The Tonnetz Environment: A Web Platform for Computer-aided "Mathemusical" Learning and Research

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Abstract: We describe the Tonnetz web environment and some of the possible applications we have developed within a pedagogical workshop on mathematics and music that has been conceived for high-school students. This web environment makes use of two geometrical representations that constitute intuitive ways of accessing some theoretical concepts underlying the equal tempered system and their possible mathematical formalizations. The environment is aimed at enhancing "mathemusical" learning processes by enabling the user to interactively manipulate these representations. Finally, we show how Tonnetz is currently being adapted in order to lead computer-based experiences in music perception and cognition that will be mainly carried at universities. These experiences will explore the way in which geometrical models could be implicitly encoded during the listening process. Their outcome may reinforce educational strategies for learning mathematics through music.

1 INTRODUCTION

Within the panoply of computer-based approaches on music education, there is a common agreement that web environments constitute an excellent device to support music learning processes (Meredith, 2016; Conway, 2020). The `Tonnetz` environment[1] is a web application based on previous research on computer-aided topological music analysis (Bigo et al., 2013; Bigo et al., 2014; Bigo and Andreatta, 2016) and fostering mathematical and computational thinking through music.

In this paper, we first provide a very short overview of the most active research axes of the Structural Music Information Research Project (SMIR project in short) within which the web environment has been developed. After discussing the "mathemusical" dynamics underlying the SMIR project, we focus in particular on some computational models of harmonic spaces that have been implemented in `Tonnetz`. This environment enables an interactive exploration of a great variety of geometric spaces for music-theoretical, analytical and compositional purposes. We will also present a current research project, supported by the CNRS, which constitutes the cognitive component of the SMIR project. This project, entitled Processes and Learning Techniques of Mathemusical Knowledge (ProAppMaMu project in short), aims to explore the link between geometric representations and cognitive sciences by following some previous work carried out by researchers in neurosciences and experimental psychology and by focusing on the interplay between structural mathematics and computational models.

After some considerations on music cognition, we show how `Tonnetz` is currently being adapted to lead computer-based educational experiences for music perception and cognition. These experiences are planned to be systematically conducted at universities and might shed some new light on the way in which geometrical models are implicitly encoded in the listening process, and eventually suggest how to reinforce the learning techniques in mathematics through music.

[1] We henceforth use `Tonnetz` for our open-source web environment and *Tonnetz* for the neo-Riemannian representation. The web environment is available at: https://guichaoua.gitlab.io/web-hexachord/. We recommend to open it with `Chrome`.

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2 THE SMIR PROJECT AND ITS "MATHEMUSICAL" DYNAMICS

The SMIR project, hosted by the *Institut de Recherche Mathématique Avancée* (IRMA) of the University of Strasbourg, is currently carried out in collaboration with musicologists from the *Groupe de Recherche Expérimentale sur l'Acte Musical* (GREAM) and computer science researchers from the Music Representations Team at IRCAM in Paris. Ongoing research includes axes such as mathematical morphology, generalized *Tonnetze*, formal concept analysis and computational music analysis (Agon et al., 2018), persistent homology and automatic classification of musical styles (Bigo and Andreatta, 2019), category theory and transformational music analysis (Popoff, Andreatta and Ehresmann, 2018), homometry and phase retrieval in music analysis (Mandereau et al., 2011), or tiling musical problems and Fuglede spectral conjecture (Andreatta, 2015).

All these topics share a common trait, which positions music at the core of mathematical formalization and computational modelling. Their dynamics is captured by the diagram in Figure 1, which shows how to navigate between music and mathematics by using computer science as an interface in the formalization ascending process (from music to mathematics) and the application descending process (from mathematics back to music). This dynamical "mathemusical" process is detailed deeper in Figure 2. Notice that some musical problems make use of computational models in an ascending formalization process, before necessarily reaching the state of general theorems in the mathematical realm. This is typically the case of NP-complete music-theoretical problems for which, unfortunately, it is not possible to exhibit a constructive algorithm. Some well-known compositional problems belong to this class. It is the case, for instance, of the enumeration of all-interval twelve-tone series, the estimation of Z-related chords, or the enumeration of Hamiltonian paths and cycles in music-theoretical graphs, such as the *Tonnetz* or its possible extensions (Cannas and Andreatta, 2018).

The most interesting case for "mathemusical" research occurs however when a musical problem reaches the state of a mathematical theorem after the formalization or generalization process and becomes integrated within a computer-aided environment to be used by composers or music theorists in different situations within the field of music. One of the most celebrated theorems that one may find in the literature is Milton Babbitt's "hexachordal theorem", stating that a hexachord and its complement always have the same intervallic content (Blau, 1999). `Tonnetz`, and in particular its circular representation associated to the equal-tempered system, can be easily adapted to this special musical problem. It becomes an interactive platform for visualising the intervallic content of two generic collections of pitches related by complementation. Figure 3 shows the environment as it has been integrated in an online article retracing the history of this famous combinatorial problem (Bayette, 2019). The reader may try to discover the possible relations between the intervallic content of a given collection of pitches and its complement by exploring this special adaptation[2].

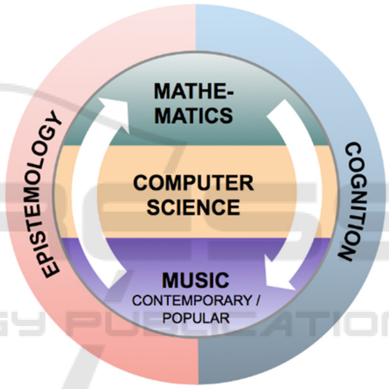

Figure 1: The "mathemusical" dynamics at the core of the SMIR project.

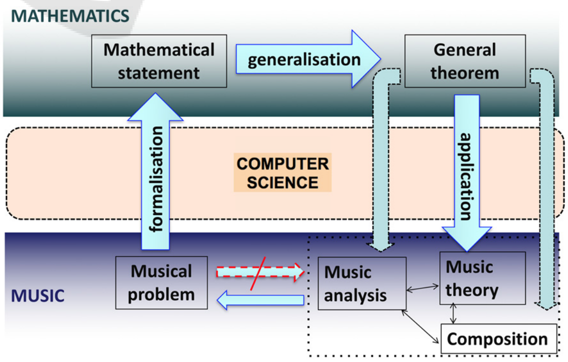

Figure 2: A more detailed perspective of Figure 1 showing the different steps of the "mathemusical" dynamics.

[2] Available at: https://guichaoua.gitlab.io/web-hexachord/hexachordTheorem. We recommend to visualize it with `Chrome`.

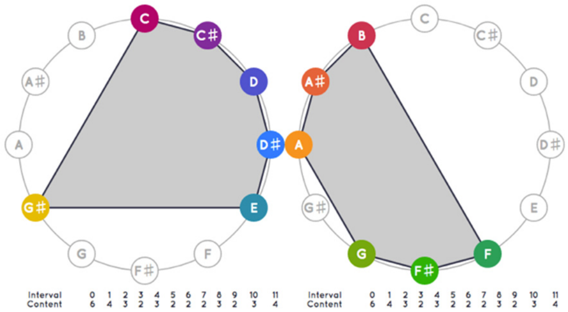

Figure 3: Adaptation of `Tonnetz` for interactively visualizing Babbitt's hexachordal theorem.

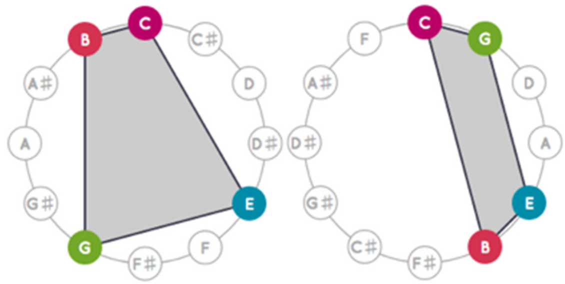

Figure 4: Two possible representations of the C7 chord in `Tonnetz`. On the left, within the chromatic circle; on the right, within the cycle of fifths.

3 THE TONNETZ GEOMETRY-BASED WEB ENVIRONMENT

`Tonnetz` makes use of two kinds of geometric representations: a circular representation and a tiling of the plane by triangles (alternatively by hexagons). We go now into some details of these geometric representations by emphasizing their relevance in supporting music learning and teaching via computer models.

3.1 The Circular Representation

As in the case of the `HexaChord` computer environment[3], which was the inspirational source for this web application, two circular representations are available to the user: the chromatic circle and the cycle of fifths representation. These representations are extensively used in both American and European research around the pitch-class set theory (Andreatta, Bardez, and Rahn, 2008). Although both circles are algebraically equivalent (since they are related by an affine transformation), they highlight different properties of chord collections and their associated intervallic content with respect of symmetry, as shown in Figure 4.

An interesting case, that can be taken as an exercise for advanced maths students, occurs when the two representations are exactly the same for a specific pitch collection. The reason of this equivalence goes deep into algebraic properties of the cyclic groups underlying the previous geometric representations. Apparently simple musical problems may give rise to deep algebraic investigations that can be easily modelled within an interactive environment such as `Tonnetz`.

3.2 Two Ways of Tiling the Space

Beyond the circles, `Tonnetz` incorporates further geometric representations: the tiling of the plane by triangles and by hexagons. These representations are widely spread within the neo-Riemannian music analysis (Gollin and Rehding, 2010). The triangulation of the 2-dimensional Euclidean space offers a first geometrical perspective on musical transformations between two given chords. According to the simplicial approach previously introduced by Louis Bigo and Moreno Andreatta (2015), triangles are 2-simplexes and they are glued together, as shown in Figure 5, by a self-assembly process when they share a common edge.

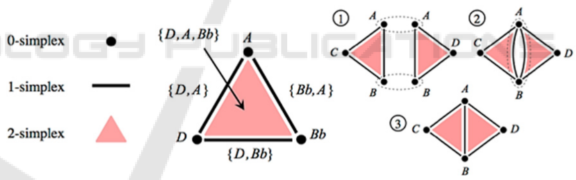

Figure 5: On the left, the first three *n*-simplexes; on the right, the self-assembly process. Adapted from Bigo and Andreatta (2015).

In the standard *Tonnetz* space, triangles represent major and minor chords, as Figure 6 puts in evidence, but one may use any 3-note chord as a 2-simplex generating the triangular tessellation of the plane via the self-assembly process. The structural properties of the resulting non-standard *Tonnetz* space will be different depending on the properties of the generating 2-simplexes (Catanzaro 2011). This fact leads to a panoply of topological cases, from the torus (as in the case of the standard *Tonnetz*) to non-connected spaces (for instance, when the space is generated by only augmented triads).

[3] Available at: https://louisbigo.com/hexachord

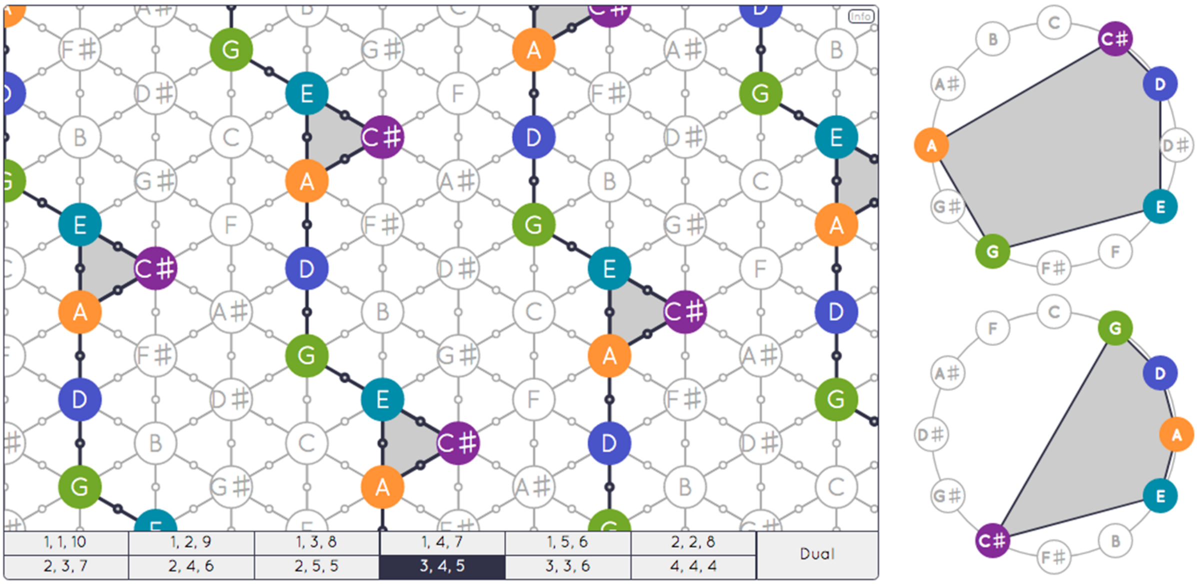

Figure 6: Multiple visualizations of pentachord {C#, D, E, G, A} within the `Tonnetz`. On the left, its representation within the standard *Tonnetz*. On the right, the circular representations associated to this chord.

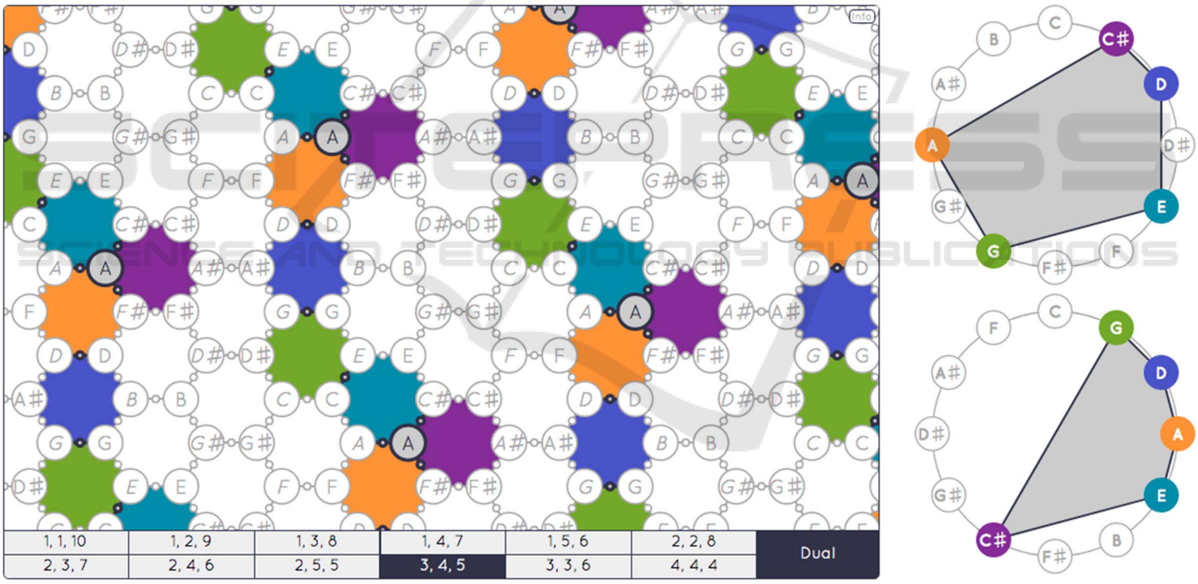

Figure 7: Multiple visualizations of pentachord {C#, D, E, G, A,} within the `Tonnetz`, but represented this time in the dual space of the standard *Tonnetz*.

Any triangulation of the plane is associated to its dual one: the graph obtained by replacing every 2-simplex with a 0-simplex (i.e., a point) and by connecting two points when they corresponding 2-simplexes share a common 1-simplex (i.e., an edge). Figure 7 shows how the pentachord of Figure 6 is now represented in the dual space, also called "chicken-wire" torus in the neo-Riemannian music theory (Douthett and Steinbach, 1998).

In this new visual representation, the vertices are now major triads (normal font with the name of the chord root) or minor ones (italic font with the name of the chord root). Now, for any given collection of pitches, if the node corresponding to A major is illuminated, it means that the collection contains the A major triad as a subset. One may notice that each hexagon of the dual space represents a single note, more precisely the note which is common to all the triads which are associated to the nodes of the hexagonal surface. For instance, pitch class F# belongs to the F# major, F# minor, D major, D# minor, B major, and B minor triads.

4 USING TONNETZ IN OUTREACH AND EDUCATIONAL CONTEXTS

`Tonnetz` has taken part of several large public events, such as the LaLaLab exhibition "Mathematics and Music" created by IMAGINARY and premiered at Heidelberg in May 2019. This exhibition was curated with the advice of several "mathemusical" researchers around the world and with the contribution of more than twenty artists and scientists working in the field. As shown in Figure 8, `Tonnetz` was presented by means of tactile screens and keyboards enabling the public to interactively explore the different geometric spaces available through the environment.

Of course, `Tonnetz` can be used in quite more controlled scenarios with educational purposes. It has been recently integrated into a pedagogical workshop which was conceived by following the new high-school programs of the French National Education. These programs introduce the topic of mathematics and music at the high-school level (typically for 15 years-old students). They put a special emphasis on the way in which several harmonic structures have been formalized, from the Pythagorean scales to the equal-tempered system, and on the use of computer science to provide computational models of musical knowledge. In this workshop, designed within the ProAppMaMu project, we have tried to reinforce the interplay between standard pedagogical tools (flyers, slide presentations, audio and video extracts) and an interactive manipulation of `Tonnetz`. Figure 9 shows the typical set-up of such a workshop in a high-school class, where the students can individually explore further contents of the presentation through our web environment, which is available in their computer screen.

The conception of a traditional paper support accompanying the "mathemusical" workshop has been done in synergy with Marie Marty, who is a designer from the Ecole Estienne of Design in Paris, specialist in scientific drawings. She has also created a series of animations for enriching the brochure support. Some topics within the official programs of the French National Education were selected for their potential to be also depicted in a traditional flyer, as shown in Figure 10. The aim was to describe, in an visually appealing way, the main features which are available in `Tonnetz`. Figure 11 highlights a portion of the paper flyer that can be "augmented" during a workshop.

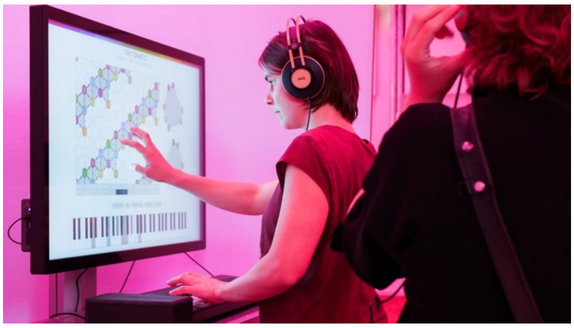

Figure 8: A visitor interacting with `Tonnetz` and a keyboard at the LaLaLab exhibition in Heidelberg (reproduced with the kind permission of IMAGINARY).

Figure 9: A typical set-up of a "mathemusical" workshop of the ProAppMaMu project.

Figure 10: The flyer for enriching the workshop.

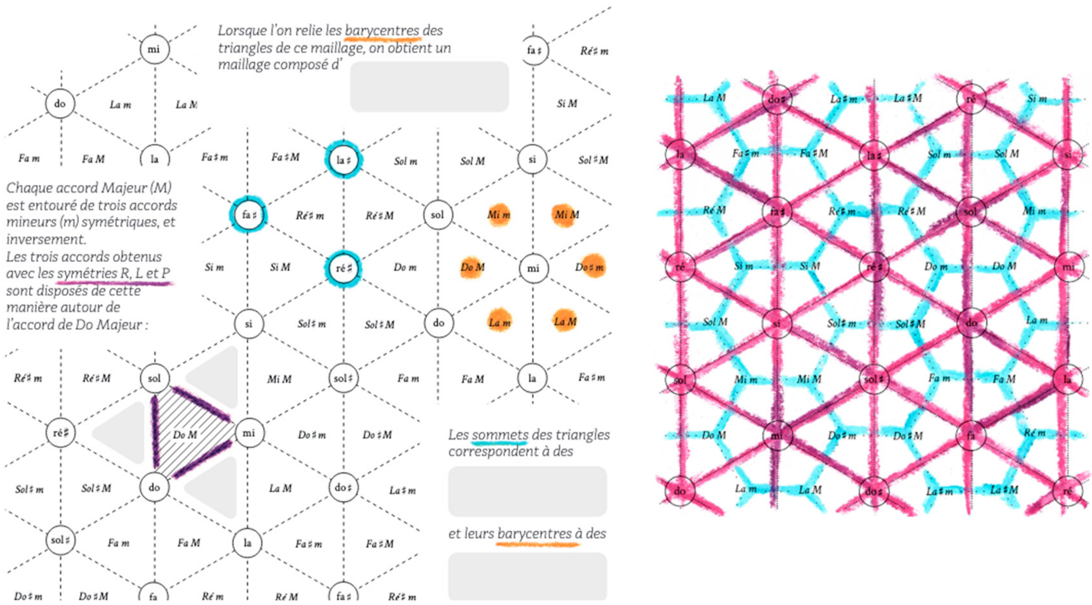

Figure 11: On the left, the topic “the *Tonnetz* system” as contained in the brochure (third inner panel); on the right, a still of its matching gif animation for slide presentations.

Both the animations and the computer-aided visualisations have been integrated into a pedagogical film (entitled “Musique et mathématiques: histoire d’une rencontre”) that aims to artistically summarize the theoretical content of the high-school workshop. The two main characters of this film, which appear in Figure 12, are playing on a stage (which is a keyboard) surrounded by several animations generated with `Tonnetz`. The dialogue between these characters provides at the same time a journey through the history of the relations between maths and music and a condensed presentation of the main concepts introduced in the workshop via the web environment[4].

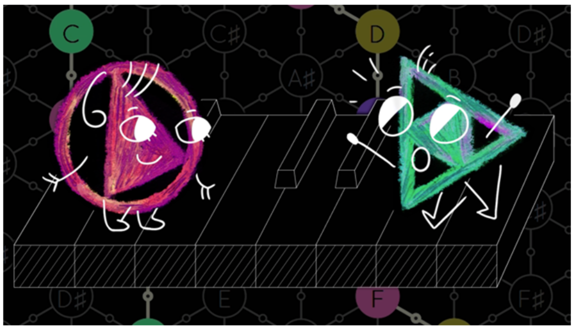

Figure 12: Screenshot of the pedagogical film “Musique et mathématiques: histoire d’une rencontre”.

[4] The film is hosted at the CNRS AudiDiMath outreach platform: http://video.math.cnrs.fr/musique-et-mathematiques/

5 PERCEPTUAL AND COGNITIVE IMPLICATIONS

So far, the main outcomes of the SMIR project have been the online uploading of the `Tonnetz` environment, its adaptation within the contexts of art and popular music, and a detailed dissemination plan trough “mathemusical” workshops and exhibitions, suitable for pedagogical applications. Besides all these achievements, the SMIR project disclosed new interesting research questions belonging to the area of psychology and cognition. These questions concern the way in which individuals process the information carried by the geometrical representations discussed in the previous sections:

- Do the structures implicitly encoded by `Tonnetz` have some perceptual foundation, somehow innate in human cognition?
- How could we grasp plausible connections between its geometrical representations and a multimodal (in our particular context, auditory and visual) musical perception?
- Which strategies would be the most suitable and efficient for helping the listeners become more

aware and familiar with such structures under the effect of such stimuli?

ProAppMaMu aims at answering these questions by means of perceptual tests. Moreover, by focusing on the interplay between structural mathematics and computational models, the project will explore the way in which geometrical models of music theory might be used to reinforce the learning techniques within the context of mathematical education.

5.1 Transformational Music Theory and Cognition

One of the main goals of systematic musicology during the last decades has been to explore the links between human cognitive capabilities to process musical signals and the theoretical foundations of music (Leman, 1995). As music develops quite differently within each human culture, the theoretical representations conceived by Western music theorists are far from standing as universals. This fact does not mean, however, that Western representations are completely arbitrary. Consequently, a search for some objectification in this sense is highly desirable. Moreover, a better comprehension of the cognitive foundations and implications of such representations would help to improve the pedagogical strategies in musical and “mathemusical” educational contexts.

Many music representations can be traced back to two main schemes, which are embedded in Western music notation: the horizontal flow of time and the vertical ordering of pitches, as outlined by the image schema theory (Johnson, 1987). Although such schemas may seem to be almost trivial within our culture, the way individuals interact with them is, cognitively speaking, quite complex. For instance, there is no one-to-one relationship between the vertical distribution of pitches and the plethora of cognitive strategies individuals adopt for pitch recognition (Letailleur, Bisesi, and Legrain, 2020). Concerning time, some composers have twisted the straight left-to-right musical timeline while drafting their oeuvre (Besada and Pagán Cánovas, 2020; Besada, Barthel-Calvet, and Pagán Cánovas, 2021).

All these examples rely on relatively simple music features (pitch and time), which are unfolded throughout a unidimensional image schema. The standard *Tonnetz* is conversely much more complex, as pitches are placed in the crossroads of three different linear directions, giving rise to geometrical shapes matching with particular chords. Previous research in the field of musical psychology has encompassed these kinds of problems from diverse approaches. A key related topic is the study of perceived distance when hearing pitch collections in tonal and post-tonal music (Rogers and Callender, 2006; Bisesi, 2017). Concerning our particular research, Carol L. Krumhansl (1998) compared three neo-Riemannian models of triadic distance with psychological data. She surmised that the perceived distance between chord roots is akin to the dual standard *Tonnetz*. Andrew J. Milne and Simon Holland (2016) have shown through experimental data that spectral pitch-class distances and the *Tonnetz* have high correlations. This evidence led them to provide a psychoacoustical explanation for perceived triadic distance. By contrast, very few is known about how individuals actually apprehend such abstract structures.

The standard *Tonnetz* probably stands as the most exploited model of transformational music theory. This branch of music theory was formally stablished by David Lewin (1987; 1993) and privileges the analysis of the way musical objects transform over time instead of focusing on their intrinsic features. As previously defined, the aforementioned gluing self-assembly process of the *Tonnetz* unfolds along three different directions which are respectively orthogonal to the three linear directions of individual pitches. Three different transformations (those that generate the space) relate each triangle (i.e., each triad) with each contiguous neighbour. As the standard *Tonnetz* is a cyclic and connected space (the aforementioned torus), any pair of triangles can be related by means of a finite series of combinations of these only three transformations generating the space.

As highlighted above, how music evolves over time is a key issue in transformational music theory. Consequently, these kinds of transformations have been sometimes described as metaphors of motion by music theorists (Attas, 2009). Musical software like `Tonnetz` allows to perceptually reify this metaphor, and therefore to observe and potentially describe the way a user conceives them.

5.2 Adapting Tonnetz for Empirical Psychology Experiences

We are currently developing a variant of the standard `Tonnetz`, in order to collect quantitative and qualitative data for discussing the cognitive features of human interaction with its environment. Several chord sequences will be submitted to listeners of different expertise, in the double format of sounds and coloured patterns. All these sequences are parsimonious (i.e. any pair of contiguous triads share two of their three respective pitches), but the last triad does not necessarily follow a parsimonious movement, as it happens for instance in the sequence of Figure 13. There, pitch names on the vertices will be erased to avoid potential bias, and the first six chords are associated with red (for major chords) and blue triangles (for minor chords). When the last chord sounds, six new triangles are illuminated in a third colour (yellow). Participants are asked to choose, according to their subjective intuition, the yellow triangle that fits best with the sound stimulus. The task will be presented twice: for the first time, the participant is unaware of what the *Tonnetz* is, whereas the second attempt is performed after a short video tutorial. The device stores, for each sequence, the user's response time and the selected triangle matching with its triad for a quantitative comparison with the correct one. In addition, a follow-on questionnaire will be aimed at qualitatively grasping the mental strategies brought into play when elucidating the answer.

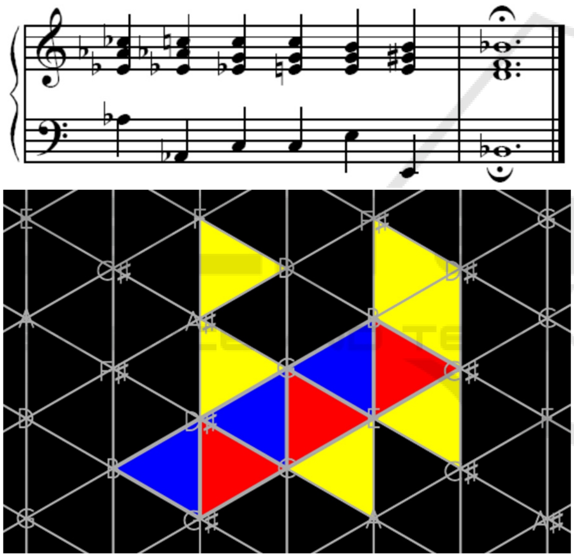

Figure 13: Top: A musical sequence among those planned for the cognitive experiences with our adaptation of `Tonnetz`. Bottom: Partial screenshot when executing the sequence. We provide the pitch names for a better understanding of this figure, although they will not be shown during the tests.

We currently test the viability of the protocol with a few individuals. Our goal is to run it with different populations. First, we are going to compare subjects whose cognitive styles are supposed to be very contrasting from each other. Following the E-S theory (Baron-Cohen, Knickmeyer, and Belmonte, 2005), we may distinguish between "the capacity to respond to feeling states of other individuals" (empathizing cognitive style) and the "capacity to respond to regularities of objects and events" (systemizing cognitive style). The E-S theory has been applied to the musical domain, (Kreutz, Schubert, and Mitchel, 2008) leading to the development of music empathizing (ME) and music systemizing (MS) scales as emerging from principal component analysis procedures. Questions aimed at distinguish between the two groups concerned thoughts about the emotional state of the composer or the performers when listening to the music, the importance ascribed to physics and acoustics of musical instruments, music structure, or the different layers of instruments and voices.

In our study, we are going adapt the ME-MS scales to the participants' mother languages (Schubert et al., 2014) for a population of trained musicians and scientific professionals to explore some potential specificities in achieving a "mathemusical" goal. While no significant difference between these groups is awaited at a first stage without training, we expect to disclose a number of correlations between the preferred cognitive style and the amount and specificity of the improvements, after some training at a second stage. In a follow-on experiment, a group of university students will be involved, adhering to a very similar protocol. In the future, we plan to consider the dual representation of the *Tonnetz*, as well as musical sequences related to non-standard *Tonnetze* or even more general related structures (Tymoczko, 2012).

6 CONCLUSIONS

In this paper we have shown how geometry-based and interactive web environments such as `Tonnetz` can be useful for supporting "mathemusical" learning and research activities. Within the larger SMIR project, the main goal was to develop a dissemination plan through several "mathemusical" workshops and exhibitions, suitable for pedagogical applications. Our activities primarily targeted high-school and university students, but were also conceived for larger audiences.

Besides this outreach achievement, the SMIR project disclosed new interesting research questions belonging to the areas of psychology and cognition. These questions are currently addressed by the ProAppMaMu project, focusing on how the more complex geometrical representations implemented in `Tonnetz` are processed during an active listening. In order to tackle this cognitive component of the SMIR project, we are adapting this web environment by integrating a series of perceptual tests which are aimed at better understanding how these geometrical representations are processed by the listeners. These tests will start by observing small populations with different musical and scientific skills, but will soon be generalized to larger groups. Our experimental approach may hopefully shed some light on the way in which some geometrical models of music theory could be used to reinforce educational strategies, even those beyond music education. They will surely enhance existing learning techniques within the exciting context of a musically driven mathematical education.

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