Categorizing Quantities using an Interactive Fuzzy Membership Function

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Abstract: In this paper, we investigate how an interactive version of the membership function from the Fuzzy Logic Theory can be used to categorize quantitative data. This function is simple and similar to a line chart and provides an explicit mapping of the categorization process. We first review the requirements for such quantitative values partitioning process and provide the fuzzy logic mathematical foundations related to the membership function. We then report on the implementation of the interactive function for several quantitative datasets case studies (e. g., age, temperature, speed). We expect this interactive function to provide more control over the categorization process, as well as a way to make the categorization more explicit.

1 INTRODUCTION

When reasoning on quantitative values—like people's age—analysts tend to use categories like YOUNG and OLD. Such a categorization process aims at mimicking the logic of human thoughts and reasoning (Clifford et al., 1975). In the situation where such categories are not in the dataset, it is likely they are subjective, from the analyst's head, based on familiarity with the domain or prior knowledge. Such a process raises the following issues:

- *An explicit mapping for those categories is missing:* the mapping function between quantities and categories should be clearly defined.

- *The mapping properties may vary across analysis sessions and analysts:* the mapping should be consistent and may not change.

- *The transfer between analysts and a user should be possible:* by some means of communication like a legend or a visual representation.

Besides, there are also some alternative automatic categorization methods and clustering. Those methods categorize quantities based on the distribution of quantitative values, e. g., KNN (Cover and Hart, 1967), K-Means (MacQueen et al., 1967), DBSCAN (Ester et al.,). However, those approaches do not provide enough control over the mapping process to let humans customize and capture uncertainty.

In this paper, our goal is to address those issues with an interactive visualization that captures such knowledge. It relies upon Fuzzy Logic Theory, created back in the 1960s to model domains with imprecise information (Zadeh, 1965; Pedrycz and Gomide, 2007) which we argue provides the theoretical framework to address the above issues. In particular, we rely upon a visual representation from this theory called *membership function* which is a line chart of the mapping function between a quantitative scale and categories, which makes this mapping *explicit*, which *captures* various points of view using a confidence factor and finally which visual can easily be *transferred*.

Our main contribution is that we designed an interactive version of fuzzy membership function as an interactive chart (Fig. 1), which is able to help users categorize quantitative values with confidence belonging to categories. We provide an implementation that demonstrates how it supports the categorization of multiple datasets. Our implementation exposes many parameters of the function that can be adjusted to generate new shapes of membership function for fine-grained categorization. The generated categories from this membership function can be assigned names that are then used as category names in the updated dataset. We conclude with an experimental design for further evaluation of this tool to validate its usability. We expect this technique to be not only used as a categories-generation tool, but also as an interactive legend (Riche et al., 2010) to communicate the generated categories properties and let the user adjust them dynamically when analyzing a visual representation.

2 RELATED WORK

This paper focuses on how to categorize quantities with fuzzy logic theory and quantify those generated categories with membership degree. In this section, we review papers in probabilistic classification, fuzzy visualization, and uncertainty visualization.

2.1 Visualization of Probabilistic Classification

(Alsallakh et al., 2014) proposed a visual interactive analysis tool to evaluate the effectiveness of classifiers to help machine learning experts find out the possible reasons for incorrect classification. This tool is able to emphasize the classification probabilities of items and make relations with a false negative and false positive. Similar to Alsallakh's study, Cao and Lin proposed UnTangle Map (Cao et al., 2015; Lin et al., 2014) by a web of the connected triangles, which is able to make the efficient relation between data items and their probabilistic labels. Based on RadViz, Seifert proposed a visual system to help understand the process of classification and results, which handles multiple classes, nominal and numeral data format (Seifert and Lex, 2009). Zhao conducted an experiment to evaluate the effectiveness and efficiency of fuzzy cluster analysis, objective questionnaires are designed to compare the accuracy and subjective questionnaires are designed to collect the experience of a user based on using four-dimensional visualization technique (Zhao et al., 2018). (Rheingans and Desjardins, 2000) is designed to visualize high-dimensional predictive results with richer representation, e.g., classification accuracy and confusion matrices to help a user understand high-dimensional data space.

2.2 Fuzzy Visualization

There exist alternatives to the membership function to capture categorization. Fuzzy representations and uncertainty visualization are suitable solutions like the Disk diagrams (Park and Park, 2010a; Park and Park, 2010b) proposed by Yeseul Park, to visualize fuzzy set operations. It can describe the complexity of fuzzy sets by showing the degree among sets with the layout of star coordinates, etc. (Zhu et al., 2018) extended the circular disk diagram layouts to improve sets membership analysis, by using color opacity and optimized layout to convey fuzzy sets membership and reveal the uncertain owner-member relationship. Besides, they also designed a computational framework by combining physical simulation and geometric interpolation.

However, those works focus on communicating categories rather than supporting their creation.

2.3 Uncertainty Visualization

Research on uncertainty visualization also offers some solutions for the presentation of ambiguity while categorization. In this field, Works related to uncertainty visualization provide guidance on advanced properties in the data, i. e. incorrectness, incompleteness, and ambiguity (Dressel and Nori, 2014). Brodlie (Brodlie et al., 2012) reviewed state of the art in uncertainty visualization according to different data, for example, point data, scalar data, multi-field scalar data, and vector data. At the same time, they also concluded the uncertainty visualization in the various dimensional datasets. Skeels proposed a classification of uncertainty (Skeels et al., 2010) for information visualization. (Dong and Hayes, 2012) designed an interactive tool to help users recognize the situations and comprehend the ambiguity, which is able to evaluate the domain-dependent decision support system (DSS).

3 DEFINING CRISP AND FUZZY MEMBERSHIP FUNCTIONS

Table 1: Dataset with quantities and categories.

Name	Age	Group	Profession
Lisa	8	Children	Student
Barney	39	Adult	Engineer
Smithers	33	Adolescent	Professor
Mr. Burns	90	Senior	Retired

The challenge we address is the explicit mapping between quantities and categories. While most visualization techniques and tools usually address it during the visual encoding of data, it remains internal—or at best using a legend—without providing a fully explicit set of properties for this mapping.

To introduce our approach, we progressively introduce the definitions by first stating our challenge as finding the relationship between Q (Quantitative) and C (Categories) as a mapping function:

$$Quantity \rightarrow Category \qquad (1)$$

Quantities are the measures of counts or values, which is able to be expressed by numbers and also can be compared in terms of "more", "less", or "equal", like attribute Age showed in Table. 1. On the contrary, categories are measures of type and can be expressed by a symbol, name, or label, which is categories, like
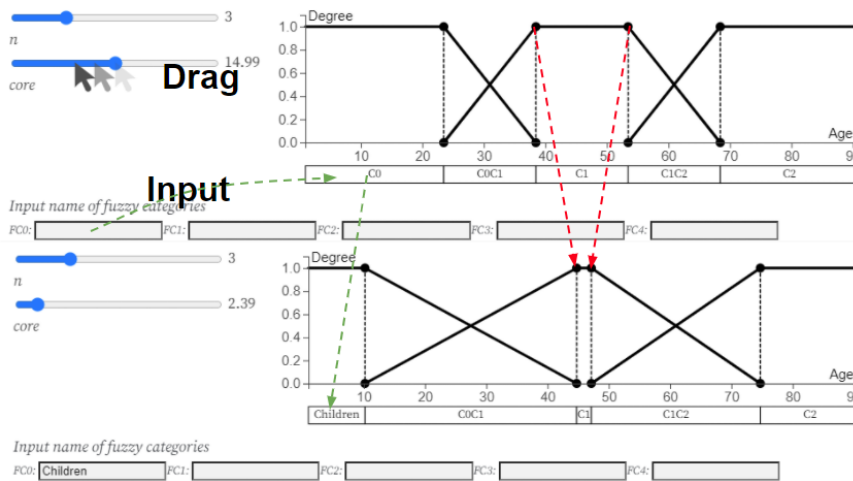
Figure 1: Manipulation of the interactive membership function. By adjusting the parameter *core*, the membership function is changed along with the new value of *core*, which is shown as the red dot line; Another interaction is defining the name of categories by inputting action and it is shown as the green dot line.

attribute *Age group* in Table. 1. Mapping function connect quantities (Q) and categories (C), e. g., the quantity "age" is separated into ordinal range, showed as $[0,:] \rightarrow < YOUNG, ADULT, OLD >$. Thus the transformation from quantities to categories is defined as Eq. (2):

$$x \rightarrow \begin{cases} YOUNG & if\, age(x) \leq 20 \\ ADULT & if\, 20 < age(x) \leq 60 \\ OLD & if\, 60 < age(x). \end{cases} \quad (2)$$

We implemented a prototype[1] of the mapping function above from quantitative scales to a domain of user-defined categories. Fig. 2 illustrates the user interface that represents the mapping of each value to categories. A user can change the value parameter *n* by adjusting the slider, which is able to change the number of bars of the mapping function. A user can also drag the vertical bars using circles so he can adjust the categories. A table below indicates the categorization result for each category.

This mapping relates to the classical sets theory, where the categorization is a *crisp* process that splits quantities into categories with a binary function: accepting or rejecting the object belonging to a category (Massad et al., 2009). An element *x* either belongs to a category or not. Those are strict perimeters and each element only belongs to a single category. Such a mapping function can be combined using intervals that often converts a whole range of values. If there is a set *W* that is not empty and a set $S \subset W$, the characteristic function of *S* will be shown as follows:

$$f_S(x) = \begin{cases} 1 & if\, x \in S \\ 0 & if\, x \notin S \end{cases} \quad (3)$$

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[1] https://observablehq.com/d/4560d69baca4663c

(a)

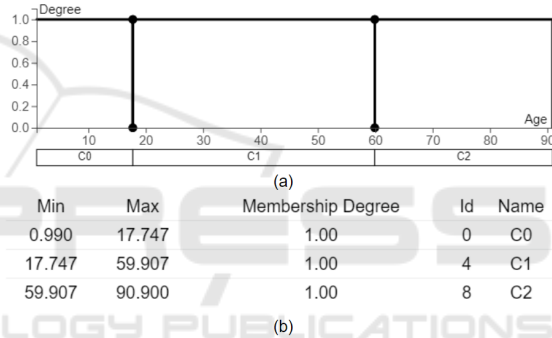| Min	Max	Membership Degree	Id	Name
0.990	17.747	1.00	0	C0
17.747	59.907	1.00	4	C1
59.907	90.900	1.00	8	C2

(b)

Figure 2: Categorization using crisp membership function, there are three crisp categories generated from the crisp membership function showed in (a) and the detail information of categories in (b).

$f_S(x)$ is the function, the domain of which is *W*. The value of $f_a(x)$ is included in set $\{0,1\}$. With $f_S(x) = 1$, it means element *x* belongs to set *S*; if $f_S(x) = 0$, it illustrates element *x* does not belong to set *S* so that this mapping function $f_S(x) \rightarrow \{0,1\}$ are able to completely represent the relationship between element *x* and set *S*.

However, *previous mapping function* has a limitation on splitting ambiguous sets. For example, it can not represent the *children*, *adolescent* and *adult*, because *adolescent* normally has overlap with *children* and *adult*. Thus, it is not any more suitable for separating quantitative values when there is ambiguity or there are more than one quantitative scale. In order to solve the problem that does not have sharp boundaries while categorizing, Zadeh proposed the membership degree that means an element can partially belong to a set (Zadeh et al., 1996).

In order to implement the fuzzy membership func-

tion into interactive categorization, we implemented the second prototype[2] to map quantities to categories based on fuzzy logic. Fig. 3(a) shows the interface for a user to adjust parameters. In this figure, x-axis represents quantitative values and y-axis shows the distribution of membership degree(μ). Compared to the first prototype, this one is able to be adjusted among fuzzy logic parameters *core* and *n* by manipulating the sliders. Besides, a user also can adjust *core* by dragging small black circles on the fuzzy membership function and make them different from other fuzzy categories. Finally, he can also define the name of generated categories by inputting texts so that a user can generate any fuzzy categories that they want with membership values and specific names. The membership values and names are illustrated in Fig. 2(b).

Fuzzy category is defined with relaxing belonging constraints, which proposes an intermediate membership value as interval $[0, 1]$. This value represent the degree how many possibility element *x* belongs to set *A*. Thus, the membership function can be described as $f_a(x) \rightarrow [0, 1]$. The value of $f_a(x)$ means the membership degree of membership function. With $f_a(x) = 1$, it represents the complete belongingness while $f_a(x) = 0$ shows the complete non-belongingness. Let $X = \{x_1, x_2, ..\}$ denote the set X of elements x_i. A fuzzy category is then a pair (X, μ). So fuzzy category is denoted as $\{\mu(x_1)/x_1, ..., \mu(x_n)/x_n\}$. The *core* and *support* respectively represent all elements that belong to a set are denoted $Core(A) = \{x \in X | A(x) = 1\}$ and all the elements which membership degree is > 0 such as $Supp(A) = \{x \in X | A(x) > 0\}$.

Any function $f_a(x) \rightarrow [0, 1]$ is membership function framework, instead of accurate representation. The real membership function depends on its shapes. Those membership functions with specific shapes are composed of specific properties and they are chosen based on different data and applied fields, e. g., triangular, trapezoidal, Gaussian and sigmoidal functions. In this paper, we chose the trapezoidal membership function and it is given by:

$$f(x) = \begin{cases} 0 & if \quad x < a \\ (x-a)/(b-a) & if \quad a \leq x < b \\ 1 & if \quad b \leq x < c \\ (d-x)/(d-c) & if \quad c \leq x < c \\ 0 & if \quad x > c \end{cases} \quad (4)$$

Where, a, b, c, d are the parameters of trapezoidal membership function. The membership functions can be displayed over a line chart in which x-axis is the quantitative value and the y-axis is the membership degrees. Each line is a category, and their membership degrees are from 0 or 1.

(a)
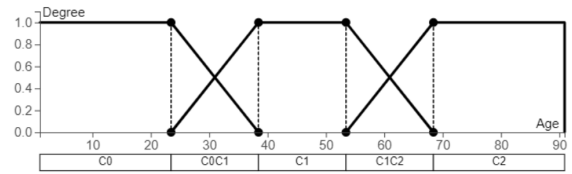
Min	Max	Membership Degree	Id	Name
0.990	23.468	1.00	0	C0
23.467	38.453	0.50	2	C0C1
38.453	53.438	1.00	4	C1
53.438	68.422	0.50	6	C1C2
68.422	90.900	1.00	8	C2

(b)

Figure 3: The mapping function based on fuzzy logic theory generates both crisp categories and fuzzy categories in (a) and all those categories and its information are in (b).

4 INTERACTIVE MEMBERSHIP FUNCTION

4.1 Interaction

We introduce how to use the interactive membership function in this section. There are two general parameters of membership function: *n* and *core* are shown as the sliders in Fig. 1. In this visual tool, a user can adjust the two parameters to generate different shapes of membership functions. By adjusting parameters in the control panel, all categories generated would be changed with the same value of *core*.

If a user would like to operate it more specifically, he can build a new shape of membership function by dragging those little circles showed in Fig. 4(a), for example, A user can change the parameter *core* of specific categories by dragging black circle showed in the left of Fig. 4(a). And then, the membership function is able to be changed as the right of Fig. 4(a), the first category in this membership function is changed. Finally, this membership function also offers the naming method. By inputting texts in the input fields shown in Fig. 4(b), the categories are able to be related to input labels and appear as new categories in the underlying dataset.
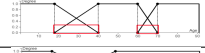
4.2 Implementation Details

This membership function is implemented in Observable, using library D3 (Bostock et al.,) and is bundled as a JavaScript ES6 module compliant with modern Web standards. An online prototype [3] is available along with the datasets described in this paper. Addi-

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Table 2: Summary of case studies illustrating interesting configurations constructed in the three case studies datasets.

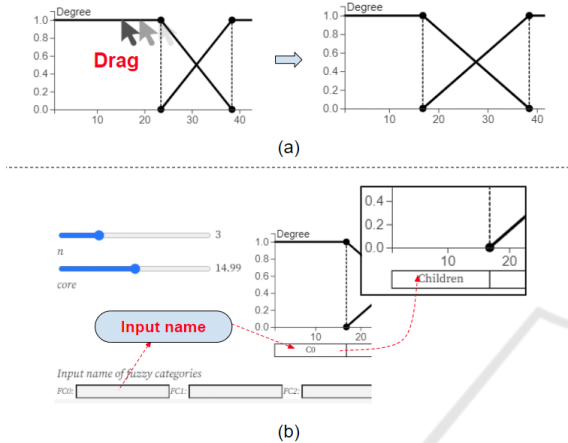| Data	FUZZYCUT	# of sets	Name	Interval
Age		5	Children, Teenager, Adult, Less old, Old	[0.99, 90.90]
Temperature		3	Cold, Warm, Hot	[-12.98, 68.58]
Speed		5	Low, Low-Middle, Middle, Mid-High, High	[0,104.04]

Figure 4: The top circles on membership function are able to be dragged to change the parameter *core* of a specific category in (a); The labels of categories are defined by inputting texts in input field in (b).

tional datasets can be added using a JSON file specification that captures the configurations that a user-defined. The parameters of configurations include the parameters of specific membership function and categories' names are showed as follows:

```
{
  "title": "Temperature",
  "attribute": "0",
  "parameters": [{
      "n": "3",
      "Core": "13.6",
      "Support": "27.2",
      "names": [
        "Cold",
        "Warm",
        "Hot"
      ]}]}
```

Such configuration enables the rapid export and sharing of the parameters, so they can be used in a different tool and can further be analyzed to understand the categorization process of multiple users.

5 CASE STUDIES

In this section, we introduce three typical quantitative datasets and the categorization we generated using our interactive visual tool. The datasets we used include age datasets, temperature datasets, and taxi speed datasets.

5.1 Case Study with Age Dataset

A typical age dataset consists of a semi-bounded interval and rounded values, there exists a minimum value but no fixed maximum value. The dataset we used, in this case, is The Simpsons, a popular U.S. TV Series. The dataset includes 24 characters with an age attribute. Age range from 0 to 90. The dataset is shown in Fig. 5 where ages are represented on the x-axis. The y-axis shows the membership degree for every category so that a user can easily check categories and their membership degrees. The table list below shows the categories and their properties. Membership values and the range of categories are shown in the table shown in Fig. 5(b), which is corresponding to the membership function and has 5 categories. We can see that the membership degrees of category 1 and category 3 are equal to 0.5, and other categories are equal to 1. The name of every category is shown on the table where each character now has an age category.

5.2 Case Study with Temperature Dataset

We use a temperature dataset of the United State in this second case study. As such temperature value does not have a clear maximum value and minimum value, the data is non-bounded and continuous. The dataset we used consists of temperatures from 357 divisions in the USA for 12 months. The categorization is shown in Fig. 6. Fig. 6(a) represents the fuzzy membership function and Fig. 6(b) is the table list of categories. In Fig. 6(a), temperature degrees are distributed along with the x-axis and the y-axis are membership values for each category. In this prototype, temperature values
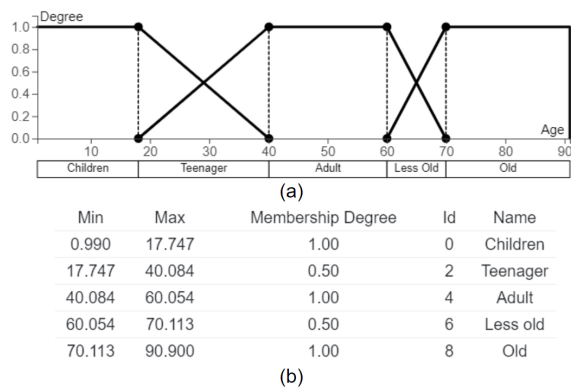
(a)

Min	Max	Membership Degree	Id	Name
0.990	17.747	1.00	0	Children
17.747	40.084	0.50	2	Teenager
40.084	60.054	1.00	4	Adult
60.054	70.113	0.50	6	Less old
70.113	90.900	1.00	8	Old

(b)

Figure 5: The age data are from 0 to 90, there are three main categories with name *Chidren*, *Adult* and *Old*. Meanwhile, two fuzzy categories created and they are named with *Teenager* and *Less old*.

are separated into three categories, which are from -12.98 to 68.56. The name of those three categories are *Cold*, *Warm* and *Hot*. But the *Warm* category is derived from *Cold* and *Hot*, membership degree of which is 0.5.
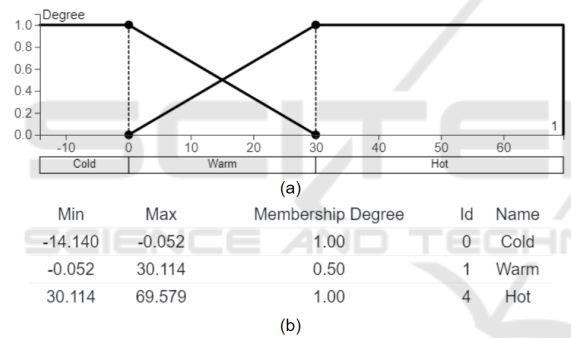

(a)

Min	Max	Membership Degree	Id	Name
-14.140	-0.052	1.00	0	Cold
-0.052	30.114	0.50	1	Warm
30.114	69.579	1.00	4	Hot

(b)

Figure 6: Temperature data are separated with two main categories *Cold* and *Hot*. The intermediate category *Warm* is created.

5.3 Case Study with Taxi Speed Dataset

We conduct the third case using a taxi speed dataset. The speed data are semi-rounded and continuous values types. Traffic analysts usually are interested in characterizing taxi drivers' characteristics, which is helpful for them to identify driving behaviors. Obviously, speed is a very essential parameter to define categories, as it reflects many characteristics of drivers. The prototype of taxi speed is showed in Fig. 7(a) and the table list is showed in Fig. 7(b). In this case, speed is separated into five categories and their main categories are *Low*, *Middle* and *High*. Besides, there are two fuzzy categories with membership degrees less than 1, named *Low-Middle* and *Mid-High*. Other categories, more domain-specific, could have been used,

e.g. *Slow*, *Fast* by editing the label input field in the prototype.

6 DISCUSSION AND PERSPECTIVES

In this paper, we designed an interactive tool to categorize quantities into categories. By using Fuzzy Logic, it enables splitting quantitative values into categories with a degree called membership degrees. We are able to present uncertainties with membership degrees to better capture user knowledge and demonstrated its application using three datasets included in an interactive prototype.

The summary of case studies is shown in Table 2. The results suggest that, in most situations, users are likely to categorize quantities into 3 - 5 categories with particular names for each category. First, naming generated categories is usually based on social basic knowledge, e. g., the name of age categories, which includes *Children*, *Adult* and *Old*. But when the categories are complicated, the name of categories tend to be *Very ...* or *Super ...* There is also another naming method, which combines the basic categories, e. g., the name *Mid-High* is derived from *Middle* and *High* in speed data. Our tool supports an extensive number of categories, up to several dozens. Beyond this number, it may however become difficult to read the category's names and interact with them.

We identified the following limits in our current work. **It does not consider the distribution of quantitative values.** The trapezoid membership function can not consider the real distribution of the dataset, instead of building a membership function based on the maximum value, minimum value, and the number of categories. The distribution is not considered in this method. **Only categorize dataset with clear**
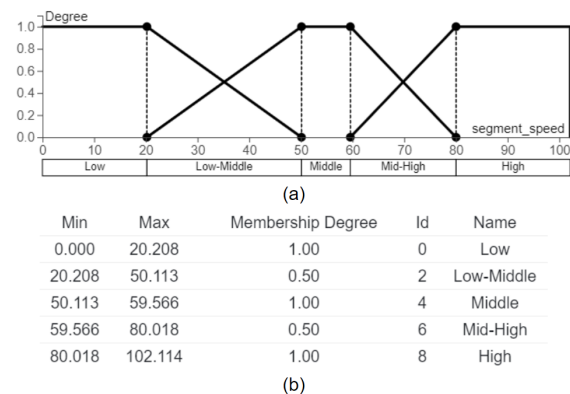

(a)

Min	Max	Membership Degree	Id	Name
0.000	20.208	1.00	0	Low
20.208	50.113	0.50	2	Low-Middle
50.113	59.566	1.00	4	Middle
59.566	80.018	0.50	6	Mid-High
80.018	102.114	1.00	8	High

(b)

Figure 7: Taxi speed are separated with three main categories and two fuzzy categories. Those two categories are named *Low-Middle* and *Mid-High*.

maximum value and minimum value. Currently, we can only use the maximum value existed in the dataset or dataset with a clear boundary. For example, in the age dataset, the maximum value is 90, so the maximum value in this prototype is 90, it is totally based on the range of the dataset. **Limitation in categorizing quantities with rounded values**: When we categorize age dataset, the results of categories are not integer, which leads to the abnormal categories e. g., category *Children* is from 0.99 years old to 17.97 years old.

Our main perspective is to validate our approach using a formal evaluation process to assess how the fuzzy membership function helps a user categorizing quantities. The evaluation includes three steps: **1) manual categories creation; 2) using fuzzy membership function and 3) post-study questionnaire.**

In manual categories creation, investigating *How does a user create categories* for datasets such as the characters in The Simpsons would be a major question to be answered as in this paper we only reported on typically used manually created to illustrate our approach. In this evaluation step, a user will be asked by questionnaire to focus on the number of categories, the intervals, and the name of categories, and we will log the results for remote analysis.

Our study protocol will be as follows (using the prototype shown in Fig. 3(a)): **Step 1:** we introduce FUZZYCUT to participants and tell them how to use and operate FUZZYCUT which will last 10 minutes. In this part, the user is able to know how to change parameters of fuzzy membership function and interact with FUZZYCUT to generate and adjust categories. **Step 2:** those parameters, categories, and their intervals are saved remotely by recording every interaction. **Step 3:** all those information collected from a user are organized and analyzed to support a formalized function, which is very important for future work to develop specific mapping function because the collected information will be the major proof of mapping specific quantities. A post-study questionnaire will collect feedback, the content of which aims to illustrate if this tool influences participants' original intent, and how those categories change after using this tool.

7 CONCLUSION

This paper introduced an interactive visualization technique to assist a user in categorizing quantities into categories. We relied upon a well-known function, the fuzzy membership function from fuzzy logic theory, which we implemented as an interactive prototype. We illustrated its use for 3 case studies: age, temperature, and taxi speed data. As the prototype enables an ex-

plicit mapping of the categorization function, we plan to use it to trace the process a user follows when creating categories. In particular to understand if there is consensus between groups of users regarding the choice of categories values interval, labeling, and confidence. We also plan to use the interactive function to communicate this mapping, e. g., as an interactive legend (Park and Park, 2010b) for both visual communication and exploration of datasets.

REFERENCES

Alsallakh, B., Hanbury, A., Hauser, H., Miksch, S., and Rauber, A. (2014). Visual methods for analyzing probabilistic classification data. *IEEE transactions on visualization and computer graphics*, 20(12):1703–1712.

Bostock, M., Ogievetsky, V., and Heer, J. D³ data-driven documents. 17(12):2301–2309.

Brodlie, K., Osorio, R. A., and Lopes, A. (2012). A review of uncertainty in data visualization. In *Expanding the frontiers of visual analytics and visualization*, pages 81–109. Springer.

Cao, N., Lin, Y.-R., and Gotz, D. (2015). Untangle map: Visual analysis of probabilistic multi-label data. *IEEE transactions on visualization and computer graphics*, 22(2):1149–1163.

Clifford, H. T., Stephenson, W., Clifford, H., and Stephenson, W. (1975). *An introduction to numerical classification*, volume 240. Academic Press New York.

Cover, T. and Hart, P. (1967). Nearest neighbor pattern classification. *IEEE transactions on information theory*, 13(1):21–27.

Dong, X. and Hayes, C. C. (2012). Uncertainty visualizations: Helping decision makers become more aware of uncertainty and its implications. *Journal of Cognitive Engineering and Decision Making*, 6(1):30–56.

Dressel, J. and Nori, F. (2014). Certainty in heisenberg's uncertainty principle: revisiting definitions for estimation errors and disturbance. *Physical Review A*, 89(2):022106.

Ester, M., Kriegel, H.-P., Sander, J., and Xu, X. Dbscan: A density-based algorithm for discovering clusters in large spatial databases with noise. In *Proc. 1996 Int. Conf. Knowledge Discovery and Data Mining (KDD'96)*, pages 226–231.

Lin, Y.-R., Cao, N., Gotz, D., and Lu, L. (2014). Untangle: visual mining for data with uncertain multi-labels via

triangle map. In *2014 IEEE International Conference on Data Mining*, pages 340–349. IEEE.

MacQueen, J. et al. (1967). Some methods for classification and analysis of multivariate observations. In *Proceedings of the fifth Berkeley symposium on mathematical statistics and probability*, volume 1, pages 281–297. Oakland, CA, USA.

Massad, E., Ortega, N. R. S., de Barros, L. C., and Struchiner, C. J. (2009). *Fuzzy logic in action: Applications in epidemiology and beyond*, volume 232. Springer Science & Business Media.

Park, Y. and Park, J. (2010a). Disk diagram: An interactive visualization technique of fuzzy set operations for the analysis of fuzzy data. *Information Visualization*, 9(3):220–232.

Park, Y. and Park, J. (2010b). Interactive visualization of fuzzy set operations. In *Visualization and Data Analysis 2010*, volume 7530, page 753002. International Society for Optics and Photonics.

Pedrycz, W. and Gomide, F. (2007). *Fuzzy systems engineering: toward human-centric computing*. John Wiley & Sons.

Rheingans, P. and Desjardins, M. (2000). Visualizing high-dimensional predictive model quality. In *Proceedings Visualization 2000. VIS 2000 (Cat. No. 00CH37145)*, pages 493–496. IEEE.

Riche, N. H., Lee, B., and Plaisant, C. (2010). Understanding interactive legends: a comparative evaluation with standard widgets. In *Computer graphics forum*, volume 29, pages 1193–1202. Wiley Online Library.

Seifert, C. and Lex, E. (2009). A novel visualization approach for data-mining-related classification. In *2009 13th International Conference Information Visualisation*, pages 490–495. IEEE.

Skeels, M., Lee, B., Smith, G., and Robertson, G. G. (2010). Revealing uncertainty for information visualization. *Information Visualization*, 9(1):70–81.

Zadeh, L. A. (1965). Fuzzy sets. *Information and control*, 8(3):338–353.

Zadeh, L. A., Klir, G. J., and Yuan, B. (1996). *Fuzzy sets, fuzzy logic, and fuzzy systems: selected papers*, volume 6. World Scientific.

Zhao, Y., Luo, F., Chen, M., Wang, Y., Xia, J., Zhou, F., Wang, Y., Chen, Y., and Chen, W. (2018). Evaluating multi-dimensional visualizations for understanding fuzzy clusters. *IEEE transactions on visualization and computer graphics*, 25(1):12–21.

Zhu, L., Xia, W., Liu, J., and Song, A. (2018). Visualizing fuzzy sets using opacity-varying freeform diagrams. *Information Visualization*, 17(2):146–160.