Uniformly Regular Triangulations for Parameterizing Lyapunov Functions

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Abstract: The computation of Lyapunov functions to determine the basins of attraction of equilibria in dynamical systems can be achieved using linear programming. In particular, we consider a CPA (continuous piecewise affine) Lyapunov function, which can be fully described by its values at the vertices of a given triangulation. The method is guaranteed to find a CPA Lyapunov function, if a sequence of finer and finer triangulations with a bound on their degeneracy is considered. Hence, the notion of (h,d)-bounded triangulations was introduced, where h is a bound on the diameter of each simplex and d a bound on the degeneracy, expressed by the so-called shape-matrices of the simplices. However, the shape-matrix, and thus the degeneracy, depends on the ordering of the vertices in each simplex. In this paper, we first remove the rather unnatural dependency of the degeneracy on the ordering of the vertices and show that an (h,d)-bounded triangulation, of which the ordering of the vertices is changed, is still (h,d^*)-bounded, where d^* is a function of d, h, and the dimension of the system. Furthermore, we express the degeneracy in terms of the condition number, which is a well-studied quantity.

1 INTRODUCTION

Lyapunov stability theory is of essential importance in dynamical systems and control theory and is studied in practically all textbooks and monographs on linear and nonlinear systems, cf. e.g. (Zubov, 1964; Yoshizawa, 1966; Hahn, 1967) or (Sastry, 1999; Vidyasagar, 2002; Khalil, 2002) for a more modern treatment. The canonical candidate for a Lyapunov function for a physical system is its (free) energy. In particular, a dissipative physical system must approach the state of a local minimum of the energy.

For general dynamical systems, however, there is no analytical method to obtain a Lyapunov function. For this reason, various methods for the numerical generation of Lyapunov functions have emerged. To name a few, in (Vannelli and Vidyasagar, 1985; Valmorbida and Anderson, 2017) the numerical generation of rational Lyapunov functions was studied, in (Parrilo, 2000; Chesi, 2011; Anderson and Papachristodoulou, 2015) sum-of-squared (SOS) polynomial Lyapunov functions were parameterized using semi-definite optimization, see also (Ratschan and She, 2010; Kamyar and Peet, 2015) for other approaches using polynomials, and in (Giesl, 2007) a Zubov type PDE was approximately solved using collocation. For more numerical approaches cf. the review (Giesl and Hafstein, 2015b).

In (Julian et al., 1999; Marinósson, 2002) linear programming was used to parameterize continuous and piecewise affine (CPA) Lyapunov functions. In this approach, a subset of the state space is first triangulated, i.e. subdivided into simplices, and then a number of constraints are derived for a given nonlinear system, such that a feasible solution to the resulting linear programming problem allows for the parametrization of a CPA Lyapunov function for the system.

In (Hafstein, 2004; Hafstein, 2005; Giesl and Hafstein, 2014) it was proved that this approach always succeeds in computing a Lyapunov function for a general nonlinear system with an exponentially stable equilibrium, if the simplices are sufficiently small and non-degenerate. The proof of this fact used the concept of (h,d)-bounded triangulations, see Definition 3.1, where $h>0$ is an upper bound on the diameters of the simplices and $d>0$ quantifies the degeneracy of the simplices. For the definition of (h,d)-

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bounded triangulations one must consider triangulations, of which the order of the vertices of each simplex has been fixed.

The first main contribution of this paper is to show that if $\mathcal{T}$ is an (h,d)-bounded triangulation in $\mathbb{R}^n$, $n \geq 2$, then any triangulation consisting of the same simplices as $\mathcal{T}$, but with a different ordering of the vertices, is an (h,d^*)-bounded triangulation with

$$d^* = d(1+d\sqrt{n-1}).$$

Thus, the property that a triangulation is (h,d)-bounded depends essentially on the simplices of the triangulation $\mathcal{T}$, and not the ordering of the vertices of the simplices. Note that the case $n=1$ is trivial and any ordering of the vertices gives an (h,d)-bounded triangulation of the line.

The second main contribution is a characterization of (h,d)-bounded triangulations using the condition number of the shape-matrices of the simplices, cf. Definition 4.2. The advantage of this characterization is that the condition number of a matrix is a more familiar concept than the degeneracy as defined in Definition 3.1.

The paper is organized as follows. After introducing some notations, we define triangulations, CPA functions, shape-matrices of simplices, and (h,d)-bounded triangulations in Section 2. In Section 3 we outline the algorithm to compute CPA Lyapunov functions and explain the relevance of (h,d)-bounded triangulations for the algorithm. In Section 4 we prove our main results in Proposition 4.1, 4.4 and 4.5, before we give conclusions in Section 5.

1.1 Prerequisites and Notation

$\mathbb{N}_0$ denotes the set $\{0,1,2,\ldots,\}$. For a vector $\mathbf{x} \in \mathbb{R}^n$ and $p \geq 1$ we define the norm $\|\mathbf{x}\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$. We also define $\|\mathbf{x}\|_\infty = \max_{i\in\{1,\ldots,n\}} |x_i|$. We will repeatedly use the norm equivalence relation

$$\|\mathbf{x}\|_p \leq \|\mathbf{x}\|_q \leq n^{q^{-1}-p^{-1}} \|\mathbf{x}\|_p \quad \text{for } p > q.$$

The *induced matrix norm* $\|\cdot\|_p$ is defined by $\|A\|_p = \max_{\|\mathbf{x}\|_p=1} \|A\mathbf{x}\|_p$. Clearly $\|A\mathbf{x}\|_p \leq \|A\|_p\|\mathbf{x}\|_p$. For a matrix A we write A^{T} for its transpose. Recall that $\|A\|_1 = \|A^{\mathrm{T}}\|_\infty = \max_i \|\mathbf{a}_i\|_1$, where $\mathbf{a}_i$ are the column vectors of A, and the norm equivalences

$$\frac{1}{\sqrt{n}}\|A\|_p \leq \|A\|_2 \leq \sqrt{n}\,\|A\|_p$$

for $A \in \mathbb{R}^{n\times n}$ and $p \in \{1,\infty\}$. The *condition number* κ_p of a nonsingular matrix $A \in \mathbb{R}^{n\times n}$ with respect to the norm $\|\cdot\|_p$ is defined as $\kappa_p(A) := \|A\|_p\|A^{-1}\|_p$.

We utilize a bold-face font for (column) vectors, e.g. $\mathbf{x} \in \mathbb{R}^{n\times 1} = \mathbb{R}^n$. For a vector $\mathbf{x}$ we write x_i or $[\mathbf{x}]_i$ for its ith component. We denote by $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ the standard orthonormal basis of $\mathbb{R}^n$ and by I the identity matrix. We denote the interior of a set $\mathcal{S} \subset \mathbb{R}^n$ by $\mathcal{S}^\circ$ and its closure by $\overline{\mathcal{S}}$.

If $B \in \mathbb{R}^{n\times n}$ and $\mathbf{u},\mathbf{v} \in \mathbb{R}^n$, then

$$\mathbf{u}\mathbf{v}^{\mathrm{T}}B\mathbf{u}\mathbf{v}^{\mathrm{T}}B = (\mathbf{v}^{\mathrm{T}}B\mathbf{u})\mathbf{u}\mathbf{v}^{\mathrm{T}}B$$

because $\mathbf{v}^{\mathrm{T}}B\mathbf{u} \in \mathbb{R}$. From this simple observation the very useful Sherman-Morrison lemma on the rank 1 correction $A + \mathbf{u}\mathbf{v}^{\mathrm{T}}$ of an invertible matrix A follows, cf. (Sherman and Morrison, 1950).

Lemma 1.1 (Sherman-Morrison). *Let $A \in \mathbb{R}^{n\times n}$ be invertible and $\mathbf{u},\mathbf{v} \in \mathbb{R}^n$. Then*

$$\left(A+\mathbf{u}\mathbf{v}^{\mathrm{T}}\right)^{-1} = A^{-1} - \frac{A^{-1}\mathbf{u}\mathbf{v}^{\mathrm{T}}A^{-1}}{1+\mathbf{v}^{\mathrm{T}}A^{-1}\mathbf{u}},$$

provided $1+\mathbf{v}^{\mathrm{T}}A^{-1}\mathbf{u} \neq 0$. Furthermore, we have the following identity:

$$\det\left(A+\mathbf{u}\mathbf{v}^{\mathrm{T}}\right) = \left(1+\mathbf{v}^{\mathrm{T}}A^{-1}\mathbf{u}\right)\det A.$$

The determinant identity can be seen from

$$(I+\mathbf{x}\mathbf{v}^{\mathrm{T}})\mathbf{x} = (1+\mathbf{v}^{\mathrm{T}}\mathbf{x})\mathbf{x}$$

and

$$(I+\mathbf{x}\mathbf{v}^{\mathrm{T}})\mathbf{z} = \mathbf{z} \quad \text{for } \mathbf{z} \in \mathbb{R}^n \text{ with } \mathbf{v}^{\mathrm{T}}\mathbf{z} = 0.$$

Thus, the eigenvalues of $(I+\mathbf{x}\mathbf{v}^{\mathrm{T}})$ are $1+\mathbf{v}^{\mathrm{T}}\mathbf{x}$ and $(n-1)$-times 1, we have $\det(I+\mathbf{x}\mathbf{v}^{\mathrm{T}}) = 1+\mathbf{v}^{\mathrm{T}}\mathbf{x}$, and it follows with $\mathbf{x} = A^{-1}\mathbf{u}$ that

$$\begin{aligned}\det\left(A+\mathbf{u}\mathbf{v}^{\mathrm{T}}\right) &= \det A \cdot \det\left(I+A^{-1}\mathbf{u}\mathbf{v}^{\mathrm{T}}\right) \\ &= \left(1+\mathbf{v}^{\mathrm{T}}A^{-1}\mathbf{u}\right)\det A.\end{aligned}$$

2 TRIANGULATIONS AND CPA FUNCTIONS

In this section we will introduce triangulations and CPA functions as well as the definition of (h,d)-bounded triangulations.

Definition 2.1. *We define the following* :

i) The **convex-combination** of vectors $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_m \in \mathbb{R}^n$, denoted

$$\mathrm{co}\{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_m\},$$

is the set of all sums

$$\sum_{i=0}^m \lambda_i \mathbf{x}_i, \quad \text{where} \quad \sum_{i=0}^m \lambda_i = 1$$

and $\forall i: 0 \leq \lambda_i \leq 1$.

ii) The vectors $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_m \in \mathbb{R}^n$ are said to be **affinely-independent** if

$$\sum_{i=0}^{m} \lambda_i \mathbf{x}_i = \mathbf{0} \quad \text{and} \quad \sum_{i=0}^{m} \lambda_i = 0$$

implies $\lambda_0 = \lambda_1 = \cdots = \lambda_m = 0$.

iii) If $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_m \in \mathbb{R}^n$ are affinely-independent, then the set $S = \operatorname{co}\{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_m\}$ is called an m-**simplex**. The vectors $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_m$ are called the **vertices** of S. The set of vertices for an m-simplex is sometimes denoted by $\operatorname{ve} S = \{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_m\}$. In $\mathbb{R}^n$ an n-simplex is often referred to as just a simplex.

iv) For an m-simplex S, define its **diameter** as:

$$\operatorname{diam}(S) := \max_{\mathbf{x}, \mathbf{y} \in S} \|\mathbf{x} - \mathbf{y}\|_2.$$

We now define a *triangulation*. For our purposes it is advantageous to have the order of the vertices of every simplex in the triangulation fixed, similar to (Giesl and Hafstein, 2015a). The reason for this becomes clear when we introduce shape-matrices of simplices. For an n-tuple of vertices $C = (\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n)$ we define $\operatorname{co} C = \operatorname{co}\{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n\}$.

Definition 2.2 (Triangulation). *Let I be a set of indices. A* triangulation $\mathcal{T} = \{S_\nu\}_{\nu \in I}$ *in $\mathbb{R}^n$ is a set of n-simplices S_ν with ordered vertices $C_\nu = (\mathbf{x}_0^\nu, \mathbf{x}_1^\nu, \ldots, \mathbf{x}_n^\nu)$ for all $\nu \in I$, such that*

$$S_\mu \cap S_\nu = \operatorname{co} \operatorname{ve} S_\mu \cap \operatorname{co} \operatorname{ve} S_\nu = \operatorname{co}(\operatorname{ve} S_\mu \cap \operatorname{ve} S_\nu) \quad (1)$$

for all $\mu, \nu \in I$. The domain of $\mathcal{T}$ is defined as

$$\mathcal{D}_\mathcal{T} := \bigcup_{\nu \in I} S_\nu$$

and its complete set of vertices is denoted by

$$\mathcal{V}_\mathcal{T} := \bigcup_{\nu \in I} \operatorname{ve} S_\nu.$$

Further, we define the diameter of $\mathcal{T}$ as

$$\operatorname{diam}(\mathcal{T}) := \sup_{S \in \mathcal{T}} \operatorname{diam}(S).$$

Given a triangulation $\mathcal{T}$ a continuous and piecewise affine function, i.e. CPA function, can be defined by fixing its values at $\mathcal{V}_\mathcal{T}$.

Definition 2.3 (CPA function). *Let $\mathcal{T}$ be a triangulation in $\mathbb{R}^n$. We denote by* $\mathrm{CPA}[\mathcal{T}]$ *the set of all continuous functions*

$$V : \mathcal{D}_\mathcal{T} \to \mathbb{R}$$

that are affine on each simplex $S_\nu \in \mathcal{T}$, i.e. for each $S_\nu \in \mathcal{T}$ there exists a vector $\mathbf{w}_\nu \in \mathbb{R}^n$ and a number $a_\nu \in \mathbb{R}$ such that

$$V(\mathbf{x}) = \mathbf{w}_\nu^{\mathrm{T}} \mathbf{x} + a_\nu \quad \forall \mathbf{x} \in S_\nu.$$

Let $V \in \mathrm{CPA}[\mathcal{T}]$ and $\mathbf{x} \in \mathcal{D}_\mathcal{T}$. Then there is a simplex $S = \operatorname{co}(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n) \in \mathcal{T}$ such that $\mathbf{x} \in S$. Further, $\mathbf{x}$ has a unique representation as the convex combination of the vertices of S, i.e. there are unique numbers $\lambda_i^\mathbf{x} \in [0,1]$, $i = 0, 1, \ldots, n$, such that

$$\mathbf{x} = \sum_{i=0}^{n} \lambda_i^\mathbf{x} \mathbf{x}_i \quad \text{and} \quad \sum_{i=0}^{n} \lambda_i^\mathbf{x} = 1.$$

It is not difficult to see that

$$V(\mathbf{x}) = \sum_{i=0}^{n} \lambda_i^\mathbf{x} V(\mathbf{x}_i).$$

Hence, each $V \in \mathrm{CPA}[\mathcal{T}]$ is completely determined by its values in the vertex set $\mathcal{V}_\mathcal{T}$.

To have concrete examples of triangulations useful for the CPA algorithm we recall the definition of the standard triangulation $\mathcal{T}_{\mathrm{std}}$ as given in (Albertsson et al., 2020); for a graphical representation see Figure 1.

Definition 2.4 (The Standard Triangulation of $\mathbb{R}^n$). *The* Standard Triangulation *is a triangulation $\mathcal{T}_{\mathrm{std}} = \{S_\nu\}_{\nu \in I}$ with indices $\nu = (\mathbf{z}, \sigma, \mathbf{J}) \in \mathbb{N}_0^n \times Sym(n) \times \{-1,+1\}^n =: I$ and vertices $C_\nu = (\mathbf{x}_0^\nu, \mathbf{x}_1^\nu, \ldots, \mathbf{x}_n^\nu)$ given by:*

$$\mathbf{x}_k^\nu = R_\mathbf{J} \left(\mathbf{z} + \sum_{l=1}^{k} \mathbf{e}_{\sigma(l)} \right) = R_\mathbf{J} \mathbf{z} + R_\mathbf{J} \mathbf{u}_k^\sigma. \quad (2)$$

Here, $\mathbf{J} = (J_1, J_2, \ldots, J_n)^{\mathrm{T}} \in \{-1,+1\}^n$ and $R_\mathbf{J} = \operatorname{diag}(\mathbf{J}) \in \mathbb{R}^{n \times n}$ is a matrix corresponding to the reflection specified by $\mathbf{J} \in \{-1,+1\}^n$. Further, $Sym(n)$ denotes the set of the permutations $\sigma : \{1,2,\ldots,n\} \to \{1,2,\ldots,n\}$ and

$$\mathbf{u}_k^\sigma = \sum_{l=1}^{k} \mathbf{e}_{\sigma(l)}.$$

We now define the shape-matrix of a simplex, of which the vertices are in a particular order. This is needed to define (h,d)-bounded triangulations. Then we explain the importance of shape-matrices in computing CPA Lyapunov functions in Section 3.

Definition 2.5. *For an n-simplex S of a triangulation with vertices $C_\nu = (\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n)$ its* **shape-matrix** *X_S is defined by*

$$X_S := \begin{pmatrix} (\mathbf{x}_1 - \mathbf{x}_0)^{\mathrm{T}} \\ (\mathbf{x}_2 - \mathbf{x}_0)^{\mathrm{T}} \\ \vdots \\ (\mathbf{x}_n - \mathbf{x}_0)^{\mathrm{T}} \end{pmatrix} \in \mathbb{R}^{n \times n}.$$

Notice, that because S in the definition of a shape-matrix is an n-simplex, its vertices $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n$ are affinely independent vectors, so the shape matrix X_S is nonsingular.

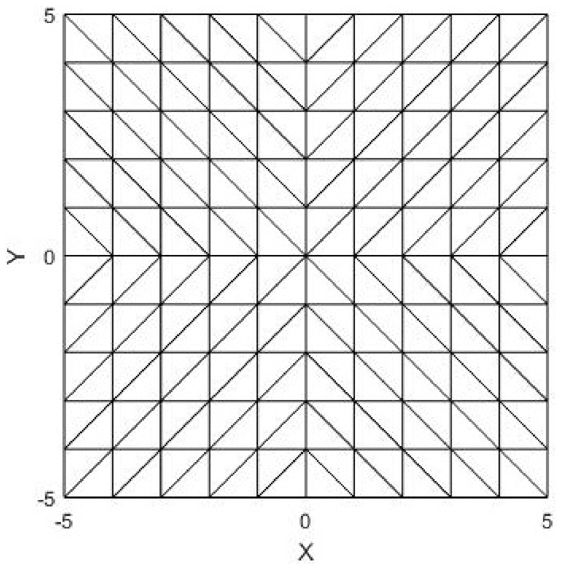

Figure 1: The standard triangulation $\mathcal{T}_{\text{std}}$ in $\mathbb{R}^2$ on $[-5,5]^2$.

Remark 2.6. *Important for computing CPA Lyapunov functions is not the shape-matrix itself but the quantity $\|X_S^{-1}\|_p$, where usually $p = 2$, but for some applications $p = 1$ or $p = \infty$ are more appropriate. Because all norms on the finite-dimensional vector space $\mathbb{R}^{n\times n}$ are equivalent there is no fundamental difference between these norms. It is tempting to assume that the quantity $\|X_S^{-1}\|_p$ could be related to the determinant of X_S, because $(n!)^{-1}|\det X_S|$ is well known to be the volume of the simplex and does not depend on the choice of $\mathbf{x}_0$ or the order of the differences $\mathbf{x}_i - \mathbf{x}_0$ in the shape-matrix. However, as e.g. shown in (Golub and van Loan, 2013, §2.6.3), there is no correlation between $\|X_S^{-1}\|_p$ and $|\det X_S|$.*

Let us give a short discussion on $\det X_S$, because we will need it later.

Remark 2.7. *That $|\det X_S|$ does not depend on the choice of $\mathbf{x}_0$ or the order of the differences $\mathbf{x}_i - \mathbf{x}_0$ in the shape-matrix follows from a volume argument, but can also be seen algebraically. Let $\mathbf{x}_i^a = (1, \mathbf{x}_i^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{R}^{n+1}$ be the vectors $\mathbf{x}_i \in \mathbb{R}^n$ augmented with 1 in the first position. Define $X^a = \left(\mathbf{x}_0^a\, \mathbf{x}_1^a \cdots \mathbf{x}_n^a\right)^{\mathrm{T}} \in \mathbb{R}^{(n+1)\times(n+1)}$. Defining $A = (a_{ij}) \in \mathbb{R}^{(n+1)\times(n+1)}$ through*

$$\begin{aligned} &a_{ii} = 1 \quad \text{for } i = 1, \ldots, n+1, \\ &a_{i1} = -1 \quad \text{for } i = 2, \ldots, n+1, \text{ and} \\ &a_{ij} = 0 \quad \text{otherwise,} \end{aligned}$$

gives

$$\begin{aligned} AX^a &= (\mathbf{x}_0^a, \mathbf{x}_1^a - \mathbf{x}_0^a, \ldots, \mathbf{x}_n^a - \mathbf{x}_0^a)^{\mathrm{T}} \\ &= \begin{pmatrix} 1 & \mathbf{x}_0^{\mathrm{T}} \\ 0 & (\mathbf{x}_1 - \mathbf{x}_0)^{\mathrm{T}} \\ \vdots & \vdots \\ 0 & (\mathbf{x}_n - \mathbf{x}_0)^{\mathrm{T}} \end{pmatrix}. \end{aligned}$$

Laplace expansion on the first row of A and the first column of AX^a gives

$$\det A = 1 \quad \text{and} \quad \det(AX^a) = 1 \cdot \det X_S,$$

i.e. $\det X^a = \det X_S$. Choosing a different base vector $\mathbf{x}_0$ and rearranging the order of the differences $\mathbf{x}_i - \mathbf{x}_0$ corresponds to choosing a permutation matrix $P \in \mathbb{R}^{(n+1)\times(n+1)}$, cf. the discussion after Definition 4.2, and considering the matrix APX^a. Since

$$|\det APX^a| = |\det X^a| = |\det X_S|$$

the proposition follows.

3 CONSTRUCTION OF CPA LYAPUNOV FUNCTIONS

Let us explain in detail why the quantity $\|X_S^{-1}\|_p$ is of so much interest in our application of computing CPA Lyapunov functions. To prove that the algorithm in (Giesl and Hafstein, 2014) always succeeds in computing a CPA Lyapunov functions for any system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, $\mathbf{f} \in C^2(\mathbb{R}^n, \mathbb{R}^n)$, with an exponentially stable equilibrium at the origin, one uses the fact that there exists a C^2 Lyapunov function W for the system. This function W is used to prove that the linear programming problem in the algorithm has a feasible solution for a suitable triangulation.

In the proof in (Giesl and Hafstein, 2014) W is approximated on $S = \mathrm{co}(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n)$ by its interpolation W_{CPA} on S: With

$$\mathbf{x} = \sum_{i=0}^{n} \lambda_i^{\mathbf{x}} \mathbf{x}_i \in S$$

as the unique convex combination of the vertices, we set

$$W_{\text{CPA}}(\mathbf{x}) = \sum_{i=0}^{n} \lambda_i^{\mathbf{x}} W(\mathbf{x}_i).$$

While this obviously approximates the values of W well on a simplex S with a small diameter $h := \mathrm{diam}(S)$, e.g. using

$$\begin{aligned} |W(\mathbf{x}) - W_{\text{CPA}}(\mathbf{x})| &\leq \sum_{i=0}^{n} \lambda_i^{\mathbf{x}} |W(\mathbf{x}) - W_{\text{CPA}}(\mathbf{x}_i)| \\ &\leq h \cdot \max_{\mathbf{z}\in S} \|\nabla W(\mathbf{z})\|_2, \end{aligned}$$

this is not sufficient for the proof, because we additionally need ∇W_{CPA} to closely approximate ∇W at the vertices $\mathbf{x}_i$, cf. the proof of Theorem 5 in (Giesl and Hafstein, 2014).

It is not difficult to show that ∇W_{CPA} is the constant vector $X_S^{-1}\mathbf{w}$, where

$$\mathbf{w} = \begin{pmatrix} W(\mathbf{x}_1) - W(\mathbf{x}_0) \\ W(\mathbf{x}_2) - W(\mathbf{x}_0) \\ \vdots \\ W(\mathbf{x}_n) - W(\mathbf{x}_0) \end{pmatrix},$$

for all $\mathbf{x} \in S^\circ$, cf. Remark 9 in (Giesl and Hafstein, 2014). From this one obtains by Taylor expansion, cf. (19) in (Giesl and Hafstein, 2014),

$$[\mathbf{w} - X_S \nabla W(\mathbf{x}_0)]_i = (\mathbf{x}_i - \mathbf{x}_0)^{\mathrm{T}} H_W(\mathbf{z}_i)(\mathbf{x}_i - \mathbf{x}_0) \quad (3)$$

for $i = 1, 2, \ldots, n$, where H_W is the Hessian matrix of W and the $\mathbf{z}_i$ are points in S.

Now

$$\begin{aligned} \|\nabla W_{\text{CPA}} - \nabla W(\mathbf{x}_i)\|_p &\leq \|X_S^{-1}\mathbf{w} - \nabla W(\mathbf{x}_0)\|_p \\ &\quad + \|\nabla W(\mathbf{x}_0) - \nabla W(\mathbf{x}_i)\|_p \end{aligned}$$

and the term $\|\nabla W(\mathbf{x}_0) - \nabla W(\mathbf{x}_i)\|_p$ is small if the diameter h of the simplex S is small because $W \in C^2$. The term $\|X_S^{-1}\mathbf{w} - \nabla W(\mathbf{x}_0)\|_p$, however, is not necessarily small even though the diameter h of the simplex S is small. But we have

$$\|X_S^{-1}\mathbf{w} - \nabla W(\mathbf{x}_0)\|_p \leq \|X_S^{-1}\|_p \|\mathbf{w} - X_S \nabla W(\mathbf{x}_0)\|_p,$$

and the ith entry of the vector $\mathbf{w} - X_S \nabla W(\mathbf{x}_0)$ can be bounded using (3),

$$\begin{aligned} |[\mathbf{w} - X_S \nabla W(\mathbf{x}_0)]_i| &= |(\mathbf{x}_i - \mathbf{x}_0)^{\mathrm{T}} H_W(\mathbf{z}_i)(\mathbf{x}_i - \mathbf{x}_0)| \\ &\leq h^2 \cdot \sup_{\mathbf{z} \in S} \|H_W(\mathbf{z})\|_2. \end{aligned}$$

Because of this, the proof in (Giesl and Hafstein, 2014) that the CPA method always succeeds in computing a Lyapunov function if one exists, uses a sequence of finite triangulations $\mathcal{T}_k$ where the simplices become smaller, i.e. $h \to 0$ as $k \to \infty$, but also such that $h^2 \cdot \|X_S^{-1}\|_p \to 0$ as $k \to \infty$, or, as a sufficient condition, that $h \cdot \|X_S^{-1}\|_p \leq d$ is bounded.

Now note that when we scale down the simplex S, i.e. multiply the vertices of S with a number $0 < s < 1$, then $\operatorname{diam}(sS) = s \operatorname{diam}(S)$ and $\|X_{sS}^{-1}\|_p = s^{-1}\|X_S^{-1}\|_p$. This leads to the following strategy of obtaining a suitable sequence of triangulations $\mathcal{T}_k$ for proving that the algorithm in (Giesl and Hafstein, 2014) succeeds in computing a Lyapunov function on any compact set $\mathcal{C}$, that is contained in the basin of attraction of the equilibrium at the origin. For simplicity we ignore some adaptations that have to be made close to the equilibrium, but do not change the main idea of the proof:

We have that $\operatorname{diam}(\mathcal{T}_{\text{std}}) = \sqrt{n}$ and from Remark 2 in (Hafstein and Valfells, 2017) we know that $\sup_{S \in \mathcal{T}_{\text{std}}} \|X_S^{-1}\|_p \leq 2$ for $p = 1, 2, \infty$. Fix a constant s fulfilling $0 < s < 1$ and define

$$\mathcal{T}_k := \{s^k S_{\mathrm{v}} : (s^k S_{\mathrm{v}}) \cap \mathcal{C}^\circ \neq \emptyset\}$$

for $k \in \mathbb{N}_0$. Then for each $k \in \mathbb{N}_0$, $\mathcal{T}_k$ consists of a finite number of simplices, we have

$$\operatorname{diam}(\mathcal{T}_k) = s^k \sqrt{n}$$

and

$$\sup_{s^k S \in \mathcal{T}_k} \operatorname{diam}(s^k S) \|X_{s^k S}^{-1}\|_p \leq s^k \sqrt{n} \cdot s^{-k} 2 = 2\sqrt{n}.$$

Thus

$$\operatorname{diam}(\mathcal{T}_k) \to 0$$

as $k \to \infty$ and

$$\sup_{s^k S \in \mathcal{T}_k} \operatorname{diam}(s^k S) \|X_{s^k S}^{-1}\|_p \leq 2\sqrt{n} =: d$$

is bounded. Hence, it follows that W_{CPA} and ∇W_{CPA} approximate W and ∇W arbitrarily close on $\mathcal{C}$ for sufficiently large k.

With identical argumentation an arbitrary sequence of triangulations $\mathcal{T}_k$ such that

- $\operatorname{diam}(\mathcal{T}_k) \to 0$ for $k \to 0$ and
- there is a bound d such that $\sup_{S \in \mathcal{T}_k} \operatorname{diam}(S) \|X_S^{-1}\|_p \leq d$ holds for all $k \in \mathbb{N}_0$

can be used in the proof in (Giesl and Hafstein, 2014).

The situation explained above led to the definition of the degeneracy of a triangulation and (h,d)-bounded triangulations in (Giesl and Hafstein, 2015a).

Definition 3.1. *We define the* **degeneracy** *of the triangulation $\mathcal{T}$ to be the quantity*

$$\sup_{S \in \mathcal{T}} \operatorname{diam}(S) \|X_S^{-1}\|_2,$$

*where X_S is the shape-matrix of S. We say that the triangulation $\mathcal{T}$ is (h,d)-***bounded** *for constants $h, d > 0$, if $\operatorname{diam}(\mathcal{T}) < h$ and the degeneracy of $\mathcal{T}$ is bounded by d, i.e. $\sup_{S \in \mathcal{T}} \operatorname{diam}(S) \|X_S^{-1}\|_2 \leq d$.*

4 MAIN RESULTS

As explained in the last section, the algorithm seeks to find a sequence of triangulations $\mathcal{T}_k$ such that each triangulation $\mathcal{T}_k$ is (h_k, d)-bounded, where $h_k \to 0$ as $k \to \infty$ and $d > 0$ is a constant independent of k.

Our first main result is Proposition 4.1, which shows that the concept of (h,d)-bounded triangulations can equivalently be formulated in terms of the norm and the condition number of the shape-matrices of the triangulation.

Proposition 4.1. *Let $S = \mathrm{co}(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n)$ be a simplex and X_S be its corresponding shape-matrix. Then*

$$\frac{1}{n}\|X_S\|_2 \leq \mathrm{diam}(S) \leq 2\sqrt{n}\,\|X_S\|_2$$

and

$$\frac{1}{n}\kappa_2(X_S) \leq \mathrm{diam}(S)\|X_S^{-1}\|_2 \leq 2\sqrt{n}\,\kappa_2(X_S).$$

Proof. Fix $i, j \in \{0, 1, \ldots, n\}$ such that $\mathrm{diam}(S) = \|\mathbf{x}_i - \mathbf{x}_j\|_2$ and $k \in \{1, 2, \ldots, n\}$ such that $\|X_S\|_\infty = \|\mathbf{x}_k - \mathbf{x}_0\|_1$. Then we have

$$\begin{aligned}\|X_S\|_\infty &= \|\mathbf{x}_k - \mathbf{x}_0\|_1 \leq \sqrt{n}\,\|\mathbf{x}_k - \mathbf{x}_0\|_2 \\ &\leq \sqrt{n}\,\|\mathbf{x}_i - \mathbf{x}_j\|_2 = \sqrt{n}\,\mathrm{diam}(S)\end{aligned}$$

and

$$\begin{aligned}\mathrm{diam}(S) &= \|\mathbf{x}_i - \mathbf{x}_j\|_2 \leq \|\mathbf{x}_i - \mathbf{x}_0\|_2 + \|\mathbf{x}_j - \mathbf{x}_0\|_2 \\ &\leq \|\mathbf{x}_i - \mathbf{x}_0\|_1 + \|\mathbf{x}_j - \mathbf{x}_0\|_1 \leq 2\|X_S\|_\infty.\end{aligned}$$

Hence,

$$\begin{aligned}\|X_S\|_2 &\leq \sqrt{n}\,\|X_S\|_\infty \leq n\,\mathrm{diam}(S) \\ &\leq 2n\|X_S\|_\infty \leq 2n\sqrt{n}\,\|X_S\|_2\end{aligned}$$

and

$$\begin{aligned}\kappa_2(X_S) &= \|X_S\|_2\|X_S^{-1}\|_2 \leq n\,\mathrm{diam}(S)\|X_S^{-1}\|_2 \\ &\leq 2n\sqrt{n}\,\kappa_2(X_S).\end{aligned}$$

□

Thus, a triangulation $\mathcal{T}$ is (h,d)-bounded for some constants $h, d > 0$, if and only if there exists constants $h^*, d^* > 0$ such that

$$\|X_S\|_2 \leq h^* \quad \text{and} \quad \kappa_2(X_S) \leq d^* \quad \text{for all } S \in \mathcal{T},$$

where X_S is the shape-matrix corresponding to the simplex $S \in \mathcal{T}$. In either case we define it to be uniformly regular:

Definition 4.2 (Uniformly regular triangulations)**.** *A triangulation $\mathcal{T}$ in $\mathbb{R}^n$ consisting of simplices with ordered vertices is said to be* uniformly regular *if there exist constants $h, d > 0$ such that*

$$\mathrm{diam}(S) \leq h \quad \textit{and} \quad \mathrm{diam}(S)\|X_S^{-1}\|_2 \leq d$$

for all $S \in \mathcal{T}$, or equivalently if there exist constants $h^*, d^* > 0$ such that

$$\|X_S\|_2 \leq h^* \quad \textit{and} \quad \kappa_2(X_S) \leq d^*$$

for all $S \in \mathcal{T}$. Here X_S denotes the shape-matrix of the simplex S.

We will now prepare our second main result, showing that a uniformly regular triangulation does not depend on the order of the vertices in Proposition 4.4. Given a permutation $\alpha \in \mathrm{Sym}(n)$ of the numbers $\{1, 2, \ldots, n\}$, the *permutation matrix* $P_\alpha \in \mathbb{R}^{n\times n}$ is defined through

$$P_\alpha \mathbf{e}_k = \mathbf{e}_{\alpha(k)} \quad \text{for } k = 1, 2, \ldots, n.$$

It is not difficult to see that $P_\alpha^{-1} = P_\alpha^{\mathrm{T}}$ and $\|P_\alpha\|_p = \|P_\alpha^{-1}\|_p = 1$ for $p \in \{1, 2, \infty\}$. Note that left-multiplication by P_α permutes the rows- and right-multiplication permutes the columns of a vector or a matrix, e.g. with $\mathbf{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$ and $\mathbf{x}_\alpha = \left(x_{\alpha(1)}, x_{\alpha(2)}, \ldots, x_{\alpha(n)}\right)^{\mathrm{T}}$ we have $P_\alpha \mathbf{x} = \mathbf{x}_\alpha$ and $\mathbf{x}^{\mathrm{T}} P_\alpha = \mathbf{x}_\alpha^{\mathrm{T}}$.

We have the following simple result.

Lemma 4.3. *Let $X, P, Q \in \mathbb{R}^{n\times n}$ be matrices, P, Q nonsingular, and $\|\cdot\|$ any sub-multiplicative matrix norm. If*

$$\|Q\| = \|Q^{-1}\| = \|P\| = \|P^{-1}\| = 1,$$

then

$$\|QXP\| = \|X\|.$$

In particular,

$$\|P_\alpha X P_\beta\|_p = \|X\|_p \quad \textit{for } p \in \{1, 2, \infty\}$$

and for any permutation matrices $P_\alpha, P_\beta \in \mathbb{R}^{n\times n}$.

Proof. The first statement follows immediately from

$$\begin{aligned}\|X\| &= \|Q^{-1}QXPP^{-1}\| \leq \|Q^{-1}\|\|QXP\|\|P^{-1}\| \\ &= \|QXP\| \leq \|Q\|\|X\|\|P\| = \|X\|.\end{aligned}$$

The second statement follows immediately from the comments above the lemma. □

In the next proposition we show one of our main results, namely that if a triangulation in $\mathbb{R}^n$, $n \geq 2$, is (h,d)-bounded for some particular ordering of the vertices of the simplices, then it is (h,d^*)-bounded for any ordering with $d^* = d(1 + d\sqrt{n-1})$. The case $n = 1$ is trivial with $d^* = d$.

Proposition 4.4. *Let $\mathcal{T} = \{\mathrm{co}\,C_\nu\}_{\nu\in I}$ be an (h,d)-bounded triangulation in $\mathbb{R}^n$, $n \geq 2$, and let $\mathcal{T}^* = \{\mathrm{co}\,C_\nu^*\}_{\nu\in I}$ be a triangulation consisting of the same simplices as $\mathcal{T}$, but with a (possibly) different ordering of the vertices. Then $\mathcal{T}^*$ is (h,d^*)-bounded, where $d^* = d\left(1 + d\sqrt{n-1}\right)$.*

Proof. Let $S = \mathrm{co}(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n) \in \mathcal{T}$, with shape-matrix $X_S = (\mathbf{x}_1 - \mathbf{x}_0, \mathbf{x}_2 - \mathbf{x}_0, \ldots, \mathbf{x}_n - \mathbf{x}_0)^{\mathrm{T}}$. Then $S = \mathrm{co}(\mathbf{x}_{\beta(0)}, \mathbf{x}_{\beta(1)}, \ldots, \mathbf{x}_{\beta(n)})$ is also a simplex in $\mathcal{T}^*$, where β is a permutation of $\{0, 1, \ldots, n\}$. If $\beta(0) = 0$, then the shape-matrix X_S^* of S in $\mathcal{T}^*$ has the same rows as the shape-matrix X_S of S in $\mathcal{T}$, just in a (possibly) different order. Then it follows immediately by Lemma 4.3 that $\|(X_S^*)^{-1}\|_2 = \|X_S^{-1}\|_2$ and then

$$\operatorname{diam}(S)\|(X_S^*)^{-1}\|_2 \leq d \leq d(1+d\sqrt{n-1}) =: d^*.$$

If $\beta(0) \neq 0$, then there is an $i \in \{1,2,\ldots,n\}$ such that $\beta(i) = 0$. Define $\alpha \in \mathrm{Sym}_n$ through $\alpha(i) = \beta(0)$ and $\alpha(k) = \beta(k)$ for $k \neq i$ and denote by P_α the permutation matrix defined through $P_\alpha \mathbf{e}_k = \mathbf{e}_{\alpha(k)}$. Then we have

$$X_S^* = \underbrace{R_i P_\alpha X_S}_{=:A} + \mathbf{u}\underbrace{(\mathbf{x}_0 - \mathbf{x}_{\alpha(i)})^{\mathrm{T}}}_{=:\mathbf{v}^{\mathrm{T}}}, \tag{4}$$

where

$$R_i := I - 2\mathbf{e}_i\mathbf{e}_i^{\mathrm{T}} \quad \text{and} \quad \mathbf{u} := \sum_{\substack{k=1 \\ k\neq i}}^{n} \mathbf{e}_k.$$

To show (4) we first calculate the left-hand side to be

$$X_S^* = \begin{pmatrix} (\mathbf{x}_{\beta(1)} - \mathbf{x}_{\beta(0)})^{\mathrm{T}} \\ \vdots \\ (\mathbf{x}_{\beta(i-1)} - \mathbf{x}_{\beta(0)})^{\mathrm{T}} \\ (\mathbf{x}_{\beta(i)} - \mathbf{x}_{\beta(0)})^{\mathrm{T}} \\ (\mathbf{x}_{\beta(i+1)} - \mathbf{x}_{\beta(0)})^{\mathrm{T}} \\ \vdots \\ (\mathbf{x}_{\beta(n)} - \mathbf{x}_{\beta(0)})^{\mathrm{T}} \end{pmatrix} = \begin{pmatrix} (\mathbf{x}_{\alpha(1)} - \mathbf{x}_{\alpha(i)})^{\mathrm{T}} \\ \vdots \\ (\mathbf{x}_{\alpha(i-1)} - \mathbf{x}_{\alpha(i)})^{\mathrm{T}} \\ (\mathbf{x}_0 - \mathbf{x}_{\alpha(i)})^{\mathrm{T}} \\ (\mathbf{x}_{\alpha(i+1)} - \mathbf{x}_{\alpha(i)})^{\mathrm{T}} \\ \vdots \\ (\mathbf{x}_{\alpha(n)} - \mathbf{x}_{\alpha(i)})^{\mathrm{T}} \end{pmatrix}.$$

For the right-hand side of (4) we have

$$\begin{aligned} A &= R_i P_\alpha \begin{pmatrix} (\mathbf{x}_1 - \mathbf{x}_0)^{\mathrm{T}} \\ \vdots \\ (\mathbf{x}_n - \mathbf{x}_0)^{\mathrm{T}} \end{pmatrix} \\ &= R_i \begin{pmatrix} (\mathbf{x}_{\alpha(1)} - \mathbf{x}_0)^{\mathrm{T}} \\ \vdots \\ (\mathbf{x}_{\alpha(n)} - \mathbf{x}_0)^{\mathrm{T}} \end{pmatrix} \\ &= \begin{pmatrix} (\mathbf{x}_{\alpha(1)} - \mathbf{x}_0)^{\mathrm{T}} \\ \vdots \\ (\mathbf{x}_{\alpha(i-1)} - \mathbf{x}_0)^{\mathrm{T}} \\ -(\mathbf{x}_{\alpha(i)} - \mathbf{x}_0)^{\mathrm{T}} \\ (\mathbf{x}_{\alpha(i+1)} - \mathbf{x}_0)^{\mathrm{T}} \\ \vdots \\ (\mathbf{x}_{\alpha(n)} - \mathbf{x}_0)^{\mathrm{T}} \end{pmatrix} \end{aligned}$$

and

$$\mathbf{u}\mathbf{v}^{\mathrm{T}} = \begin{pmatrix} (\mathbf{x}_0 - \mathbf{x}_{\alpha(i)})^{\mathrm{T}} \\ \vdots \\ (\mathbf{x}_0 - \mathbf{x}_{\alpha(i)})^{\mathrm{T}} \\ \mathbf{0}^{\mathrm{T}} \\ (\mathbf{x}_0 - \mathbf{x}_{\alpha(i)})^{\mathrm{T}} \\ \vdots \\ (\mathbf{x}_0 - \mathbf{x}_{\alpha(i)})^{\mathrm{T}} \end{pmatrix}$$

This shows (4). Now

$$\begin{aligned} |\det A| &= |\det(R_i P_\alpha X_S)| = |\det R_i| \cdot |\det P_\alpha| \cdot |\det X_S| \\ &= 1 \cdot 1 \cdot |\det X_S| = |\det X_S| \end{aligned}$$

and by Remark 2.7 we have $|\det X_S| = |\det X_S^*| \neq 0$.

Note that $1 + \mathbf{v}^{\mathrm{T}} A^{-1}\mathbf{u} \neq 0$. Indeed, otherwise we have

$$\begin{aligned} 0 &= \mathbf{u}(1 + \mathbf{v}^{\mathrm{T}} A^{-1}\mathbf{u}) \\ &= (A + \mathbf{u}\mathbf{v}^{\mathrm{T}})A^{-1}\mathbf{u} \\ &= X_S^* A^{-1}\mathbf{u}, \end{aligned}$$

which is a contradiction because X_S^* and A^{-1} are invertible and $\mathbf{u} \neq \mathbf{0}$.

Thus we obtain by Lemma 1.1 that

$$\left|1 + \mathbf{v}^{\mathrm{T}} A^{-1}\mathbf{u}\right| = \left|\frac{\det(A + \mathbf{u}\mathbf{v}^{\mathrm{T}})}{\det A}\right| = \frac{|\det X_S^*|}{|\det X_S|} = 1.$$

Further, again by Lemma 1.1, we obtain that

$$\left\|(X_S^*)^{-1}\right\|_2 = \left\|A^{-1} - \frac{A^{-1}\mathbf{u}\mathbf{v}^T A^{-1}}{1+\mathbf{v}^T A^{-1}\mathbf{u}}\right\|_2 \leq \|A^{-1}\|_2\left(1+\|A^{-1}\|_2\|\mathbf{u}\mathbf{v}^T\|_2\right).$$

It is easy to see that

$$\begin{aligned}\|\mathbf{u}\mathbf{v}^T\|_2 &= \max_{\|\mathbf{x}\|_2=1}\|\mathbf{u}\mathbf{v}^T\mathbf{x}\|_2 \\ &\leq \max_{\|\mathbf{x}\|_2=1}|\mathbf{v}^T\mathbf{x}|\,\|\mathbf{u}\|_2 \\ &\leq \|\mathbf{v}\|_2\sqrt{n-1} \\ &= \|\mathbf{x}_0-\mathbf{x}_{\alpha(i)}\|_2\sqrt{n-1} \\ &\leq \operatorname{diam}(S)\sqrt{n-1},\end{aligned}$$

and by Lemma 4.3 we have $\|A^{-1}\|_2 = \|X_S^{-1}P_\alpha^{-1}R_i^{-1}\|_2 = \|X_S^{-1}\|_2$, from which

$$\left\|(X_S^*)^{-1}\right\|_2 \leq \|X_S^{-1}\|_2\left(1+\|X_S^{-1}\|_2\operatorname{diam}(S)\sqrt{n-1}\right) \leq \|X_S^{-1}\|_2\left(1+d\sqrt{n-1}\right)$$

and then

$$\operatorname{diam}(S)\left\|(X_S^*)^{-1}\right\|_2 \leq d\left(1+d\sqrt{n-1}\right) =: d^*$$

follows.

Since the simplex $S \in \mathcal{T}^*$ was arbitrary, we have shown that $\mathcal{T}^*$ is (h,d^*)-bounded. □

The following proposition is a direct consequence of Proposition 4.4.

Proposition 4.5. *Assume $\mathcal{T}_k$, $k \in \mathbb{N}_0$, is a sequence of triangulations in $\mathbb{R}^n$, such that $\mathcal{T}_k$ is (h_k,d_k)-bounded, and $h_k \to 0$ as $k \to \infty$ and $d_k \leq d$ for all $k \in \mathbb{N}_0$. Let $\mathcal{T}_k^*$, $k \in \mathbb{N}_0$, be a sequence of triangulations such that $\mathcal{T}_k^*$ consists of the simplices of $\mathcal{T}_k$, but with a (possibly) different ordering of the vertices of the simplices. Then there are constants d_k^*, d^* such that $\mathcal{T}_k^*$ is (h_k,d_k^*)-bounded, $k \in \mathbb{N}_0$, and $d_k^* \leq d^*$ for all $k \in \mathbb{N}_0$.*

Proof. The case $n=1$ is trivial and the case $n \geq 2$ is obvious from Proposition 4.4 with $d_k^* = d_k(1+d_k\sqrt{n-1})$ and $d^* = d^*(1+d^*\sqrt{n-1})$. □

We have shown that one can talk about an (h,d)-bounded triangulation $\mathcal{T} = \{\operatorname{co}C_\nu\}_{\nu\in I}$, where $C_\nu = \{\mathbf{x}_0,\mathbf{x}_1,\ldots,\mathbf{x}_n\}$ for every $\nu \in I$. That is, the vertices C_ν do not have to be ordered n-tuples and can be sets. The understanding is then that no matter how the vertices of the simplices are ordered, the resulting triangulation in $\mathbb{R}^n$ is (h,d)-bounded in the sense of Definition 3.1. Similarly, if $\mathcal{T} = \{\operatorname{co}C_\nu\}_{\nu\in I}$, where $C_\nu = (\mathbf{x}_0,\mathbf{x}_1,\ldots,\mathbf{x}_n)$ is an ordered n-tuple for every $\nu \in I$, we can be sure that the corresponding triangulation, where the C_νs are changed into sets, is (h,d^*)-bounded in this new sense with $d^* = d(1+d\sqrt{n-1})$.

Thus, one can define for triangulations, of which the vertices of the simplices are not necessarily ordered

Definition 4.6 (Uniformly regular triangulations)**.** *A triangulation $\mathcal{T}$ in $\mathbb{R}^n$ consisting of simplices $S_\nu = \operatorname{co}C_\nu$, $C_\nu := \{\mathbf{x}_0,\mathbf{x}_1,\cdots,\mathbf{x}_n\}$ is said to be* uniformly regular *if there exist constants $h,d>0$ such that for* any ordering of the vertices C_ν *of the simplices $S_\nu \in \mathcal{T}$ we have*

$$\operatorname{diam}(S) \leq h \quad \textit{and} \quad \operatorname{diam}(S)\|X_S^{-1}\|_2 \leq d$$

for all $S \in \mathcal{T}$, or equivalently if there exist constants $h^*, d^* > 0$ such that

$$\|X_S\|_2 \leq h^* \quad \textit{and} \quad \kappa_2(X_S) \leq d^*$$

for all $S \in \mathcal{T}$. Here X_S denotes the shape-matrix of the simplex S with respect to the ordering chosen.

5 CONCLUSIONS

Sequences of triangulations of $\mathbb{R}^n$ having uniform upper bounds on the diameters and degeneracy of the simplices are important for the CPA algorithm to compute continuous and piecewise affine (CPA) Lyapunov functions for nonlinear systems (Hafstein, 2004; Hafstein, 2005; Giesl and Hafstein, 2014).

In this paper we have eliminated the dependence of the degeneracy on the ordering of the vertices of the simplices in the triangulation. Thus, the degeneracy can be defined for the simplices as geometrical objects. Further, we have provided a characterization of the degeneracy in terms of the condition number of the shape-matrices.

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