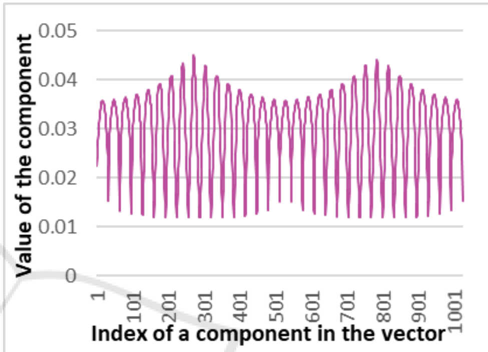

Figure 6: Plot of Eigenvector for the largest eigenvalue.

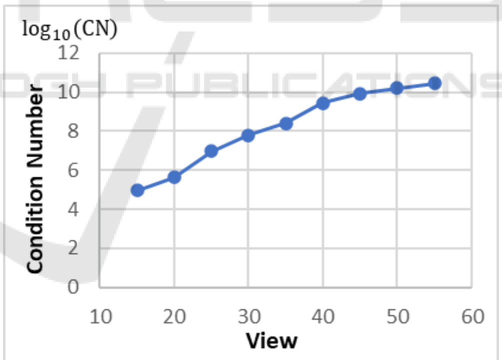

Figure 7: Plot of the condition number versus the number of views.

We computed all eigenvalues and eigenvectors by the Jacobi method (Press et al, 1992). Figure 5 shows the eigenvalues with decreasing order. Figure 6 shows the eigenvector for the largest eigenvalue.

The sensitivity of the solution to change the data vector, can be indicated by the condition number of the matrix AA^t. Condition number (CN) can be introduced by defining

$$CN(L) = \|L^{-1}\| \times \|L\|, \tag{9}$$

where L is a bounded linear operator in Hilbert space. Therefore, in this case our condition number, as shown in the Appendix, is defined by

$$CN(AA^t) = \frac{\lambda_1}{\lambda_n}, \tag{10}$$

where λ_1 is the largest eigenvalue and λ_n is the smallest eigenvalue. Figure 7 illustrates the plot of the condition number versus the number of views. From Fig. 7 we can see that the condition number increases with increasing the number of views.

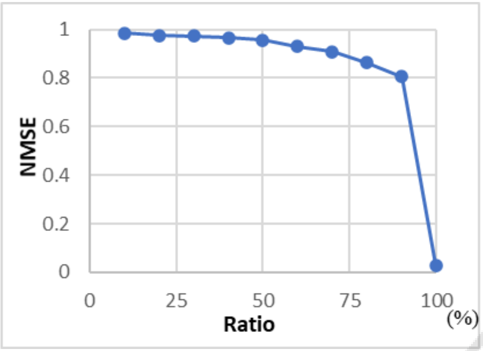

Figure 8: Plots of the normalized mean square error versus ratio.

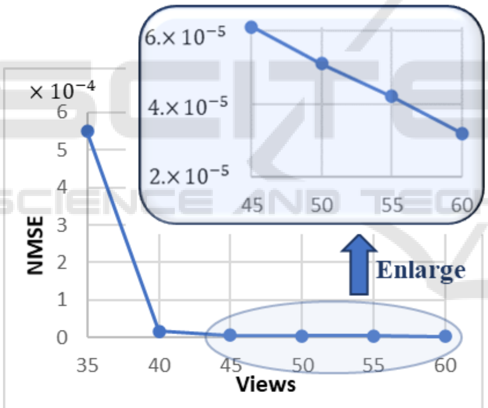

Figure 9: Plots of the normalized mean square error versus the number of views.

If the condition number is large, then the generalized solution is affected by numerical instability.

Figure 8 illustrates the plots of the normalized mean square error versus the ratio of number used in reconstruction to total number of eigenvectors. In this case, the total number of eigenvectors is 1024.

NMSE is defined by

$$\mathrm{NMSE}(k) = \frac{\|\boldsymbol{f}^k - \boldsymbol{f}\|_2^2}{\|\boldsymbol{f}\|_2^2}, \tag{11}$$

where $\boldsymbol{f}^k$ is the image which is reconstructed by using k eigenvectors and $\boldsymbol{f}$ is the original image. $\|\cdot\|_2$ indicates the ℓ^2-norm. From Fig. 8 we can see that the error decreases with increasing the number of eigenvectors.

To check the effect of the number of views in reconstructed image, we changed the number from 35 to 60. If the number of views is 60 and the number of detectors per view is 32, the matrix size is 1920×1920. The number of eigenvectors which was used in reconstruction process is 1024. Figure 9 illustrates the plots of the normalized mean square error versus the number of views. From Fig. 9 we can see that the error decreases with increasing the number of views.

4 CONCLUSIONS

By discretizing the image reconstruction problem, we applied GARDS to the problem and evaluated the image quality. In GARDS, it is important mathematically to reveal the spectrum of bounded self-adjoint operator in Hilbert space. All eigenvalues and eigenvectors were computed by Jacobi method. We showed that the condition number increases with increasing the number of views. In singular value decomposition, the condition number play an important role to solve linear systems. If the condition number was large, the accuracy of eigen values and eigen vectors was influenced by the matrix size. Also, we showed that the error decreases with increasing the number of eigenvectors and the number of views.

There were many parameters, the number of views, detectors-source pair and the pixel size of reconstructed image. The matrix size was changed by these parameters. If the size was large, computation of our algorithm consumed time to large quantities. For a large size of the matrix, especially, it is difficult to calculate all eigenvalues and eigenvectors with enough accuracy. The image quality of reconstructed image in this method is affected by these. If the matrix size is larger, it is necessary to computer all eigen values and eigen vectors by the other method, for example, Lanczos method and so on. Many numerical methods for large eigen value problems of matrix have been proposed and reported. It is worth trying to use these methods. Another idea will be to try to use parallel matrix computations. These become the future problems.

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APPENDIX

Let T be a bounded linear operator and x be an element in Hilbert space. Operator norm can be introduced by defining

$$\|T\| = \sup_{x\neq 0} \frac{\|Tx\|}{\|x\|}. \tag{12}$$

Let x_i be an eigenvector and λ_i be a corresponding eigenvalue, $i = 1, \cdots, n$. Then, T, operate on a vector x_i, can be to transform it into a scalar multiple of itself.

$$Tx_i = \lambda_i x_i. \tag{13}$$

We assume that eigenvalues are ordered in such a way as to form a decreasing sequence

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n. \tag{14}$$

From eq. (12) and (13), we obtain

$$\|T\| = \sup_{x\neq 0} \frac{\|Tx_i\|}{\|x_i\|} = \sup_{x\neq 0} \frac{\|\lambda_i x_i\|}{\|x_i\|} = \lambda_1. \tag{15}$$

From eq. (13), we obtain

$$T^{-1}Tx_i = \lambda_i T^{-1} x_i. \tag{16}$$

$$T^{-1}x_i = \frac{1}{\lambda_i} x_i. \tag{17}$$

From eq. (12) and (17), we obtain

$$\|T^{-1}\| = \sup_{x\neq 0} \frac{\|T^{-1}x_i\|}{\|x_i\|}$$
$$= \sup_{x\neq 0} \left\|\frac{x_i}{\lambda_i}\right\| \left\|\frac{1}{x_i}\right\| = \frac{1}{\lambda_n}. \tag{18}$$

Hence, from Eq. (15) and (18), we conclude

$$\|T^{-1}\| \times \|T\| = \frac{\lambda_1}{\lambda_n}. \tag{19}$$