Rough Continuity Represented by Intuitionistic Fuzzy Sets

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Abstract: Studying rough calculus was initiated by Z. Pawlak in his many papers. He originated the concept of rough real functions. Like the notion of continuity in classical analysis, the rough continuity is also a central notion in rough calculus. Relying on the Pawlak's approximation spaces on the real closed bounded intervals, first, two intuitionistic fuzzy sets are established starting from rough functions. Then, based on them, some necessary and sufficient conditions for the rough continuity in terms of intuitionistic fuzzy set theory will be presented.

1 INTRODUCTION

In 1965, Lotfi A. Zadeh initiated the fuzzy set theory (Zadeh, 1965) as a new mathematical theory to manage uncertainty. In the early 1980s, Zdzisław Pawlak established a new mathematical tool also to manage uncertainty which is called the rough set theory (RST) (Pawlak, 1982).

Let U be a nonempty reference set which is commonly called the *universe*. Any set, classical or nonclassical, is formed from the elements of the universe. They can be represented with more or less similar tools, called membership functions.

A classical or crisp set S can be represented, among other things, by its *characteristic function* $\chi_S : U \to \{0,1\}$ (Halmos, 1960; Hayden et al., 1968). Generalizing this representation, a fuzzy set F is defined by a function $\mu_F : U \to [0,1]$ which is called the *fuzzy membership function*. In rough set theory, however, the definition of a similar representation tool is somewhat more complex.

In RST, first, it is assumed that a beforehand predefined family of subsets of U is given. Namely, this set family is a partition of U generated by an equivalence relation. Any equivalence class can be viewed as a set of indiscernible objects characterized by the available information (knowledge) about them. Accordingly, in RST an equivalence relation is actually called the *indiscernible relation*.

The partition is called the *base system*, and its elements, i.e., the equivalence classes are the *base sets*. From the base sets the so–called *definable sets* are formed with the union operation.

Next, with the help of base sets, *lower* and *upper approximation sets* are formed for any $S \subseteq U$. The former is the union of all base sets which are included in S, whereas the latter is the union of all base sets which have a nonempty intersection with S. The difference of upper and lower approximation sets is the *boundary* of S. S is *exact* if its boundary is the empty set, otherwise it is *rough*.

In RST, the *rough membership function*, rm–function in short, is defined as follows. Let U be finite. Then, the rm–function is commonly defined by

$$\mu_S(u) = \frac{|[\![u]\!] \cap S|}{|[\![u]\!]|},$$

where $|\cdot|$ denotes the number of elements of a set, and $[\![u]\!]$ is the base set (equivalence class) to which $u \in U$ belongs ($|\emptyset| = 0$ by definition). This rm–function quantifies the degree of the relative overlap between the set S and a base set.

Both characteristic and fuzzy membership functions are of *a priori* nature. They have a wide range of applications, see, e.g., (Aquino et al., 2020; de Jesus Rubio, 2009; Chiang et al., 2019; Elias et al., 2020; Meda-Campaña, 2018; Hernández et al., 2020)

In contrast, in RST, initially some information about the elements of the universe is necessary to have at our disposal in order to be able to approximate a set. Thus, the rm–function is of *a posteriori* nature.

Still, an rm–function can formally be viewed as a special type of fuzzy membership function, of course,

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with many constraints owing to their derived nature (Yao and Zhang, 2000). However, the converse is not true in general ((Biswas, 2000), Example 3.1).

It is broadly accepted that the fuzzy and rough set theories are related but distinct and complementary to each other. Nevertheless, they can be combined with each other (Dubois and Prade, 1987; Dubois and Prade, 1992). Moreover, their common/ distinct features can be outlined based on their fuzzy/ rough membership functions (Chakraborty, 2011; Csajbók and Ködmön, 2020).

In Section 2 some basic notations are summarized for the sake of full clarity. Then, the notion of rough real function and its two possible representations will be described.

Section 3 contains the material which is required to establish the connection between the rough continuity and intuitionistic fuzzy sets. It is the most extensive section in the paper.

Section 4 presents the main result of the paper. It provides a necessary and sufficient conditions for the rough continuity in terms of intuitionistic fuzzy set theory.

2 ROUGH REAL FUNCTIONS

In the mid 1990s, relying on rough set theory, Pawlak originated the study of rough calculus in many papers (Pawlak, 1994; Pawlak, 1996; Pawlak, 1997). Its basic notion is the rough real function. In (Csajbók, 2020), employing Pawlak's ideas, some additional representations of rough real functions are given. Two of them, pointwise and blockwise representations, will be required in the rest of this paper.

Let U,V be two nonempty sets. A function f is denoted by $f : U \to V$, $u \mapsto f(u)$ with domain $\mathsf{Dom} f = U$ and co-domain $\mathsf{Im} f = V$. In addition, $u \mapsto f(u)$ is the assignment or mapping rule of f. For any $S \subseteq U$, $f(S) = \{f(u) \mid u \in S\} \subseteq V$ is the direct image of S. V^U denotes the set of all such functions.

If $f,g \in V^U$, the operation $f \odot g$, $\odot \in \{+,-,\cdot,/\}$, and the relation $f \boxdot g$, $\boxdot \in \{=,\neq,\leq,<,\geq,>\}$ are understood pointwise.

$\mathbb{R}$ is the set of real numbers. $\mathbb{R}^{\geq 0}$ denotes the set of nonnegative real numbers.

Let $a,b \in \mathbb{R}$ $(a \leq b)$. $[a,b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$ and $]a,b[= \{x \in \mathbb{R} \mid a < x < b\}$ denote the closed and open bounded intervals, respectively. It is easy to interpret, then, the open-closed $]a,b]$ and closed-open $[a,b[$ intervals.

By $(\cdot,\cdot)$, we mean an ordered pair.

Let $[n] = \{0,1,\ldots,n\} \subseteq \mathbb{N}$ denote a finite set of natural numbers. Accordingly, $]n] = \{1,\ldots,n\}$, $[n[= \{0,1,\ldots,n-1\}$, and $]n[= \{1,\ldots,n-1\}$.

Throughout the paper, let I be a closed bounded interval $I = [0,a]$ ($a \in \mathbb{R}^{\geq 0}$, $a > 0$).

The initial notion of the rough calculus is the following.

Definition 1. A *categorization* or *discretization* of I is the sequence $S_I = \{x_i\}_{i \in [n]} \subseteq \mathbb{R}^{\geq 0}$, where $n \geq 1$ and $0 = x_0 < x_1 < \cdots < x_n = a$. $\square$

Let I_S denote the equivalence relation generated by the categorization S_I. Let $x,y \in I$. xI_Sy if

- $x = y = x_i \in S_I$ for some $i \in [n]$, or
- $x,y \in]x_i,x_{i+1}[$ for some $i \in [n[$.

The partition I/I_S generated by I_S is the following:

$$I/I_S = \{\{x_0\},]x_0,x_1[,\{x_1\},\ldots,]x_{n-1},x_n[,\{x_n\}\},$$

where $\{x_i\} = [x_i,x_i]$ $(i \in [n])$.

The block of the partition I/I_S containing $x \in I$ is denoted by $[\![x]\!]_{I_S}$. In particular, if $x \in S_I$, $[\![x]\!]_{I_S} = \{x\}$. If $x \in [\![x]\!]_{I_S} =]x_i,x_{i+1}[$, then $\overline{[\![x]\!]}_{I_S}$ denotes the closed interval $[x_i,x_{i+1}]$. When $x \in S_I$, $[\![x]\!]_{I_S} = \overline{[\![x]\!]}_{I_S} = \{x\}$.

In terms of RST terminology, I_S is an indiscernibility relation on I. The members of I/I_S are the *base sets*. Any union of base sets are referred to as *definable sets*. By definition, $\emptyset$ is definable. Their collection is $\mathcal{D}_{I/I_S}$.

In RST, the domain and co–domain of the lower and upper approximation functions are the power set of I. In the rough calculus, however, the closed bounded intervals of the form $[0,x]$ $(x \in I)$ will only be approximated. Therefore, the lower and upper approximations sets are defined by

$$\begin{aligned}
\mathsf{l}_S([0,x]) &= \{x' \in I \mid [\![x']\!]_{I_S} \subseteq [0,x]\} \\
&= \cup\{[\![x']\!]_{I_S} \in I/I_S \mid [\![x']\!]_{I_S} \subseteq [0,x]\}; \\
\mathsf{u}_S([0,x]) &= \{x' \in I \mid [\![x']\!]_{I_S} \cap [0,x] \neq \emptyset\} \\
&= \cup\{[\![x']\!]_{I_S} \in I/I_S \mid [\![x']\!]_{I_S} \cap [0,x] \neq \emptyset\}.
\end{aligned}$$

$\mathsf{PAS}(I) = (I, I/I_S, \mathcal{D}_{I/I_S}, \mathsf{l}_S, \mathsf{u}_S)$ is called the *Pawlak approximation space*.

The boundary of $[0,x]$ is

$$\mathsf{bnd}_S([0,x]) = \mathsf{u}_S([0,x]) \setminus \mathsf{l}_S([0,x]).$$

With a slight abuse of the notations, in order to simplify the above notations, let us define the following numbers:

$$\begin{aligned}
\mathsf{l}_S(x) &= \max\{x' \in S_I \mid x' \leq x\}, \\
\mathsf{u}_S(x) &= \min\{x' \in S_I \mid x' \geq x\}.
\end{aligned}$$

Then, it is easy to check that

- if $x \in S_I$, then $\mathsf{l}_S([0,x]) = [0,\mathsf{l}_S(x)] = [0,x]$ and $\mathsf{u}_S([0,x]) = [0,\mathsf{u}_S(x)] = [0,x]$;

- if $x \notin S_I$, then $\mathsf{l}_S([0,x]) = [0, \mathsf{l}_S(x)] \subsetneq [0,x]$, and $\mathsf{u}_S([0,x]) = [0, \mathsf{u}_S(x)[\supsetneq [0,x]$;
- if $x \in S_I$, then $\mathsf{bnd}_S([0,x]) = \emptyset$; if $x \notin S_I$, then $\mathsf{bnd}_S([0,x]) =]\mathsf{l}_S(x), \mathsf{u}_S(x)[\neq \emptyset$.

The number $x \in I$ is *exact* with respect to $\mathsf{PAS}(I)$ if $\mathsf{l}_S(x) = \mathsf{u}_S(x)$, otherwise x is *inexact* or *rough* (Pawlak, 1996). Of course, $x \in I$ is exact iff $x \in S_I$.

In this context, the members of I/I_S are called the *rough numbers* with respect to $\mathsf{PAS}(I)$. In addition, the categorization points in S_I are called the *roughly isolated points* with respect to $\mathsf{PAS}(I)$.

Let $I = [0, a_I]$ and $J = [0, a_J]$ be two closed bounded intervals with $a_I, a_J \in \mathbb{R}^{\geq 0}$, $a_I, a_J > 0$. Let S_I and P_J be the categorizations of I and J, respectively, where $S_I = \{x_i\}_{i \in [n]}$ and $P_J = \{y_j\}_{j \in [m]} \subseteq \mathbb{R}^{\geq 0}$ in such a way that $m, n \geq 1$, and $0 = x_0 < x_1 < \cdots < x_n = a_I$, $0 = y_0 < y_1 < \cdots < y_m = a_J$. The corresponding Pawlak approximation spaces are $\mathsf{PAS}(I)$, $\mathsf{PAS}(J)$.

A Cartesian coordinate system whose x and y axes equipped with $\mathsf{PAS}(I)$ and $\mathsf{PAS}(J)$ is called the (S_I, P_J)*–coordinate system*, or *rough coordinate system* in short. Any function $f \in J^I$ attached to a rough coordinate system is called the *rough real function*.

In order to make the rough coordinate system easier to handle technically, the blocks of the partition I/I_S are enumerated as follows.

$$N_I : I/I_S \to [2n],$$
$$[\![x]\!]_{I_S} \mapsto \begin{cases} B_{2i} = 2i, & \text{if } \exists i \in [n]\,([\![x]\!]_{I_S} = \{x_i\}), \\ B_{2i+1} = 2i+1, & \text{if } \exists i \in [n[\,(x \in]x_i, x_{i+1}[). \end{cases}$$

The inverse of N_I is:

$$N_I^{-1} : [2n] \to I/I_S,$$
$$B_i \mapsto \begin{cases} \{x_{i/2}\}, & \text{if } i \equiv 0 \pmod 2 \\ \left]x_{\frac{i-1}{2}}, x_{\frac{i+1}{2}}\right[, & \text{if } i \equiv 1 \pmod 2 \end{cases}.$$

The equivalence classes of J/J_P can be enumerated in the same way with the help of an enumeration function N_J. They are referred to as C_j's ($j \in [2m]$).

Example 1. Figure 1 (a) depicts a rough coordinate system with $S_I = \{x_0 = 0, x_1, x_2, x_3, x_4, x_5\}$ and $P_{[0,1]} = \{y_0 = 0, y_1, y_2, y_3, y_4 = 1\}$. Figure 1 (b) presents a rough real function attached to this rough coordinate system. □

Definition 2 ((Pawlak, 1994)). Let $f \in J^I$. The *pointwise* (S_I, P_J)*–lower* and (S_I, P_J)*–upper approximations* of f are the functions

$$\underline{f} : I \to P_J,\ x \mapsto \mathsf{l}_P(f(x)) = \max\{y \in P_J \mid y \leq f(x)\},$$
$$\overline{f} : I \to P_J,\ x \mapsto \mathsf{u}_P(f(x)) = \min\{y \in P_J \mid y \geq f(x)\}.$$

f is *pointwise exact* at x if $\underline{f}(x) = \overline{f}(x)$, otherwise f is *pointwise inexact* or *rough* at x.

f is *pointwise exact on* $I' \subseteq I$ if $\underline{f}(x) = \overline{f}(x)$ for all $x \in I'$, otherwise f is *pointwise inexact* (*rough*) *on* I'. □

Definition 3. Let $f \in J^I$. The *block by block, blockwise in short,* (S_I, P_J)*–lower* and (S_I, P_J)*–upper approximations* of f are the functions

$$\underset{\leftrightarrow}{f} : I \to P_J,\ x \mapsto \mathsf{l}_P(\inf f([\![x]\!]_{I_S})),$$
$$\overset{\leftrightarrow}{f} : I \to P_J,\ x \mapsto \mathsf{u}_P(\sup f([\![x]\!]_{I_S})).$$

The function f is *blockwise exact on* B_i for some $i \in [2n]$ if $\underset{\leftrightarrow}{f}(B_i) = \overset{\leftrightarrow}{f}(B_i)$, that is, the direct images of B_i with respect to $\underset{\leftrightarrow}{f}$ and $\overset{\leftrightarrow}{f}$ are equal; otherwise f is *blockwise inexact* (*rough*) on B_i.

The function f is *blockwise exact on* I if f is blockwise exact on all $B_i \in I/I_S$, otherwise f is *blockwise inexact* (*rough*) *on* I. □

Owing to the fact that $\inf f([\![x]\!]_{I_S})$ and $\sup f([\![x]\!]_{I_S})$ are constant on every B_i, the functions $\underset{\leftrightarrow}{f}$ and $\overset{\leftrightarrow}{f}$ are constant on every B_i ($i \in [2n]$). Accordingly, using the word "blockwise" is appropriate.

Example 2. Figure 2 (a) depicts the pointwise lower and upper approximations of f. f is pointwise exact at $x^i, x^{ii}, x_2, x^{iii}, x^{iv}$, and pointwise rough at all other points.

Figure 2 (b) shows the blockwise lower and upper approximations of f. f is blockwise exact only on $B_4 = \{x_2\}$, and blockwise rough on all other blocks. □

It is easy to check the following simple but important statement.

Lemma 1. *Let $f \in J^I$ be a rough real function. Then,*

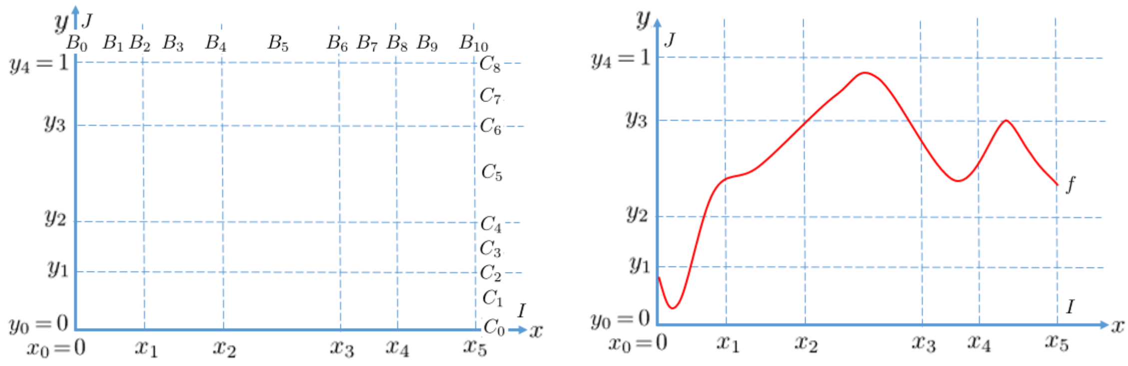
$$\underset{\leftrightarrow}{f} \leq \underline{f} \leq f \leq \overline{f} \leq \overset{\leftrightarrow}{f}$$

holds on I.

3 DERIVING INTUITIONISTIC FUZZY SETS FROM ROUGH REAL FUNCTIONS

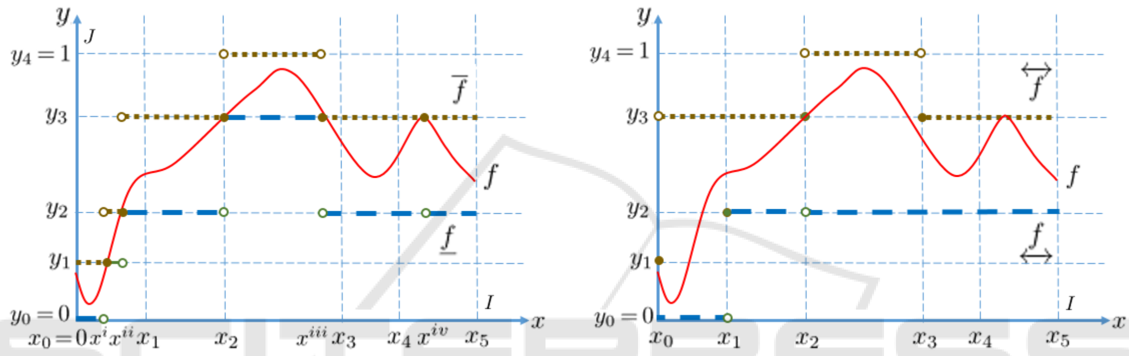
Let U be a nonempty set.

According to (Zadeh, 1965) a *fuzzy set* (FS) on U is the function $\mu \in [0,1]^U$; see also, (Klir and Yuan, 1995; Dubois and Prade, 2000; Zimmermann, 2001; Ross, 2010). μ is also called the *membership function*. $\mathcal{FS}(U)$ denotes the family of all fuzzy sets on U.



(a) (b)

Figure 1: A rough coordinate system and a rough real function.



(a) (b)

Figure 2: Pointwise and blockwise lower/upper approximations of f.

Let $\mathbb{I} = \{[a,b] \mid 0 \leq a \leq b \leq 1\}$.

Let $\mu_A, \nu_A \in \mathcal{FS}(U)$ with $\mu_A \leq \nu_A$. An *interval-valued fuzzy set* (IVFS) on U is the function $\mu_A^{IVFS} : U \to \mathbb{I}$, $u \mapsto [\mu_A(u), \nu_A(u)]$ (Gorzałczany, 1987). μ_A^{IVFS} is also denoted simply by $[\mu_A, \nu_A]$.

Let $\mu_A, \nu_A \in \mathcal{FS}(U)$ with $0 \leq \mu_A + \nu_A \leq 1$. An *intuitionistic fuzzy set* (IFS) on U is defined by the function pair $\mu_A^{IFS} = (\mu_A, \nu_A)$ (Atanassov, 1986; Atanassov, 1999; Atanassov, 2012). μ_A and ν_A are the IFS *membership* and IFS *nonmembership* functions, respectively. $\pi_A = 1 - \mu_A - \nu_A \in \mathcal{FS}(U)$ is the IFS *indeterminacy* function. The family of all intuitionistic fuzzy sets on U is denoted by $\mathcal{IFS}(U)$.

Let $\mu_A^{IFS}, \mu_B^{IFS} \in \mathcal{IFS}(U)$. Then,

- $\mu_A^{IFS} = \mu_B^{IFS}$ if $\mu_A = \mu_B$ and $\nu_A = \nu_B$;
- $\mu_A^{IFS} \subseteq \mu_B^{IFS}$ if $\mu_A \leq \mu_B$ and $\nu_A \geq \nu_B$.

It is well known that every IVFS $[\mu_A, \nu_A]$ corresponds to an IFS $(\mu_A, 1 - \nu_A)$, while every IFS (μ_A, ν_A) corresponds to an IVFS $[\mu_A, 1 - \nu_A]$ (Atanassov and Gargov, 1989; Bustince and Burillo, 1996).

There are many papers dealing with the interrelationship between rough set and intuitionistic fuzzy set theory (Rizvi et al., 2002; Cornelis et al., 2003; Zhou and Wu, 2011; Xu et al., 2014). In this paper, the starting point is the rough real functions, i.e., real functions managing them in rough coordinate systems. Thereafter, intuitionistic fuzzy sets are derived from their pointwise and blockwise representations.

For the rest of this section, let $\mathsf{PAS}(I)$ and $\mathsf{PAS}([0,1])$ be two Pawlak approximation spaces defined on the intervals I and $[0,1]$ with the categorizations $S_I = \{x_0 = 0, x_1, \ldots, x_n\}$ and $P_{[0,1]} = \{y_0 = 0, y_1, \ldots, y_m = 1\}$. In addition, let $f \in [0,1]^I$ be a rough function attached to the $(S_I, P_{[0,1]})$–coordinate system.

According to Definition 2, $\underline{f}, \overline{f} \in [0,1]^I$, that is, the pointwise $(S_I, P_{[0,1]})$–lower and upper approximations of f are fuzzy sets. Moreover, $\underline{f} \leq \overline{f}$ also holds. Hence, $f_{pw}^{IVFS} = [\underline{f}, \overline{f}]$ forms an interval–valued fuzzy set. Then, the function pair $f_{pw}^{IFS} = (\underline{f}, 1 - \overline{f})$ forms an intuitionistic fuzzy set. (The subscript "*pw*" refers to "pointwise".)

In the intuitionistic fuzzy set theory context, $\underline{f}$ and $1 - \overline{f}$ are the IFS membership and nonmembership functions, respectively, and $\pi_f^- = 1 - \underline{f} - (1 - \overline{f}) =$

$\overline{f} - \underline{f}$ is the IFS indeterminacy function.

Similarly, according to Definition 3, $\underset{\leftrightarrow}{f}, \overset{\leftrightarrow}{f} \in [0,1]^I$, that is, the blockwise $(S_I, P_{[0,1]})$–lower and upper approximations of f are also fuzzy sets, and $\underset{\leftrightarrow}{f} \leq \overset{\leftrightarrow}{f}$ holds, too. Hence, $f^{IVFS}_{bw} = \left[\underset{\leftrightarrow}{f}, \overset{\leftrightarrow}{f}\right]$ forms an interval–valued fuzzy set, and so the function pair $f^{IFS}_{bw} = \left(\underset{\leftrightarrow}{f}, 1 - \overset{\leftrightarrow}{f}\right)$ is an intuitionistic fuzzy set. (The subscript "*bw*" refers to "blockwise".)

In terms of intuitionistic fuzzy set theory, $\underset{\leftrightarrow}{f}$ and $1 - \overset{\leftrightarrow}{f}$ are the IFS membership and nonmembership functions, respectively, and $\pi^{\leftrightarrow}_f = 1 - \underset{\leftrightarrow}{f} - (1 - \overset{\leftrightarrow}{f}) = \overset{\leftrightarrow}{f} - \underset{\leftrightarrow}{f}$ is the IFS indeterminacy function.

Intuitionistic fuzzy sets f^{IFS}_{pw} and f^{IFS}_{bw} are derived from f with respect to a $(S_I, P_{[0,1]})$–coordinate system. They are called the *pointwise* and *blockwise roughly derived intuitionistic fuzzy sets*.

There are many different geometric interpretations of intuitionistic fuzzy sets. For our purposes, the so–called "unit segments" representation will be appropriate (cf. (Atanassov, 1999), Figure 1.3.). Figure 3 in this way depicts the geometric interpretations of $f^{IFS}_{pw} = (\underline{f}, 1 - \overline{f})$ and $f^{IFS}_{bw} = (\underset{\leftrightarrow}{f}, 1 - \overset{\leftrightarrow}{f})$. Accordingly, in Figure 3 (a), unit segments $\underline{f}(x)$, $\overline{f}(x) - \underline{f}(x)$, and $1 - \overline{f}(x)$ are assigned to every $x \in I$; correspondingly, in Figure 3 (b), unit segments $\underset{\leftrightarrow}{f}(x)$, $\overset{\leftrightarrow}{f}(x) - \underset{\leftrightarrow}{f}(x)$, and $1 - \overset{\leftrightarrow}{f}(x)$ are assigned to every $x \in I$.

Proposition 1. *The both inclusion relations*

$$f^{IFS}_{pw} = (\underline{f}, 1 - \overline{f}) \subseteq f^{IFS}_{bw} = (\underset{\leftrightarrow}{f}, 1 - \overset{\leftrightarrow}{f}) \quad (1)$$

$$f^{IFS}_{bw} = (\underset{\leftrightarrow}{f}, 1 - \overset{\leftrightarrow}{f}) \subseteq f^{IFS}_{pw} = (\underline{f}, 1 - \overline{f}) \quad (2)$$

fail in general.

Proof. Due to Lemma 1 , in general,

- $\underline{f} \leq \underset{\leftrightarrow}{f}$ fails in the case of Equation 1 , and
- $1 - \overset{\leftrightarrow}{f} \geq 1 - \overline{f}$ fails in the case of Equation 2 . □

Proposition 2. *Let f^{IFS}_{pw} and f^{IFS}_{bw} be the pointwise and blockwise roughly derived intuitionistic fuzzy sets. Then, $f^{IFS}_{pw}(x_i) = f^{IFS}_{bw}(x_i)$ for every $x_i \in S_I$ ($i \in [n]$) categorization point.*

Proof. $f^{IFS}_{pw} = f^{IFS}_{bw} \Leftrightarrow (\underline{f}, 1 - \overline{f}) = (\underset{\leftrightarrow}{f}, 1 - \overset{\leftrightarrow}{f}) \Leftrightarrow \underline{f} = \underset{\leftrightarrow}{f}$ and $\overline{f} = \overset{\leftrightarrow}{f}$. Then, the statement follows from the fact that $\underline{f}(x_i) = \underset{\leftrightarrow}{f}(x_i)$ and $\overline{f}(x_i) = \overset{\leftrightarrow}{f}(x_i)$ for every $x_i \in S_I$ ($i \in [n]$) categorization point. □

Example 3. According to Figures 2 (a) and 2 (b),

- $\underline{f}(x_0) = \underset{\leftrightarrow}{f}(x_0) = y_0 = 0$, $\underline{f}(x_1) = \underset{\leftrightarrow}{f}(x_1) = y_2$, $\underline{f}(x_2) = \underset{\leftrightarrow}{f}(x_2) = y_3$, $\underline{f}(x_3) = \underset{\leftrightarrow}{f}(x_3) = y_2$, $\underline{f}(x_4) = \underset{\leftrightarrow}{f}(x_4) = y_2$, $\underline{f}(x_5) = \underset{\leftrightarrow}{f}(x_5) = y_2$.
- $\overline{f}(x_0) = \overset{\leftrightarrow}{f}(x_0) = y_1$, $\overline{f}(x_1) = \overset{\leftrightarrow}{f}(x_1) = y_3$, $\overline{f}(x_2) = \overset{\leftrightarrow}{f}(x_2) = y_3$, $\overline{f}(x_3) = \overset{\leftrightarrow}{f}(x_3) = y_3$, $\overline{f}(x_4) = \overset{\leftrightarrow}{f}(x_4) = y_3$, $\overline{f}(x_5) = \overset{\leftrightarrow}{f}(x_5) = y_3$. □

Linked to the indeterminacy functions π^-_f and $\pi^{\leftrightarrow}_f$, *pointwise* and *blockwise indeterminacy regions* Π^-_f and $\Pi^{\leftrightarrow}_f$ are defined by

$$\begin{aligned}
\Pi^-_f &= \{(x,y) \mid x \in I,\ y = \underline{f}(x) = \overline{f}(x) \text{ or } \\
&\qquad y \in]\underline{f}(x), \overline{f}(x)[\text{ if } \underline{f}(x) \neq \overline{f}(x)\}; \\
\Pi^{\leftrightarrow}_f &= \Big\{(x,y) \mid x \in I,\ y = \underset{\leftrightarrow}{f}(x) = \overset{\leftrightarrow}{f}(x) \text{ or } \\
&\qquad y \in \Big]\underset{\leftrightarrow}{f}(x), \overset{\leftrightarrow}{f}(x)\Big[\text{ if } \underset{\leftrightarrow}{f}(x) \neq \overset{\leftrightarrow}{f}(x)\Big\}
\end{aligned}$$

Example 4. In Figures 3 (a) and 3 (b), the areas filled with grid pattern and solid circles depict the indeterminacy regions Π^-_f and $\Pi^{\leftrightarrow}_f$, respectively. □

Having defined the indeterminacy regions, let us define two families of functions with the help of them:

$$\begin{aligned}
G^-_f &= \{g \mid g : I \to [0,1], (x, g(x)) \in \Pi^-_f\}, \\
G^{\leftrightarrow}_f &= \{g \mid g : I \to [0,1], (x, g(x)) \in \Pi^{\leftrightarrow}_f\}.
\end{aligned}$$

Example 5. In Figures 4 (a) and 4 (b) show a function g from G^-_f and a function g' from $G^{\leftrightarrow}_f$, respectively.□

Proposition 3. *Let f be a rough real function. Then,*

1. $\Pi^-_f \subseteq \Pi^{\leftrightarrow}_f$.
2. $\Pi^-_f = \Pi^{\leftrightarrow}_f$ *if and only if* $f^{IFS}_{pw} = f^{IFS}_{bw}$.

Proof. 1. Case $x \in I$, $\underline{f}(x) \neq \overline{f}(x)$. Applying Lemma 1 , $]\underline{f}(x), \overline{f}(x)[\subseteq \Big]\underset{\leftrightarrow}{f}(x), \overset{\leftrightarrow}{f}(x)\Big[$, and so

$$\begin{aligned}
&\{(x,y) \mid x \in I, y \in]\underline{f}(x), \overline{f}(x)[\} \\
&\subseteq \Big\{(x,y) \mid x \in I, y \in \Big]\underset{\leftrightarrow}{f}(x), \overset{\leftrightarrow}{f}(x)\Big[\Big\}
\end{aligned}$$

Case $x \in I$, $\underline{f}(x) = \overline{f}(x)$. If $x' \in S_I$, then

$$f(x') = \underline{f}(x') = \overline{f}(x') = \underset{\leftrightarrow}{f}(x') = \overset{\leftrightarrow}{f}(x').$$

Thus, $(x', f(x')) \in \{(x,y) \mid x \in I, y = \underline{f}(x) = \overline{f}(x)\}$, and $(x', f(x')) \in \Big\{(x,y) \mid x \in I, y = \underset{\leftrightarrow}{f}(x) = \overset{\leftrightarrow}{f}(x)\Big\}$ also holds.

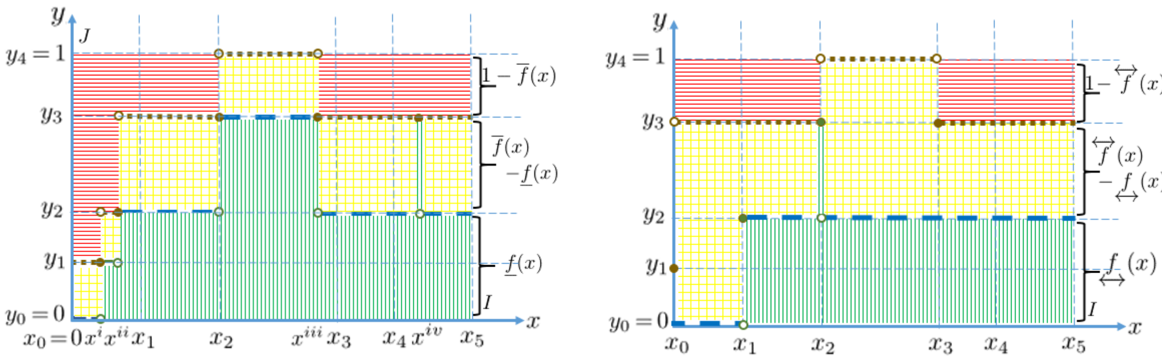


Figure 3: Geometric interpretations of $f_{pw}^{IFS} = (\underline{f}, 1-\overline{f})$ and $f_{bw}^{IFS} = (\underset{\leftrightarrow}{f}, 1-\overset{\leftrightarrow}{f})$.

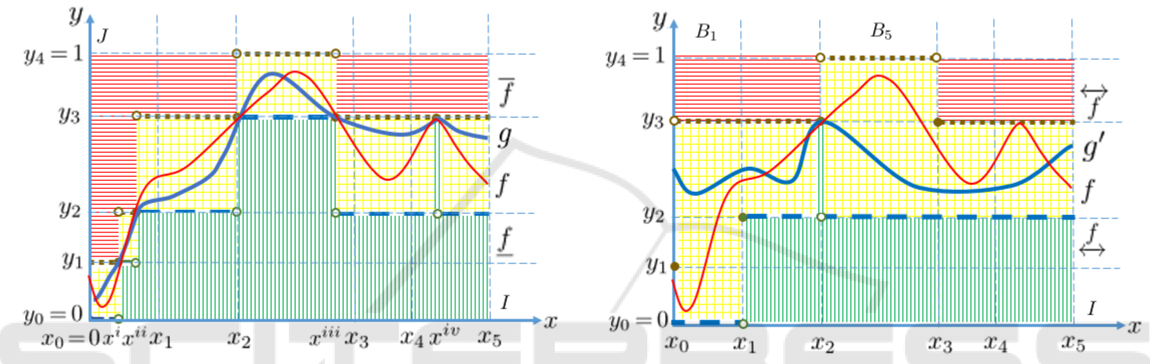


Figure 4: A function g from G_f^-, and a function g' from $G_f^{\leftrightarrow}$.

Let $x' \notin S_I$. It may occur that

$$f(x') = \underline{f}(x') = \overline{f}(x') = \underset{\leftrightarrow}{f}(x') = \overset{\leftrightarrow}{f}(x').$$

Then, $(x', f(x')) \in \{(x,y) \mid x \in I, y = \underline{f}(x) = \overline{f}(x)\}$ and $(x', f(x')) \in \left\{(x,y) \mid x \in I, y = \underset{\leftrightarrow}{f}(x) = \overset{\leftrightarrow}{f}(x)\right\}$ holds at the same time.

If $\underset{\leftrightarrow}{f}(x') \neq \overset{\leftrightarrow}{f}(x')$, then $(x', f(x')) \in \{(x,y) \mid x \in I, y = \underline{f}(x) = \overline{f}(x)\}$, and, due to Lemma 1 , $(x', f(x')) \in \left\{(x,y) \mid x \in I, y \in \left]\underset{\leftrightarrow}{f}(x), \overset{\leftrightarrow}{f}(x)\right[\right\}$ also holds.

2. $(\Rightarrow)$ It should be proved that "$\Pi_f^- = \Pi_f^{\leftrightarrow}$ implies $f_{pw}^{IFS} = f_{bw}^{IFS}$". Instead, its contrapositive form "$f_{pw}^{IFS} \neq f_{bw}^{IFS}$ implies $\Pi_f^- \neq \Pi_f^{\leftrightarrow}$" will be proved.

On one hand, $f_{pw}^{IFS} \neq f_{bw}^{IFS} \Leftrightarrow (\underline{f}, 1-\overline{f}) \neq (\underset{\leftrightarrow}{f}, 1-\overset{\leftrightarrow}{f}) \Leftrightarrow \underline{f} \neq \underset{\leftrightarrow}{f}$ or $\overline{f} \neq \overset{\leftrightarrow}{f}$. On the other hand, according to point *1.* of this Proposition, $\Pi_f^- \neq \Pi_f^{\leftrightarrow} \Leftrightarrow \Pi_f^{\leftrightarrow} \not\subseteq \Pi_f^-$.

Let us assume that $\underline{f} \neq \underset{\leftrightarrow}{f}$ (the case $\overline{f} \neq \overset{\leftrightarrow}{f}$ can be proved similarly). Then, there is an $x' \in I$ in such a way that $\underset{\leftrightarrow}{f}(x') < \underline{f}(x')$. Hence, there is an $h \in G_f^{\leftrightarrow}$ in such a way that $\underset{\leftrightarrow}{f}(x') < h(x') < \underline{f}(x')$.

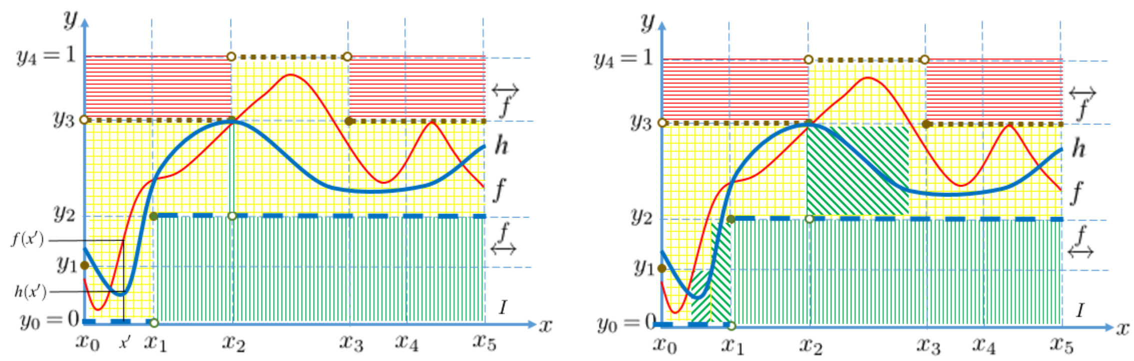
If $\underset{\leftrightarrow}{f}(x') < h(x') < \underline{f}(x') < \overline{f}(x') \leq \overset{\leftrightarrow}{f}(x')$, then it is straightforward that $(x', h(x')) \in \left]\underset{\leftrightarrow}{f}(x'), \overset{\leftrightarrow}{f}(x')\right[$, but $(x', h(x')) \notin]\underline{f}(x'), \overline{f}(x')[$. In other words, $(x', h(x')) \in \Pi_f^{\leftrightarrow}$, but $(x', h(x')) \notin \Pi_f^-$, i.e., $\Pi_f^{\leftrightarrow} \not\subseteq \Pi_f^-$ satisfies.

If $\underset{\leftrightarrow}{f}(x') < h(x') < \underline{f}(x') = \overline{f}(x') \leq \overset{\leftrightarrow}{f}(x')$, of course, $(x', h(x')) \in \left]\underset{\leftrightarrow}{f}(x'), \overset{\leftrightarrow}{f}(x')\right[$ also holds. In addition, $h(x') \neq \underline{f}(x') = \overline{f}(x')$. Therefore, in this case, $(x', h(x')) \in \Pi_f^{\leftrightarrow}$, but $(x', h(x')) \notin \Pi_f^-$, i.e., $\Pi_f^{\leftrightarrow} \not\subseteq \Pi_f^-$ also satisfies.

$(\Leftarrow)$ It is straightforward. □

Example 6. In Figures 3 (a), (b), it can be observed that the area of the pointwise indeterminacy region Π_f^- is included in the area of the blockwise indeterminacy region $\Pi_f^{\leftrightarrow}$ in accordance with Proposition 3 *1.*

As shown in Figure 3, $f_{pw}^{IFS} \neq f_{bw}^{IFS}$, in particular,



(a) (b)

Figure 5: $\Pi^{-}_{f} \neq \Pi^{\leftrightarrow}_{f}$.

both $\underset{\leftrightarrow}{f} \leq \underline{f}$ and $\overline{f} \leq \overset{\leftrightarrow}{f}$ satisfy. It can be seen that $\Pi^{-}_{f} \subseteq \Pi^{\leftrightarrow}_{f}$ but $\Pi^{-}_{f} \neq \Pi^{\leftrightarrow}_{f}$ in accordance with Proposition 3 2. □

Example 7. Figure 2 (a) shows that the graph of f intersects the horizontal line segments $y = y_1$ at x^i, and $y = y_2$ at x^{ii}. In Figure 5 (a), $x' \in \,]x^i, x^{ii}[$ in such a way that $y_0 = \underset{\leftrightarrow}{f}(x') < h(x') < \underline{f}(x') = y_1$. Moreover, $h(x') \in \left]\underset{\leftrightarrow}{f}(x'), \underline{f}(x')\right[\; = \;]y_0, y_1[$ and $f(x') \in \,]\underline{f}(x'), \overline{f}(x')[=]y_1, y_2[$.

In other words, the points $(x', f(x'))$ and $(x', h(x'))$ are on the vertical line segment $x = x'$. More precisely, $(x', h(x'))$ is between the points (x', y_0) and (x', y_1), and $(x', f(x'))$ is between the points (x', y_1) and (x', y_2).

Moving on for any $x \in \,]x^i, x^{ii}[$ and suitable functions $h \in G^{\leftrightarrow}_{f}$, the points $(x, h(x))$'s form the rectangular area $]x^i, x^{ii}[\, \times \,]y_0, y_1[$.[1] This area is filled with diagonal up pattern in Figure 5 (b). It is belongs to $\Pi^{\leftrightarrow}_{f}$ but does not belong to Π^{-}_{f}. The area filled with diagonal down pattern can be derived similarly. □

Corollary 1. *Let f be a rough real function. Then,*

1. $G^{-}_{f} \subseteq G^{\leftrightarrow}_{f}$.
2. $G^{-}_{f} = G^{\leftrightarrow}_{f}$ *if and only if* $f^{IFS}_{pw} = f^{IFS}_{bw}$.

Proof. These statements immediately follow from Proposition 3 . □

In Figure 4 (b), it can be observed that $g' \in G^{\leftrightarrow}_{f}$, but $\underset{\leftrightarrow}{f} \neq \underset{\leftrightarrow}{g'}$, $\overset{\leftrightarrow}{f} \neq \overset{\leftrightarrow}{g'}$. This is because $\underset{\leftrightarrow}{f} = y_0 < y_2 = \underset{\leftrightarrow}{g'}$ on $B_1 =]x_0, x_1[$, and $\overset{\leftrightarrow}{f} = y_4 > y_3 = \overset{\leftrightarrow}{g'}$ on $B_5 =]x_2, x_3[$. In other words, $f^{IFS}_{bw} \neq g^{IFS}_{bw}$. This observation motivates the following definition.

[1] Here, $\times$ denotes the Cartesian product operation.

Definition 4. Let f^{IFS}_{pw} and f^{IFS}_{bw} be pointwise and blockwise roughly derived IFSs.

- f^{IFS}_{pw} is *roughly strong* if $f^{IFS}_{pw} = g^{IFS}_{pw}$ for all $g \in G^{-}_{f}$, otherwise f^{IFS}_{pw} is *roughly weak*.
- f^{IFS}_{bw} is *roughly strong* if $f^{IFS}_{bw} = g^{IFS}_{bw}$ for all $g \in G^{\leftrightarrow}_{f}$, otherwise f^{IFS}_{bw} is *roughly weak*. □

In the case of f^{IFS}_{pw}, Figure 4 (a) suggests that f^{IFS}_{pw} is always roughly strong. That is what the following proposition is about.

Proposition 4. *For any pointwise roughly derived IFS f^{IFS}_{pw}, f^{IFS}_{pw} is roughly strong, that is, $f^{IFS}_{pw} = g^{IFS}_{pw}$ for all $g \in G^{-}_{f}$.*

Proof. Let $g \in G^{-}_{f}$. Then, $(x, g(x)) \in \Pi^{-}_{f}$.

Case $x \in I$, $g(x) \in \,]\underline{f}(x), \overline{f}(x)[$. In this case,

$$\begin{aligned} g(x) &\in \;]\underline{f}(x), \overline{f}(x)[=]\mathrm{l}_P(f(x)), \mathrm{u}_P(f(x))[\\ &= \;]\max\{y \in P_{[0,1]} \mid y \leq f(x)\}, \\ &\qquad \min\{y \in P_{[0,1]} \mid y \geq f(x)\}[\\ &= \;]y_j, y_{j+1}[\\ &= \;]\max\{y \in P_{[0,1]} \mid y \leq g(x)\}, \\ &\qquad \min\{y \in P_{[0,1]} \mid y \geq g(x)\}[\\ &= \;]\underline{g}(x), \overline{g}(x)[, \end{aligned}$$

where $y_j, y_{j+1} \in P_{[0,1]}$ for some $j \in [m[$.

That is,

$$f^{IFS}_{pw}(x) = (\underline{f}(x), 1 - \overline{f}(x)) = (\underline{g}(x), 1 - \overline{g}(x)) = g^{IFS}_{pw}(x)$$

satisfies for all such $x \in I$ that $\underline{f}(x) \neq \overline{f}(x)$.

Case $x \in I$, $g(x) = \underline{f}(x) = \overline{f}(x)$. Then, $g(x) = f(x) = \overline{f}(x) = y_j$, where $y_j \in P_{[0,1]}$ for some $j \in [m]$. And so $g(x) = y_j = \underline{g}(x) = \overline{g}(x)$.

That is, $f^{IFS}_{pw}(x) = g^{IFS}_{pw}(x)$ also holds for all such $x \in I$ that $\underline{f}(x) = \overline{f}(x)$. □

Proposition 5. *Let f_{bw}^{IFS} be a blockwise roughly derived IFS.*

f_{bw}^{IFS} is roughly strong if and only if $f_{pw}^{IFS} = f_{bw}^{IFS}$.

Proof. ($\Rightarrow$) On the contrary, let us assume that $f_{pw}^{IFS} \neq f_{bw}^{IFS}$. Then, such a function $g \in G_f^{\leftrightarrow}$ will be constructed for which $g_{bw}^{IFS} \neq f_{bw}^{IFS}$ holds. However, it contradicts the condition that f_{bw}^{IFS} is roughly strong.

$f_{pw}^{IFS} \neq f_{bw}^{IFS} \Leftrightarrow (\underline{f}, 1-\overline{f}) \neq (\underset{\leftrightarrow}{f}, 1-\overset{\leftrightarrow}{f}) \Leftrightarrow \underline{f} \neq \underset{\leftrightarrow}{f}$ or $\overline{f} \neq \overset{\leftrightarrow}{f}$. It us assumed that $\underline{f} \neq \underset{\leftrightarrow}{f}$, the case $\overline{f} \neq \overset{\leftrightarrow}{f}$ can be proved similarly. However, due to Proposition 2 , $\underline{f}(x_i) = \underset{\leftrightarrow}{f}(x_i)$ satisfies for every $x_i \in S_I$ ($i \in [n]$) categorization point. Then, there should be an open interval $B_i = \left]x_{\frac{i-1}{2}}, x_{\frac{i+1}{2}}\right[\in I/I_S$, where $i \equiv 1 \pmod 2$ ($i \in [2n[$) in such a way that $\underline{f} \neq \underset{\leftrightarrow}{f}$, i.e., $\underset{\leftrightarrow}{f} < \underline{f}$ on B_i.

The case $\underline{f} = \overline{f}$ on B_i is not possible, because it would imply that $\underline{f} = \overline{f} = \underset{\leftrightarrow}{f} = \overset{\leftrightarrow}{f}$ on B_i. However, it contradicts the condition that $\underset{\leftrightarrow}{f} < \underline{f}$ on B_i.

When $\underline{f} \neq \overline{f}$ on B_i, let $g \in G_f^{\leftrightarrow}$ with the constraint that $g(x) \in]\underline{f}, \overline{f}[$ on B_i. It is possible, because $]\underline{f}, \overline{f}[\subseteq \left]\underset{\leftrightarrow}{f}, \overset{\leftrightarrow}{f}\right[$ on B_i. Then, $\underset{\leftrightarrow}{f} < \underline{f} = \underset{\leftrightarrow}{g}$ on B_i, and so $f_{bw}^{IFS} \neq g_{bw}^{IFS}$ which is the requested contradiction.

($\Leftarrow$) According to Proposition 4 , f_{pw}^{IFS} is roughly strong, and so f_{bw}^{IFS} is roughly strong as well. □

4 ROUGH CONTINUITY AND ROUGHLY DERIVED INTUITIONISTIC FUZZY SETS

Rough continuity is a central notion in rough calculus like the continuity in the classical real analysis.

Let I and J two real intervals with categorizations S_I and P_J as they are given above.

Definition 5 ((Pawlak, 1996)). A rough real function $f \in J^I$ is (S_I, P_J)*–continuous* or *roughly continuous* at x if

$$f(\overline{[\![x]\!]}_{I_S}) \subseteq \overline{[\![f(x)]\!]}_{J_p}.$$

Otherwise, f is (S_I, P_J)*–discontinuous* or *roughly discontinuous* at $x \in I$.

f is (S_I, P_J)*–continuous* (*roughly continuous*) on $I' \subseteq I$ if f is (S_I, P_J)–continuous at every point of I'. Otherwise, f is *not roughly continuous on* I'. □

Proposition 6 ((Csajbók, 2019)). *A rough real function $f \in J^I$ is (S_I, P_J)–continuous at every $x \in S_I$ roughly isolated point.*

Definition 6 ((Csajbók, 2019)). The (S_I, P_J)-discontinuity types of $f \in J^I$ are defined as follows.

The rough discontinuity of f is called

(1) the *rough jump discontinuity of the first kind* if it is derived from touching a straight line $y = y_j$ for some $j \in [m]$;

(2) the *rough jump discontinuity of the second kind* if it is derived from intersecting a straight line $y = y_j$ for some $j \in]m[$;

(3) any other type of discontinuity is called the *rough jump discontinuity of the third kind.* □

Example 8. Figure 6 (a) depicts rough jump discontinuities of the first and second type.

- f has the rough jump discontinuity of the *first kind* at x^{vi} because it is derived from touching the straight line $y = y_3$ at x^{vi}:

$$\begin{aligned} f(\overline{[\![x^{vi}]\!]}_{I_S}) &= f([x_4, x_5]) \subseteq]y_2, y_3] \\ &\not\subseteq \{y_3\} = \overline{[\![f(x^{vi})]\!]}_{J_p}. \end{aligned}$$

- f has the rough jump discontinuities of the *second kind* at x^i, x^{ii}, and x^{iv} because they are derived from intersecting the line segments $y = y_1$, $y = y_2$, and $y = y_3$, respectively. For instance,

$$\begin{aligned} f(\overline{[\![x^{i}]\!]}_{I_S}) &= f([x_0, x_1]) \subseteq]y_0, y_3] \\ &\not\subseteq \{y_1\} = \overline{[\![f(x^{i})]\!]}_{J_p}. \end{aligned}$$

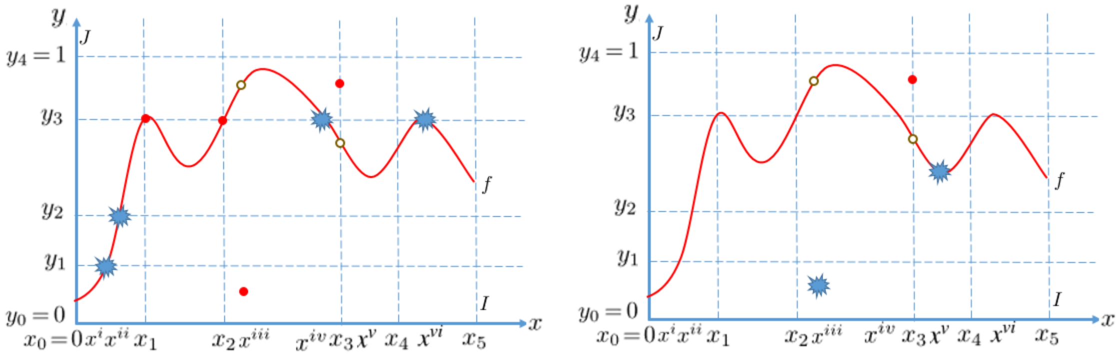
The discontinuities at x^{ii} and x^{iv} can be showed in similar way.

It should be noted that f touches at x_1 and intersects at x_2 the line segment $y = y_3$ but f is still continuous at both points. It can be seen that the contact point is (x_1, y_3) and the intersection point is (x_2, y_3), that is, their both coordinates are categorizations points.

In Figure 6 (b), f has rough jump discontinuities of the *third kind* at x^{iii} and x^v:

$$\begin{aligned} f(\overline{[\![x^{iii}]\!]}_{I_S}) &= f([x_2, x_3]) \subseteq]y_2, y_4[\\ &\not\subseteq [y_0, y_1] = \overline{[\![f(x^{iii})]\!]}_{J_p}; \\ f(\overline{[\![x^{v}]\!]}_{I_S}) &= f([x_3, x_4]) \subseteq]y_2, y_4[\\ &\not\subseteq [y_2, y_3] = \overline{[\![f(x^{iii})]\!]}_{J_p}. \end{aligned}$$

Although, f is roughly continuous at x_3 pursuant to Proposition 6 , it may cause rough jump discontinuities of the third kind in blocks $]x_2, x_3[$ and/or $]x_3, x_4[$, specially, in $]x_3, x_4[$ at x^v. □



(a) (b)

Figure 6: Rough discontinuities.

Proposition 7 ((Csajbók, 2019)). *A rough real function $f \in J^I$ is (S_I, P_J)–continuous on I if and only if f does not have rough jump discontinuity of any kind.*

Proposition 8. *A rough real function $f \in J^I$ is (S_I, P_J)–continuous on I if and only if the blockwise roughly derived IFS f_{bw}^{IFS} is roughly strong.*

Proof. ($\Rightarrow$) Since f is roughly continuous on I, then $f(\llbracket x \rrbracket_{I_S}) \subseteq \llbracket f(x) \rrbracket_{J_p}$ for all $x \in I$.

It is straightforward that on $C_j (\in J/J_P,\ j \in [2m])$, $\underset{\leftrightarrow}{f} = \overset{\leftrightarrow}{f} = y_{j/2}$ if $j \equiv 0 \pmod 2$, and $\underset{\leftrightarrow}{f} = y_{\frac{j-1}{2}}$, $\overset{\leftrightarrow}{f} = y_{\frac{j+1}{2}}$ if $j \equiv 1 \pmod 2$. Thus,

- $f(x) = y_{j/2}$ on $\llbracket x \rrbracket_{I_S}$ if $f(\llbracket x \rrbracket_{I_S}) = \llbracket f(x) \rrbracket_{J_p} = C_j$ for some $j \in [2m], j \equiv 0 \pmod 2$;
- $y_{\frac{j-1}{2}} \leq f(x) \leq y_{\frac{j+1}{2}}$ on $\llbracket x \rrbracket_{I_S}$ if $f(\llbracket x \rrbracket_{I_S}) \subseteq \llbracket f(x) \rrbracket_{J_p} = C_j$ for some $j \in [2m], j \equiv 1 \pmod 2$.

It means, considering the definition of $\Pi_f^{\leftrightarrow}$ and $G_f^{\leftrightarrow}$, that for all $g \in G_f^{\leftrightarrow}$, $f_{bw}^{IFS} = g_{bw}^{IFS}$, i.e., f_{bw}^{IFS} is roughly strong.

($\Leftarrow$) On the the contrary, let us assume that f is roughly discontinuous for some $x \in I$. Since f is roughly continuous in every roughly isolated point, see, Proposition 6, x belongs to an open interval $B_i = \llbracket x \rrbracket_{I_S} = \left]x_{\frac{i-1}{2}}, x_{\frac{i+1}{2}}\right[\in I/I_S$ for some $i \in [2n[$, $i \equiv 1 \pmod 2$.

First, it can be stated that $f(\llbracket x \rrbracket_{I_S}) \cap \llbracket f(x) \rrbracket_{J_p} \neq \emptyset$, because $x \in \llbracket x \rrbracket_{I_S}$, and so $f(x) \in f(\llbracket x \rrbracket_{I_S}), \llbracket f(x) \rrbracket_{J_p}$.

Moreover, there must be an $x \neq x' \in \llbracket x \rrbracket_{I_S}$ in such a way that $f(x) \neq f(x')$ and $f(x') \in f(\llbracket x \rrbracket_{I_S})$ but $f(x') \notin \llbracket f(x) \rrbracket_{J_p}$, otherwise $f(\llbracket x \rrbracket_{I_S}) \subseteq \llbracket f(x) \rrbracket_{J_p}$ would be, which contradicts the assumption that f discontinuous at x.

It may occur that $|\llbracket f(x) \rrbracket_{J_p}| = 1$. It happens when $f(x) = \underline{f}(x) = \overline{f}(x)$, that is when f touches or intersects a horizontal line segment $y = y_j$ at x.

Case $|\llbracket f(x) \rrbracket_{J_p}| = 1$. Then, $\llbracket f(x) \rrbracket_{J_p} = \{y_j\}$ for some $y_j \in P_J$, $j \in [m]$. Moreover, let us recall that $f(x') \neq f(x) = y_j$.

First, let us assume that f touches the line segment $y = y_j$ at x, i.e., $f(x) = \underline{f}(x) = \overline{f}(x) = y_j$.

Let $g \in G_f^{\leftrightarrow}$ with the constraint that $g(x) = y_j$ on $\llbracket x \rrbracket_{I_S}$. Such a function g exists, because $f(x) = y_j \neq f(x')$, and so

- $\underset{\leftrightarrow}{f} < y_j \leq \overset{\leftrightarrow}{f}$ if $f(x') < y_j$, or
- $\underset{\leftrightarrow}{f} \leq y_j < \overset{\leftrightarrow}{f}$ if $f(x') > y_j$

hold on $\llbracket x \rrbracket_{I_S}$.

Then, $\underset{\leftrightarrow}{f} < \underset{\leftrightarrow}{g} = y_j$ on $\llbracket x \rrbracket_{I_S}$ if $f(x') < y_j$, while $\overset{\leftrightarrow}{g} = y_j < \overset{\leftrightarrow}{f}$ on $\llbracket x \rrbracket_{I_S}$ if $f(x') > y_j$, i.e., $f_{bw}^{IFS} \neq g_{bw}^{IFS}$ which contradicts the condition that f_{bw}^{IFS} is roughly strong.

Secondly, let us assume that f intersects the line segment $y = y_j$ at x. In this case, $j \in]m[$, and $f(x) = \underline{f}(x) = \overline{f}(x) = y_j$. Hereinafter, the proof is similar to the previous case.

Case $|\llbracket f(x) \rrbracket_{J_p}| > 1$. Then, $\llbracket f(x) \rrbracket_{J_p} = \left]y_{\frac{i-1}{2}}, y_{\frac{i+1}{2}}\right[$ for some $j \in [2m[$, $j \equiv 1 \pmod 2$. Moreover, let us recall that $f(x') \notin \left[y_{\frac{i-1}{2}}, y_{\frac{i+1}{2}}\right]$.

First, if $f(x') < y_{\frac{i-1}{2}}$, let $g \in G_f^{\leftrightarrow}$ with the constraint that $g(x) = y_{\frac{i-1}{2}}$ on $\llbracket x \rrbracket_{I_S}$. Such a function g exists, because $\underset{\leftrightarrow}{f} < y_{\frac{i-1}{2}} \leq \overset{\leftrightarrow}{f}$ holds on $\llbracket x \rrbracket_{I_S}$.

Then, $\underset{\leftrightarrow}{f} < \underset{\leftrightarrow}{g} = y_{\frac{i-1}{2}}$ on $\llbracket x \rrbracket_{I_S}$, i.e., $f_{bw}^{IFS} \neq g_{bw}^{IFS}$, which contradicts the condition that f_{bw}^{IFS} is roughly strong.

Secondly, if $f(x') > y_{\frac{i+1}{2}}$, let $g \in G_f^{\leftrightarrow}$ with the constraint that $g(x) = y_{\frac{i+1}{2}}$ on $\overline{[\![x]\!]}_{I_S}$. Such a function g exists, because $\underset{\leftrightarrow}{f} \leq y_{\frac{i+1}{2}} < \overset{\leftrightarrow}{f}$ holds on $\overline{[\![x]\!]}_{I_S}$.

Then, $\underset{\leftrightarrow}{g} = y_{\frac{i+1}{2}} < \overset{\leftrightarrow}{f}$ on $\overline{[\![x]\!]}_{I_S}$, i.e., $f_{bw}^{IFS} \neq g_{bw}^{IFS}$, which contradicts the condition that f_{bw}^{IFS} is roughly strong. □

Corollary 2. *A rough real function $f \in J^I$ is (S_I, P_J)–continuous on I if and only if $\Pi_f^- = \Pi_f^{\leftrightarrow}$.*

Proof. f is roughly continuous
$\Leftrightarrow f_{bw}^{IFS}$ is roughly strong by Proposition 8
$\Leftrightarrow f_{pw}^{IFS} = f_{bw}^{IFS}$ by Proposition 5
$\Leftrightarrow \Pi_f^- = \Pi_f^{\leftrightarrow}$ by Propositions 3 2. □

Corollary 3. *A rough real function $f \in J^I$ is (S_I, P_J)–continuous on I if and only if $G_f^- = G_f^{\leftrightarrow}$.*

Proof. f is roughly continuous
$\Leftrightarrow f_{bw}^{IFS}$ is roughly strong by Proposition 8
$\Leftrightarrow f_{pw}^{IFS} = f_{bw}^{IFS}$ by Proposition 5
$\Leftrightarrow G_f^- = G_f^{\leftrightarrow}$ by Corollary 1 2. □

5 CONCLUSION AND FUTURE WORK

Rough continuity is a central notion in rough calculus. This paper has characterized the rough continuity in three different ways in terms of intuitionistic fuzzy set theory.

This characterization establishes a connection between the two theories of uncertainty management, the rough set theory and intuitionistic fuzzy set theory. It may allow the application of the means of intuitionistic fuzzy calculus in rough calculus.

In the future, the investigations can be continued in several directions. This article has addressed only one important concept of rough calculus, namely, the rough continuity. First of all, rough continuity has some additional features, such as rough discontinuity, rough Darboux property or Intermediate Value Property (IVP). The question is how they could also be captured with the help of IFS tools. Moreover, the relationships between additional notions of rough calculus and IFS can also be studied.

Classical Pawlak's rough set theory has many different generalizations. The question is whether they can be captured with IFS tools in one way or another.

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