A Measurement for Essential Conflict in Dempster-Shafer Theory

Wenjun Ma, Jieyu Zhan* and Yuncheng Jiang
Guangzhou Key Laboratory of Big Data and Intelligent Education, School of Computer Science,
South China Normal University, Guangzhou, China

Keywords: D-S Theory of Evidence, Conflict Management, Essential Conflict, Information Fusion.

Abstract: Dempster's combination rule in Dempster-Shafer (D-S) theory is widely used in data mining, machine learning, clustering and database systems. In these applications, the counter-intuitive result is often obtained with this rule when the combination of mass function is performed without checking whether original beliefs are in conflict. In this paper, a new type of conflict called *essential conflict* has been revealed with two characteristics: (i) it is an essential factor that leading to the counter-intuitive result by turning a possible state into a necessary true state or an impossible state; (ii) it cannot be corrected by the combination process of any new mass functions. After showing that the existing conflict measurements in D-S theory have the limitations to address the essential conflict and presenting a formalism about the concept of essential conflict, we propose a measurement of essential conflict between two mass functions based on the mass value and the intersection relation of their focal elements. We argue that if there exists a focal element of one mass function, such that the intersection of it and any focal element of another mass function is an empty set, then the essential conflict is caused and Dempster's combination rule is not applicable.

1 INTRODUCTION

Dempster-Shafer (D-S) theory (Dempster, 2008; Shafer, 1976) is a powerful tool for modelling and reasoning with ambiguous information in applications (Ma et al., 2013), such as information fusion (Hong et al., 2016), pattern recognition (Jiang et al., 2016), and decision making (Ma et al., 2017). In this theory, when multiple pieces of evidence for a proposition are accumulated from multiple distinct sources, they need to be combined to see how strongly they support the proposition together (Jiang and Zhan, 2017). And Dempster's combination rule is widely employed to do this. Nevertheless, many researchers challenge its validity and consistency when it is used to combine highly conflicting evidence (Destercke and Burger, 2012; Liu, 2006), which makes the use of Dempster's combination rule questionable.

To remedy this weakness, two major viewpoints have been proposed. The first viewpoint suggests that we should develop a new combination rule to replace Dempster's combination rule and redistribute the conflict, while the researches holding with the second viewpoint suggest that we should consider the conditions in which Dempster's combination rule is safe to be used and modify the belief function if the conditions are unsatisfied. Nevertheless, in the current methods, the alternative rules do not get wide acceptance in real-world application, and the proposed Dempster's rule combination conditions is controversial. (More details discuss in Section 3)

Since for both viewpoints, the fundamental question that what does *conflict* mean among evidence is important and still an open issue, in this paper, we focus on this issue and propose to study the notion of conflict from a different perspective based on the intersection of the focal elements of two original mass functions. More specifically, firstly in this paper we reveal a new type of conflict: *essential conflict*. Two characteristics of it are analysed. (1) Belief Absolutization: such conflict can turn a possible state of an original mass function into a necessary true state or an impossible state in the combination result with Dempster's combination rule; (2) Uncorrectable Assertion: such conflict cannot be corrected by the combination of new mass function. Then after presenting a formal definition of *essential conflict* and revealing the properties of conflict in the combination process, we argue essential conflict is a more importance factor to show the essence and uncorrectable disagreement between sources. Hence, a measurement of essential conflict is given between two mass functions in D-S

*Corresponding author

Ma, W., Zhan, J. and Jiang, Y.
A Measurement for Essential Conflict in Dempster-Shafer Theory.
DOI: 10.5220/0010394512821289
In *Proceedings of the 13th International Conference on Agents and Artificial Intelligence (ICAART 2021) - Volume 2*, pages 1282-1289
ISBN: 978-989-758-484-8

theory based on their focal element set and the original mass values. Also, we will analyse the properties of such new conflict measurement. Finally, examples are given to illustrate the advantages of our method.

This paper advances the state of the art in the area of D-S theory in the following aspects: (i) reveal a new type of conflict in the combination process: *essential conflict* that highly related to the counter-intuitive results with Dempster's combination rule; (ii) give a formal definition to represent the essential conflict; (iii) design a measurement for identifying conflict between two mass functions more effectively and less conservatively.

The rest of this paper is organised as follows. Section 2 recaps background knowledge. Section 3 discusses related work. Section 4 gives a formal definition of the essential conflict. Section 5 analyses the properties of the essential conflict. Section 6 gives a novel essential conflict measurement with desirable properties and illustration examples. Finally, Section 7 concludes the paper with future work.

2 PRELIMINARIES

This section recaps some base concepts in D-S theory.

Definition 1. *(Shafer, 1976) Let $\Theta = \{\omega_1, \ldots, \omega_n\}$ be a set of exhaustive and mutually exclusive elements (i.e.,* states of the world*), called a frame of discernment (or simple a frame). Function $m : 2^\Theta \to [0,1]$ is a mass function if $m(\emptyset) = 0$ and $\sum_{A \subseteq \Theta} m(A) = 1$.*

Here a mass function is *completely ignorable* if and only if $m(\Theta) = 1$, and F_m is the *focal element* set of the mass function m if for any $B \in F_m$, $m(B) > 0$.

Definition 2. *(Shafer, 1976) Let m_1 and m_2 be two mass functions from independent and fully reliable sources over a frame of discernment Θ. Then the combined mass function from m_1 and m_2 by* Dempster's combination rule*, denoted as $m_{1,2}$, is defined as:*

$$m_{1,2}(x) = \begin{cases} 0 & \text{if } x = \emptyset, \\ \frac{1}{1-k_{12}}\left(\sum_{A \cap B = x} m_1(A) m_2(B)\right) & \text{if } x \neq \emptyset, \end{cases} \quad (1)$$

where normalization constant

$$k_{12} = \sum_{A_i \cap B_j = \emptyset} m_1(A) m_2(B) < 1 \quad (2)$$

is a classical conflict coefficient to measure the conflict between the pieces of evidence.

Definition 3. *(Smets, 2005) Let m be a mass function over Θ. Its associated pignistic probability function $BetP_m : \Theta \to [0,1]$ is defined as:*

$$BetP_m(\omega) = \sum_{A \subseteq \Theta, \omega \in A} \frac{1}{|A|} \frac{m(A)}{1 - m(\emptyset)}, m(\emptyset) < 1, \quad (3)$$

where $|A|$ is the cardinality of A.

Here, when an initial mass function gives $m(\emptyset) = 0$, $\frac{m(\omega)}{1-m(\emptyset)}$ is reduced to $m(\omega)$. $BetP_m(\omega)$ is a probability measure. It tells what is the total mass value that a state ω can carry for decision making based on the corresponding evidence referred by mass function m.

Finally, although Dempster's combination rule has been used in many real world applications, it has been criticised upon some of its counter-intuitive combination results (Liu, 2006; Zadeh, 1986). Perhaps the most famous one is as follows:

Example 1 (Zadeh's counter-example (Zadeh, 1986)). *Let m_1 and m_2 be two mass functions defined on a frame of discernment $\Theta = \{a, b, c\}$ with:*

$$m_1(\{a\}) = 0.9,\ m_1(\{b\}) = 0.1,\ m_1(\{c\}) = 0;$$
$$m_2(\{a\}) = 0,\ m_2(\{b\}) = 0.1,\ m_2(\{c\}) = 0.9.$$

They mean that option a is strongly supported by the first piece of evidence but absolutely denied by the second one, option b is weekly supported by both, and option c is strongly supported by the second but absolutely denied by the first. By Definition 2, if we use Dempster's combination rule to combine the two mass functions, we have $m_{12}(\{b\}) = 1$.

That is, option b, hardly supported by each piece of evidence, turns out to be fully supported after the combination of the two mass functions. Therefore, Zadeh argued that such a result is highly violated our intuition about the evidence combination.

3 RELATED WORK

In general, there are two major viewpoints to improve Dempster's combination rule to resolve Zadeh's counter-example.

From the first viewpoint, the counter-intuitive results were caused by Dempster's combination rule, so they modified combination rule and proposed a number of new evidential combination rules to remove the defect (*e.g.*, (Chebbah et al., 2015; Deng et al., 2014; Dubois and Prade, 1988a; Elouedi and Mercier, 2011; Smarandache and Dezert, 2006; Yager, 1987)). However, without a general mechanism to accurately measure the degree of conflict other than using the conflict coefficient k (which cannot measure the conflict in the desired way), such rules are actually ad hoc because of lacking a theoretical justification. As Smets (Smets, 2007) pointed out, the pragmatic fact "our rule works fine" in some application cases is of course not a proper justification (at most a necessary condition). Moreover, since the alternative combination rules always cause higher computation complexity, give up some desirable properties (*i.e.*, associative and commutative) in combination process, and encounter new counter-intuitive behaviours in applications, such rules do not get wide acceptance in the real-world applications. (Deng, 2015; Jiang and Zhan, 2017)

From the second viewpoint, the counter-intuitive results were caused by abusing Dempster's combination rule inappropriately, so they limited the condition that the Dempster's combination rule can be used by reconstructing the mass function (*i.e.*, discounting mass function and weighted averaging mass function) (Deng et al., 2004; Dubois and Prade, 1988b; Jiang et al., 2016; Murphy, 2000; Shafer, 1976; Smets, 2000; Wang et al., 2016; Zhao et al., 2016), or introducing the open-world assumption (Deng, 2015; Jiang and Zhan, 2017; Smets, 2000), or managing conflict with conflict measurement (Daniel, 2014; Jousselme et al., 2001; Jiang, 2018; Liu, 2006; Zhao et al., 2016). For the cases of discounting mass function (Dubois and Prade, 1988b; Shafer, 1976; Smets, 2000; Zhao et al., 2016) and introducing the open-world assumption, they just evade the criticism of Dempster's combination rule by making an additional assumption, such as the assumption that the evidence cannot be all fully reliable or the frame of discernment cannot be exhaustive. Therefore, in some cases, such ideas are too conservative since they give a too strong limitation for the Dempster's combination rule. And for the cases of using weighted averaging mass function (Deng et al., 2004; Jiang et al., 2016; Murphy, 2000; Wang et al., 2016), they will cause another counter-intuitive behaviour, since the original mass function can be changed after combining with a completely ignorable mass function (Ma et al., 2019). Thus, the conflict management with conflict measurement is the most common way for the second viewpoint. Since the conflict coefficient k in Definition 2 cannot represent conflict reasonably, various conflict measurements are proposed to quantify the opposition between mass function, such as the relative coefficient Jousselme distance (Jousselme et al., 2001), non-intersection correlation coefficient (Jiang, 2018), pignistic probability distance (Liu, 2006), plausibility conflict measurement (Daniel, 2014), and so on. However, although the conflict measurements are various and fruitful, it is still inconclusive for what conflict is and where it comes from. (Jiang, 2018)

All in all, by two major viewpoints to resolve the conflict problem in D-S theory, we find that the fundamental question is what does conflict mean among evidence. Moreover, since the conflict problem in D-S theory is due to criticisms on the counter-intuitive result of applying Dempster's combination rule (e.g., Example 1), there at least exists a type of conflict that should be highly related to the counter-intuitive result. In this vein, we would like to make a claim about the relationship between such type of *conflict* and *counter-intuitive combination result* as follows:

Claim 1. *For any two combination results with Dempster's combination rule, there at least exists a type of conflict, such that its value of a counter-intuitive combination result should be higher than its value of a non-counter-intuitive combination result.*

Since such type of conflict is highly related to the counter-intuitive combination result, we would like to call it *essential conflict*.

4 FORMAL DEFINITION OF ESSENTIAL CONFLICT

Now, we show that the existing conflict measurements have the limitations to address *essential conflict*.

First, we discuss the cases in which the existing conflict measurements in D-S theory will infer a low conflict while a counter-intuitive combination result occurs. Consider the following example:

Example 2. *let m_3 and m_4 be two mass functions provided by two distinct and totally reliable sources on a frame of discernment $\Theta = \{a,b,c\}$ that:*

$$m_3(\{a\}) = m_4(\{a\}) = 0.8,\ m_3(\{b\}) = m_4(\{c\}) = 0.2.$$

Clearly, by Definitions 1 and 2, the maximum difference of mass value for Example 2 is $|m_3(\{b\}) - m_4(\{b\})| = 0.2$, the classical conflict coefficient is $k_{34} = 0.36$, and the combination result is $m_{34}(\{a\}) = 1$. Following the idea of most current conflict measurements, it should be a low conflict combination result. However, the combination result that a possible state (i.e., state a) turns into a necessary true state and some possible states (i.e., states b and c) turn into the impossible states is somehow arbitrary. This combination result strongly violates our intuition.

Therefore, in Example 2, low difference of mass value and low value of conflict coefficient k cannot indicate the Dempster's combination rule is safe to be applied. Thus, the essential conflict that highly related to a counter-intuitive combination result in Example 2 cannot fully represented by the current methods.

Second, we discuss the cases that the existing conflict measurements in D-S theory will infer a high conflict while an acceptable justification for the combination result occurs by the following example:

Example 3. *let m_5 and m_6 be two mass functions provided by two totally reliable sources on a frame of discernment $\Theta = \{a,b\}$ that:*

$$m_5(\{a\}) = m_6(\{b\}) = 0.9,\ m_5(\{b\}) = m_6(\{a\}) = 0.1.$$

Clearly, the maximum difference of mass value assign to set $\{a\}$ (or $\{b\}$) between m_5 and m_6 is $|m_5(\{a\}) - m_6(\{a\})| = 0.8$ and classical conflict coefficient is $k_{56} = 0.82$ by Definition 2. Under the current methods of conflict measurement, this difference and conflict coefficient value would warrant a verdict that *these two pieces of evidence are in high conflict and Dempster's rule should not be used*.

However, the combination result of m_5 and m_6 with Dempster's rule is $m_{56}(\{a\}) = m_{56}(\{b\}) = 0.5$. This result seems to satisfy our intuition that as both pieces of evidence are totally reliable and give different judgements about an issue, in order to make an agreement in the combination process, we make an equally concession for the judgements of both original beliefs. This phenomenon is common in real-world applications. For example, a policeman thinks that a suspect A should be the murderer of a murder crime while another policeman thinks that suspect B should be the murderer of this murder crime. If both policemen are totally reliable, we prefer to considering that they make their judgements based on different points of view and give equal attention to both suspects.

Compared Example 3 with Example 1, although two examples both have high difference of mass value and high classical conflict coefficient, we thinks the combination result of Example 3 can be justified without violating our intuition while that of Example 1 is totally unacceptable. Thus, in Example 3, high difference of mass value and high value of k cannot indicate high essential conflict value.

In other words, by Example 2 and 3, the difference of the mass values between two mass functions and classical conflict coefficient, which are two major component parts of the current methods of conflict measurement, is not the essential factor that prevents us to use Dempster's rule without causing the counter-intuitive results. As a result, a formal definition and conflict measurement for the essential conflict is required if we want to combine the evidence safely with Dempster's combination rule.

By analysing the set structures of Examples 1-3 before and after the combination result, we find that for Examples 1 and 2, there exists at least one focal element of one mass function, such that the intersection of it and any focal element of another mass function is an empty set. While for Example 3, such situation does not exist. Following this idea, we infer that such issue should be a main feature for defining essential conflicts. Formally, we have

Definition 4. *Let m_1 and m_2 be two mass functions over a frame Θ, F_1 and F_2 be the focal element sets of m_1 and m_2, respectively. Then $\Upsilon_{12} \subseteq \Theta$ is a set of essential conflict elements with the mass function m_1 and m_2 if and only if for any $\omega \in \Upsilon_{12}$, there exists $A \in F_i \wedge \omega \in A$, such that for any $B \in F_j$, we have $A \cap B = \emptyset$ ($i \neq j$ and $i,\ j \in \{1,2\}$).*

In addition, if $\Upsilon_{12} \neq \emptyset$, then m_1 and m_2 are in essential conflict.

In fact, by Definition 4, we can find that if $A \in F_i$ and $\forall B \in F_j, A \cap B = \emptyset$ ($i \neq j$ and $i,\ j \in \{1,2\}$), then for any $C \subseteq A$, we have:

$$C \cap B = \emptyset, \text{ and } \sum_{X \bigcap Y = C} m_1(X) m_2(Y) = 0.$$

In other words, the combined mass value of any subset of any $A \in \Upsilon_{12}$ is zero by applying Dempster's combination rule (*i.e.*, formula (1)). So, it means no matter how many new mass function m_i consider A as a focal element and how high mass value $m_i(A)$ is, if subset A belongs to the conflict element set (*i.e.*, $A \in \Upsilon_M$), the result of Dempster's combination rule still rules it and its more special subsets out of the set of the possible states. This is the reason why Dempster's combination rule could cause some counter-intuitive behaviours such as Example 1.

Now, we will apply Definition 4 to analyse Examples 1-3. For Example 1, after checking the focal elements of each mass function (*i.e.*, $\{a\}$ and $\{b\}$ for m_1, and $\{b\}$ and $\{c\}$ for m_2), we can find that

(1) $m_1(\{a\}) > 0$, and for any $A \subseteq \{a,b,c\} \wedge A \cap \{a\} \neq \emptyset$, we have $m_2(A) = 0$, and

(2) $m_2(\{c\}) > 0$, and for any $B \subseteq \{a,b,c\} \wedge B \cap \{c\} \neq \emptyset$, we have $m_1(B) = 0$.

Thus we have $\Upsilon_{12} = \{a,c\}$. Since $\Upsilon_{12} \neq \emptyset$, m_1 and m_2 are in essential conflict in Example 1. Similarly, for Example 2, we have $\Upsilon_{34} = \{b,c\} \neq \emptyset$. Thus, m_3 and m_4 are in essential conflict in Example 2. Finally, for Example 3, we find that for all focal elements of m_5 (*i.e.*, $\{a\}$ and $\{b\}$), there exists a focal element of m_6 (*i.e.*, $\{a\}$ and $\{b\}$), such that the intersection of them is not an empty set. Thus, we have $\Upsilon_{56} = \emptyset$. It means m_5 and m_6 are not in essential conflict.

And such results of Examples 1-3 show that our definition of essential conflict indeed reveals the relation between conflict and counter-intuitive combination result by applying Dempster's combination rule.

5 PROPERTIES OF ESSENTIAL CONFLICT

In this section, we will reveal two properties (i.e., belief absolutization and uncorrectable assertion) of the essential conflict that make it as a main factor to cause the counter-intuitive combination result by Dempster's combination rule.

Theorem 1 (Belief Absolutization). *Let m_1 and m_2 be two mass functions over a frame of discernment Θ that are in essential conflict, m_{12} be the combination result of m_1 and m_2 with Dempster's combination rule, and $BetP_{m_1}$, $BetP_{m_2}$ and $BetP_{m_{12}}$ be the pignistic probability function of m_1, m_2 and m_{12}, respectively. Then there exists $\omega \in \Theta$, such that $BetP_{m_1}(\omega) > 0$ or $BetP_{m_2}(\omega) > 0$ but $BetP_{m_{12}}(\omega) = 0$.*

Proof. Suppose F_1, F_2 and F_{12} are the focal element sets of m_1, m_2 and m_{12}, respectively. Then by Definition 4, if m_1 and m_2 are in essential conflict, there exists $A \in F_i \wedge \omega \in A$, such that for any $B \in F_j$, we have $A \cap B = \emptyset$ ($i \neq j$ and $i,\ j \in \{1,2\}$). Without loss of generality, we assume $A \in F_1$, then by Definitions 1 and 3, we have $m_1(A) > 0$ and $BetP_{m_1}(\omega) > 0$. Since $\forall B \in F_2, A \cap B = \emptyset$ and $\omega \in A$, we have for any $C \in F_1$ and any $B \in F_2$, $\omega \notin C \cap B$. Thus, $\omega \notin F_{12}$. Then, by Definitions 2 and 3, we have $BetP_{m_{12}}(\omega) = 0$. □

Since the pignistic probability function $BetP_m(\omega)$ tells what is the total mass value that a state ω can carry for decision making based on the corresponding evidence referred by mass function m, $BetP_{m_i}(\omega) > 0$ for $i \in \{1,2\}$ means one of the original mass functions m_1 and m_2 support the claim that there exists some chance that the state ω turns out to be the real state, while $BetP_{m_{12}}(\omega) = 0$ means it is impossible that ω turns out to be the real state by the combination result of m_1 and m_2. Therefore, Theorem 1 means that if the essential conflict exists for two mass functions with Dempster's combination rule, then there at least exists a possible state ω supported by the original mass function turns out to be an impossible state after the combination result. To make matters worse, if only one possible state of a frame does not belong to the set of essential conflict elements defined by Definition 4 for two mass functions, then such possible state will turn out to be a necessary true state after combination of the two mass functions by Dempster's combination rule. And this is the exactly reason why Zadeh's counter-example (i.e., Example 1) occurs.

Before discussing the property of uncorrectable assertion, we first define the concept of *correctable* in information fusion by Dempster's combination rule. In real world application of intelligent surveillance, for the reason of limited surveillance, time pressure, the scotomas of cameras, the definition and sharpness of images, disturbance of unknown factors (such as signal interference, a sudden jarring or jerking), and so on, a sensor might produce a fault evidence for a given target that disagrees with the other information sources. And the evidence combination with such fault evidence will lead to a wrong judgement about the surveillance target. However, in some cases, such wrong judgement can be corrected by the further information fusion of the additional evidences. For example, suppose the mass functions m_5 and m_6 in Example 3 are provided by two information sources in an intelligent surveillance system and the true state is b. Thus, the combination result $m_{56}(\{a\}) = m_{56}(\{b\}) = 0.5$ somehow deviated from the correct judgement made by m_6. However, if we have a new mass function m_7 provided by an additional information source with $m_7(\{a\}) = 0.1$ and $m_7(\{b\}) = 0.9$, then the combination result of m_{56} and m_7 for the state b is exactly the same as $m_6(\{b\}) = 0.9$. Thus, we can say the mistake or the deviation caused by m_5 is corrected by additional evidence combination. Formally, we can define the concept of correctable for the original combination result as follows:

Definition 5. *Let m_1 and m_2 be two mass functions over a frame of discernment Θ, m_{12} be the combination result of m_1 and m_2 with Dempster's combination rule, and $BetP_{m_1}$ and $BetP_{m_2}$ be the pignistic probability functions of m_1 and m_2, respectively. Then the combination result m_{12} is correctable if for any $\omega \in \Theta$ and $BetP_{m_i}(\omega)$ ($i \in \{1,2\}$), we can always construct an additional mass function m_3, such that for the combination result m_{123} of the mass functions m_1, m_2 and m_3, we have*

$$BetP_{m_{123}}(\omega) = BetP_{m_i}(\omega).$$

Here, the pignistic probability function $BetP_m(\omega)$ works as a probability measurement to represent the total mass value of m that a state ω can carry for decision making (Smets, 2005). Thus, $BetP_{m_{123}}(\omega) = BetP_{m_i}(\omega)$ for $i \in \{1,2\}$ means no matter what judgement the combination result m_{12} makes about the chance that a state ω turn out to be a true state, with the combination of an additional mass function m_3, the judgement of m_{12} can be corrected and m_{123} will share the same judgement with the mass functions m_i about the chance that a state ω turns out to be a true state. Thus, even the mass function m_j ($j \neq i$ and $j \in \{1,2\}$) makes a wrong judgement about the chance that a state ω turns out to be a true state, if the combination result is correctable, the influence of m_j in the combination process can be eliminated. Such property of correctable for the combination result is desirable for the real-world application, since the wrong judgement of a small portion of the information resource cannot prevent the convergence toward truth in the combination process.

Unfortunately, such property of correctable for the combination result is non-universal. Consider

Zadeh's counter-example in Example 1, the combination result of the mass functions m_1 and m_2 is $m_{12}(\{b\}) = 1$. In this situation, by Definition 2, for any mass function m_3 with $k_{123} < 1$, the combination result of m_{12} and m_3 will be $m_{123}(\{b\}) = 1$. Thus, by Definition 5, the combination result in Example 1 is uncorrectable.

Now, we will show that if two mass functions are in essential conflict, the combination result of them is uncorrectable by the following theorem.

Theorem 2 (Uncorrectable Assertion)**.** *Let m_1 and m_2 be two mass functions over a frame of discernment Θ that are in essential conflict, m_{12} be the combination result of m_1 and m_2 with Dempster's combination rule, and $BetP_{m_1}$ and $BetP_{m_2}$ be the pignistic probability function of m_1 and m_2, respectively. Then there exists a state $\omega \in \Theta$ and $BetP_{m_i}(\omega)$ ($i \in \{1,2\}$), such that for any mass function m_3, we have*

$$BetP_{m_{123}}(\omega) \neq BetP_{m_i}(\omega).$$

Proof. Suppose F_1, F_2 and F_{12} are the focal element sets of m_1, m_2 and m_{12}, respectively. Then by Definition 4, if m_1 and m_2 are in essential conflict, there exists $A \subset \Theta$, such that $A \in F_i \wedge \omega \in A$, and for any $B \in F_j$, we have $A \cap B = \emptyset$ ($i \neq j$ and $i, j \in \{1,2\}$). Without loss of generality, we assume $A \in F_1$, then by Definitions 1 and 3, we have $m_1(A) > 0$ and $BetP_{m_1}(\omega) > 0$. Since $\forall B \in F_2, A \cap B = \emptyset$ and $\omega \in A$, we have for any $C \in F_1$ and any $B \in F_2$, $\omega \notin C \cap B$. Thus, $\omega \notin F_{12}$. Then, by Definitions 2 and 3, we have $BetP_{m_{12}}(\omega) = 0$.

By Theorem 1 and the fact that the mass functions m_1 and m_2 are in essential conflict, we can find a state $\omega \in \Theta$, such that $BetP_{m_1}(\omega) > 0$ or $BetP_{m_2}(\omega) > 0$ but $BetP_{m_{12}}(\omega) = 0$. Without loss of generality, we assume $BetP_{m_1}(\omega) > 0$ and $BetP_{m_{12}}(\omega) = 0$. Then by Definition 3 and the fact that $BetP_{m_{12}}(\omega) = 0$, for any $T \subseteq \Theta$ that satisfies $\omega \in T$, we have $m_{12}(T) = 0$. Moreover, by Definition 2, for any mass function m_3, the combination mass value $m_{123}(T)$ that obtained by applying Dempster's combination rule for mass functions m_{12} and m_3 should satisfy

$$m_{123}(T) = \frac{\sum\limits_{A \bigcap B = T} m_{12}(A)m_3(B)}{1 - \sum\limits_{A_i \bigcap B_j = \emptyset} m_1(A)m_2(B)} = 0$$

Thus, by Definition 3, we have

$$BetP_{m_{123}}(\omega) = \sum_{T \subseteq \Theta, \omega \in T} \frac{1}{|T|} \frac{m(T)}{1 - m(\emptyset)} = 0.$$

By $BetP_{m_1}(\omega) > 0$ and $BetP_{m_{123}}(\omega) = 0$, we have $BetP_{m_1}(\omega) \neq BetP_{m_{123}}(\omega)$ and prove the theorem. □

In fact, Theorem 2 reveals an undesirable property of essential conflict that a wrong judgement about the true state might be always remained in the combination process no matter how many correct evidence we have collected.

The property of belief absolutization in Theorem 1 and the property of uncorrectable assertion in Theorems 2 show the insight of essential conflict that the combination process with two essential conflict mass functions will always lead to an extreme judgement (*i.e.*, necessary true or impossible) of a possible state that cannot be corrected by any further evidence. Such insight reveals the relation of *conflict* and *counter-intuitive* and this is the reason why we call such type of conflict as *essential conflict*.

6 A MEASUREMENT OF ESSENTIAL CONFLICT

In this section, we will first propose a measurement for essential conflict that we discover in the previous section. After that, we will reveal its properties. Finally, we will illustrates the advantage of our measurement by analysing some examples.

Definition 6. *Let m_1 and m_2 be two mass functions over a frame Θ, $\Upsilon_{12} \subseteq \Theta$ be a set of essential conflict elements with the mass functions m_1 and m_2. Then the degree of the essential conflict with two mass functions m_1 and m_2, denoted as $\kappa(m_1,m_2)$, is given by:*

$$\kappa(m_1,m_2) = \sum_{A,B \subseteq \Upsilon_{12}} m_1(A) + m_2(B) - m_1(A)m_2(B).$$

Here, a state set $A \subseteq \Upsilon_{12}$ means for any state $\omega \in A$, we have $\omega \in \Upsilon_{12}$, thus $\sum_{A \subseteq \Upsilon_{12}} m_i(A)$ for $i \in \{1,2\}$ means the total support degree of the evidence represented by mass function m_i for the states that ruled out in the combination process. Moreover, since the mass value of $\sum_{A,B \subseteq \Upsilon_{12}} m_1(A)m_2(B)$ has been double counted in $\sum_{A \subseteq \Upsilon_{12}} m_i(A)$ for $i \in \{1,2\}$, $-m_1(A)m_2(B)$ is required for the degree of the essential conflict with two mass functions m_1 and m_2.

Moreover, we find that the essential conflict measurement $\kappa(m_1,m_2)$ has some good properties approved in (Destercke and Burger, 2012).

Theorem 3. *Let m_1 and m_2 be two mass functions over a frame of discernment Θ, F_1 and F_2 be the focal element sets of m_1 and m_2, $\Upsilon_{12} \subseteq \Theta$ be a set of essential conflict elements with the mass function m_1 and m_2, and $\kappa(m_1,m_2)$ be the essential conflict measurement of m_1 and m_2. Then we have:*

(i) ***Symmetry***. $\kappa(m_1,m_2) = \kappa(m_2,m_1)$.

(ii) ***Extreme consistency.*** $\kappa(m_1, m_2) = 0$ *if and only if* m_1 *and* m_2 *are not in essential conflict, while* $\kappa(m_1, m_2) = 1$ *if and only if* $A \cap B = \emptyset$ *for any* $A \in F_1$ *and* $B \in F_2$.

(iii) ***Bounded.*** $0 \geq \kappa(m_1, m_2) \geq 1$.

(iv) ***Ignorance is bliss.*** *If* $m_2(\Theta) = 1$, *then* $\kappa(m_1, m_2) = 0$.

Proof. By Definition 6, we have

$$\kappa(m_1,m_2) = \sum_{A,B \subseteq \Upsilon_{12}} m_1(A) + m_2(B) - m_1(A)m_2(B) = \kappa(m_2,m_1).$$

Thus, item (i) of the theorem holds.

If m_1 and m_2 are not in essential conflict, by Definition 4, we have $\Upsilon_{12} = \emptyset$. Hence, by Definition 6, we have $\kappa(m_1,m_2) = m_1(\emptyset) + m_2(\emptyset) = 0$. If $A \cap B = \emptyset$ for any $A \in F_1$ and $B \in F_2$, then

$$\kappa(m_1,m_2) = \sum_{A \in F_1, B \in F_2} m_1(A) + m_2(B) - m_1(A)m_2(B) = 1.$$

Thus, item (ii) of the theorem holds.

By Definition 1, for any $A, B \subseteq \Upsilon_{12} \subseteq \Theta$, we have $m_1(A) \in [0,1]$, $m_2(B) \in [0,1]$, and $m_1(A) + m_2(B) \geq m_1(A)m_2(B)$. Hence by the fact that $\sum_{A \subseteq \Theta} m_1(A) = 1$, $\sum_{B \subseteq \Theta} m_1(B) = 1$, and $\Upsilon_{12} \subseteq \Theta$, we have

$$0 \geq \sum_{A \subseteq \Upsilon_{12}} m_1(A) \geq 1,\ 0 \geq \sum_{B \subseteq \Upsilon_{12}} m_2(B) \geq 1,\ \textit{and}$$

$$0 \geq \sum_{A \in F_1, B \in F_2} m_1(A)m_2(B) \geq 1.$$

Since

$$\kappa(m_1,m_2) = \sum_{A \in F_1} m_1(A) + \sum_{B \in F_1} m_2(B) - \sum_{A \in F_1, B \in F_2} m_1(A)m_2(B),$$

we have $0 \geq \kappa(m_1,m_2) \geq 1$. Thus, item (iii) of the theorem holds.

If $m_2(\Theta) = 1$, by Definition 4, we have $\Upsilon_{12} = \emptyset$. Hence, by Definition 6, we have $\kappa(m_1,m_2) = 0$. Thus, item (iv) of the theorem holds. □

Now, we will use Examples 1-3 to illustrates the effectiveness of our essential conflict measurement. For Example 1, by Definition 4, we have $\Upsilon_{12} = \{a, c\}$. Thus, by Definition 6, we have $\kappa(m_1,m_2) = 0.99$. Similarly, for Example 2, by $\Upsilon_{34} = \{b, c\}$ and Definition 6, we have $\kappa(m_3,m_4) = 0.36$. And for Example 3, by $\Upsilon_{56} = \emptyset$ and Definition 6, we have $\kappa(m_5,m_6) = 0$. Thus, we have $\kappa(m_1,m_2) > \kappa(m_3,m_4) > \kappa(m_5,m_6)$. This result satisfies our intuitions discussed in Section 4. In other words, our measurement is highly related to counter-intuitive combination result with Dempster's combination rule. Hence, compared with the current conflict measurements, our measurement indeed gives a better explanation for the conflict.

7 CONCLUSION AND FUTURE WORKS

Our goal is to study the notion of conflict in D-S theory from a new perspective about the relation of *conflict* and *counter-intuitive combination result* with Dempster's combination rule. After showing the limitations of the existing conflict measurements in handling the type of conflict (*i.e.,* essential conflict) that highly related to the counter-intuitive combination result, we give a formal definition of essential conflict by the intersection relation of the focal elements of the original mass functions during the combination process. Moreover, by revealing two core properties of the essential conflict: belief absolutization and uncorrectable assertion, we show the insight of essential conflict, that is the combination process with two essential conflict mass functions will always lead to an extreme judgement (*i.e.*, necessary true or impossible) of a possible state that cannot be corrected by any further evidence. Thus, such type of conflict is the reason that causes the counter-intuitive combination result. Finally, by proposing a formal measurement for essential conflict, analysing the properties of such measurement, and applying such measurement to address some examples, we show that our new conflict measurement indeed gives a better explanation for the relation of the conflict and the counter-intuitive combination result between two mass functions.

There are many possible extensions to our work. Maybe the most interesting one is to extend our concept of essential conflict to specific needs and real-world applications, such as multiple sensor surveillance system (Hong et al., 2016) and automated e-business negotiation (Zhan et al., 2018). Another tempting avenue is to develop an alternative combination rule that can solve the conflict situation we mentioned in this paper. Since for the Dempster's combination rule, we can only suggest to avoid the case of essential conflict, it is interesting to find out whether there exists a combination rule that can solve the essential conflict without losing the desirable properties of Dempster's combination rule. Finally, it is significant to analyse more properties and rationalities about our conflict concept in information fusion as well as the theoretical comparison with other proposed conflict concepts in (Deng, 2015; Liu, 2006; Shafer, 1976).

ACKNOWLEDGEMENTS

The works described in this paper are supported by the National Natural Science Foundation of China under Grant Nos. 61772210, 61806080, 62006085 and U1911201; Guangdong Province Universities Pearl River Scholar Funded Scheme (2018); Humanities and Social Sciences Foundation of Ministry of Education of China under Grant No. 18YJC72040002; Doctoral Startup Project of Natural Science Foundation of Guangdong Province of China under Grant No. 2018A030310529; Project of Science and Technology in Guangzhou in China under Grant Nos. 201807010043 and 202007040006.

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