Graphs with Partition Dimension 3 and Locating-chromatic Number 4

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Abstract: The characterization study of all graphs with partition dimension either 2, n−2, n−1 or n has been completely done. In the case of locating-chromatic numbers, the efforts in characterizing all graphs with locating-chromatic number either 2, 3, n−1 or n have reached to complete results. In this paper we present methods to obtain a family of graphs having partition dimension 3 or locating-chromatic number 4 by using the previous known results.

1 INTRODUCTION

The concepts of partition dimension and locating-chromatic number of connected graphs were introduced by Chartrand et al. in (Chartrand et al., 1998) and in (Chartrand et al., 2002), respectively. The locating-chromatic number for graphs is a special case of the partition dimension notion. In order to generalize these two concepts, Haryeni et al. in (Haryeni et al., 2014) enlarged the notion of the partition dimension so that it can be applied also to disconnected graphs, and Welyant et al. in (Welyant et al., 2016) enlarged the notion of locating-chromatic number for disconnected graphs.

Let $F=(V, E)$ be a (not necessarily connected) graph and $\Pi = \{S_1, S_2, ..., S_k\}$ be a partition of $V(F)$, where $S_i$ is a partition class of $\Pi$ for each $1 \leq i \leq k$. If the distance $d_F(v, S_i) < \infty$ for all $v \in V(F)$ and $S_i \in \Pi$, then the representation $r(v|\Pi)$ of $v$ with respect to $\Pi$ is $(d_F(v, S_1), d_F(v, S_2), ..., d_F(v, S_k))$. The partition $\Pi$ is a resolving partition of $F$ if every two distinct vertices $u, v \in V(F)$ have distinct representations with respect to $\Pi$, namely $r(u|\Pi) \neq r(v|\Pi)$. The partition dimension of $F$, denoted by $pd(F)$, for a connected $F$ or by $pd(F)$ for a disconnected $F$, is the cardinality of a smallest resolving partition of $F$. For a disconnected graph $F$, if there is no a resolving partition of $F$, then $pd(F) = \infty$. In addition, if the partition $\Pi$ is induced by a proper $k$-coloring $c$, then we define the color code $c_\Pi(v)$ of a vertex $v \in V(F)$ with $(d_F(v, S_1), d_F(v, S_2), ..., d_F(v, S_k))$. If all vertices of $F$ have different color codes, then $c$ is a locating coloring of $F$.

The locating-chromatic number of $F$, denoted by $\chi_L(F)$ for a connected $F$ or by $\chi_L(F)$ for a disconnected graph $F$, is the least integer $k$ such that $F$ admits a locating $k$-coloring. Otherwise, we say that $\chi_L(F) = \infty$.

Chartrand et al. in (Chartrand et al., 2000) characterized all connected graphs on $n(\geq 3)$ vertices having the partition dimension $2$, $n$, or $n−1$. Tomescu in (Tomescu, 2008) showed that there are only 23 connected graphs on $n(\geq 9)$ vertices with the partition dimension $n−2$. Further results of the partition dimension of graphs for some graph operations, namely corona product, Cartesian product and strong product, can be observed in (Rodríguez-Velazquez et al., 2016; Yero et al., 2014; Yero et al., 2010).

On the other hand, all connected graphs on $n$ vertices with locating-chromatic number $n$ or $n−1$ was characterized in (Chartrand et al., 2003). In the same paper, they also gave conditions for graph $F$ on $n(\geq 5)$ vertices with $\chi_L(F) \leq n−2$. The characterization of all graphs with locating-chromatic number 3 can be seen in (Baskoro and Asmiati, 2013) and (Asmiati and Baskoro, 2012).

Keywords: Graph, Tree, Partition Dimension, Locating-Chromatic Number, Cycle, Path.

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In this paper, motivated by the results of the characterization of all graphs with locating-chromatic number 3, we present some methods to extend those graphs so that their partition dimension is equal to 3. Furthermore, we show that these new graphs have locating-chromatic number 4. We also construct some classes of graphs by connecting some vertices in a disjoint union of paths, so that the partition dimension of these graphs remains equal to 3.

In order to present the results, we need additional notions and some known results as follows. Let $F$ be a graph and $\Pi = \{S_1, S_2, \ldots, S_k\}$ be a resolving partition of $F$. For an integer $k \geq 1$, a vertex $u \in V(F)$ is defined as $k$-distance vertex with respect to $\Pi$ if $d_F(v, S_i) = 0$ or $k$ for any $S_i \in \Pi$. Note that in the locating-chromatic number of $F$, the only possible value of $k$ is 1 and the vertex satisfies this condition is called a dominant vertex.

**Definition 1.1.** (Haryeni et al., 2019) Let $F$ be a graph and $\Pi = \{S_1, S_2, \ldots, S_k\}$ be a minimum resolving partition of $F$. Two distinct vertices $p, q \in V(F)$ in $S_i$ for some $i \in [1, k]$ are called independent vertices with respect to $\Pi$ if there exist two distinct integers, namely $l$ and $m$ which different from $i$, such that $d_F(p, S_i) = d_F(q, S_i) \neq d_F(p, S_l) = d_F(q, S_m)$. Furthermore, if there exists a minimum resolving partition of $F$ such that any two vertices in the same class partition are independent, then $F$ is called an independent graph. Otherwise, $F$ is a dependent graph.

**Definition 1.2.** (Haryeni et al., 2019) Let $F$ be a graph and $B \subseteq V(F)$ where $B = (b_1, b_2, \ldots, b_k)$. We denote $F(B; (n_1, n_2, \ldots, n_k))$ as a hair graph of $F$ with respect to $B$ which is obtained from $F$ by attaching a path $P_{n_i}$ with $n_i \geq 2$ vertices to a root vertex $b_i$ for all $i \in [1, k]$. Furthermore, the set of all hair graphs obtained from the graph $F$ is denoted by $Hair(F)$.

**Theorem 1.3.** (Haryeni et al., 2019) Let $F$ be a graph with a finite partition dimension. For any $H \in Hair(F)$, then

$$pdd(H) \leq \begin{cases} pdd(F), & \text{if } F \text{ is independent,} \\ pdd(F) + 1, & \text{if } F \text{ is dependent.} \end{cases}$$

**Proposition 1.4.** (Haryeni et al., 2019) For any $n \geq 3$, a path $P_n$ is a dependent graph with any resolving 2-partition.

**Theorem 1.5.** (Haryeni et al., 2017) Let $F$ be a disjoint union of $m$ paths with different lengths. If $m = 1$, then $pd(F) = 2$. Otherwise, $pdd(F) = 3$.

**Corollary 1.6.** (Haryeni et al., 2017) If $H \in Hair(P_m)$ and $H \cong P_n$ for any $n \geq m$, then $pd(H) = 3$.

2 TREES WITH PARTITION DIMENSION 3 AND LOCATING-CHROMATIC NUMBER 4

Let $T$ be a tree with 3 dominant vertices $x, y$ and $z$, and $aP_b = (a, x, u_1, u_2, u_3, y, v_1, v_2, \ldots, v_{s-1}, v_s = z, b)$ be a path with $r, s$ odd. If $r, s > 1$, then define $u^* = u_{[s+1]}, u^{**} = u_{[s+1]}, v^* = v_{[s+1]}$, and $v^{**} = v_{[s+1]}$. Note that all internal vertices $u_i$ and $v_j$ of $T$ excluding $u^*, u^{**}, v^*$ and $v^{**}$ have degree 2. In the following result, Baskoro and Asmiati (Baskoro and Asmiati, 2013) characterized all trees with locating-chromatic number 3.

**Lemma 2.1.** (Baskoro and Asmiati, 2013) In any tree $T$ with $\chi^L(T) = 3$, the color code of any vertex is $(a_1, a_2, a_3)$ where $\{a_1, a_2, a_3\} = \{0, 1, k\}$ for some $k \geq 1$.

**Theorem 2.2.** (Baskoro and Asmiati, 2013) Let $T$ be a tree on $n(\geq 3)$ vertices. The value $\chi^L(T) = 3$ iff $T$ is isomorphic to either $P_3, P_4, S_2, S_2$, or any subtree in Figure 1 containing a path $aP_b$. Theorem 2.3. Let $F$ be a graph other than a path with $\chi^L(F) = 3$. Then, $F$ is always independent.

**Proof.** Let $c: V(F) \rightarrow \{1, 2, 3\}$ be any 3-coloring on graph $F$. Let $\Pi = \{S_1, S_2, S_3\}$ be the partition of $F$ induced by $c$. We will show that any two vertices $p, q \in S_a$ for some $a \in [1, 3]$ are independent vertices with respect to $\Pi$. If $p$ and $q$ are in different partition classes, then certainly $p$ and $q$ are independent. Now assume that $p$ and $q$ are in the same class, say $p, q \in S_1$. This implies that $c_1(x) = (0, c_2, c_3)$ and $c_1(y) = (0, d_2, d_3)$. By Lemma 2.1, then we have $c_1(x) = (0, c_2, 1)$ and $c_1(y) = (0, d_2, 1)$, or $c_1(x) = (0, 1, c_3)$ and $c_1(y) = (0, 1, d_3)$, or $c_1(x) = (0, 1, c_3)$ and $c_1(y) = (0, d_2, 1)$, or $c_1(x) = (0, c_2, 1)$ and
Figure 1: Any subtree $T$ containing a path $aP_5$ with $\chi_L(T) = 3$.

Now, we present the following results.

Let $c(y) = (0,1,d_3)$. Note that $c$ is a locating coloring of $F$ so that $c(x) \neq c(y)$. Therefore, for the previous four cases we can conclude that $c_2 - d_2 \neq c_3 - d_3$. This implies that any two vertices $x, y \in S_a$ of $F$ are independent vertices so that $F$ is an independent graph with respect to the partition $\Pi$.

By Theorems 2.2 and 2.3, we show that any hair graph of tree $T$ in Theorem 2.2 has partition dimension 3 and locating-chromatic number 4.

Corollary 2.4. Let $T$ be either a path $P_3$ or $P_4$, a double star $S_{1,2}$ or $S_{2,2}$, or a subtree in Figure 1 containing a path $aP_5$. For all $H \in Hair(T)$ where $H \cong P_m$, then $pd(H) = 3$ and with $\chi_L(H) = 4$.

Proof. If $T$ is isomorphic to a path, then $pd(H) = 3$ for any $H \in Hair(T)$ with $H \cong P_m$ by Corollary 1.5.

Now we suppose that $T$ is not isomorphic to a path. Since $H \cong P_m$, $pd(H) \geq 3$. By Theorems 2.2 and 2.3, then $T$ is an independent graph with locating 3-coloring. By Theorem 1.3, then $pd(H) \leq pd(T) \leq \chi_L(T) = 3$. Furthermore, since all trees $T$ with $\chi_L(T) = 3$ are only a path $P_3$ or $P_4$, a double star $S_{1,2}$ or $S_{2,2}$, or a subtree in Figure 1 containing a path $aP_5$, $\chi_L(H) \geq 4$. The coloring of $H$ with 4 colors is given in Figure 2. The color of the new vertices of $H$ are 4 and $i$ alternately, where $i$ is the color of the root vertex.

3 GRAPHS CONTAINING CYCLE WITH PARTITION DIMENSION 3 AND LOCATING-CHROMATIC NUMBER 4

In the following theorem, all graphs containing cycle with locating-chromatic number 3 have been characterized, see (Asmiati and Baskoro, 2012).

Figure 2: The locating 4-coloring of graph $H \in Hair(T)$ where $T$ depicted in Figure 1.

Theorem 3.1. (Asmiati and Baskoro, 2012) Let $F$ be any graph having a smallest odd cycle $C$. Then $\chi_L(F) = 3$ iff $F$ is a subgraph of one of the graphs in Figure 3 which every vertex $a \notin C$ of degree 3 must
be lie in a path connecting two different vertices in \( C \).

By a similar reason to Corollary 2.4, we show that for every \( H \in Hair(F) \), where \( F \) is a graph in Theorem 3.1, then \( pd(H) = 3 \) and \( \chi_L(H) = 4 \).

**Corollary 3.2.** Let \( F \) be any graph having a smallest odd cycle \( C \), where \( F \) is a subgraph of one of the graphs in Figure 3 which every vertex of degree 3 must be lie in a path connecting two different vertices in \( C \). For all \( H \in Hair(F) \), then \( pd(H) = 3 \) and \( \chi_L(H) = 4 \).

Figure 3: The four types of maximal graphs containing an odd cycle with chromatic location number 3.

For now on, for any integer \( m \geq 2 \), define the graph \( G = U_{i=1}^{m} P_{n_i} \) where \( n_1 \geq 3 \) and \( n_{i+1} = n_i + 1 \) for all \( i \in [1, m-1] \). Note that \( pd(G) = 3 \) by Lemma 1.5. In the next result, we construct some graphs obtained from disjoint union of paths \( G = U_{i=1}^{m} P_{n_i} \) so that their partition dimensions remains equal to 3. Let the set of vertices and edges of \( G \) be

\[
V(G) = \{v_{i,j}: 1 \leq i \leq m, 1 \leq j \leq n_i \} \quad \text{and} \quad E(G) = \{v_{i,j}v_{i,j+1}: 1 \leq i \leq m, 1 \leq j \leq n_i - 1 \},
\]

respectively. Let \( S_1 \), \( S_2 \) and \( S_3 \) be three subsets of \( V(G) \) where

\[
S_1 = \{v_{i,1}: 1 \leq i \leq m \}, \quad (1)
\]

\[
S_2 = \{v_{i,j}: 1 \leq i \leq m, 2 \leq j \leq i + 1 \}, \quad (2)
\]

\[
S_3 = \{v_{i,j}: 1 \leq i \leq m, i + 2 \leq j \leq n_i \}. \quad (3)
\]

By the above definitions, for three distinct vertices \( x, y, z \in V(F) \) where \( x = v_{i,1} \in S_1 \), \( y = v_{i,j} \in S_2 \) and \( z = v_{i,k} \in S_3 \) for some \( i \in [1, m], j \in [2, i + 1] \) and \( k \in [i + 2, n_i] \), we have

\[
d_G(x, S_t) = \begin{cases} 
0, & \text{if } t = 1, \\
1, & \text{if } t = 2, \\
i + 1, & \text{if } t = 3,
\end{cases}
\]

\[
d_G(y, S_t) = \begin{cases} 
0, & \text{if } t = 1, \\
j - 1, & \text{if } t = 2, \\
i + 2 - j, & \text{if } t = 3,
\end{cases}
\]

\[
d_G(z, S_t) = \begin{cases} 
k - 1, & \text{if } t = 1, \\
k - i - 1, & \text{if } t = 2, \\
0, & \text{if } t = 3.
\end{cases}
\]

Now, define new graphs \( G' = G \cup E_1 \cup E_2 \) and \( G \subseteq G'' \subseteq G' \), where \( E_1 \) and \( E_2 \) are two sets of additional edges connecting some vertices of \( F \) as follows.

\[
E_1 = \{v_{i,j}v_{i+1,j}: 1 \leq i \leq m - 1, 1 \leq j \leq n_i\}
\]

\[
E_2 = \{v_{i,j}v_{i+1,j+1}: 1 \leq i \leq m - 1, 1 \leq j \leq n_i - 1\}
\]

By the above definitions, then \( V(G'') = V(G') = V(G) \). Let \( S_1, S_2 \) and \( S_3 \) be three subsets of \( V(G'') \) similar to the equations in (1), (2) and (3), respectively. Therefore, for three distinct vertices \( x, y, z \in V(G'') \) where \( x = v_{i,1} \in S_1 \), \( y = v_{i,j} \in S_2 \) and \( z = v_{i,k} \in S_3 \) where \( i \in [1, m], j \in [2, i + 1] \) and \( k \in [i + 2, n_i] \), we have

\[
d_G'(x, S_2) = \min\{d_G''(v_{i,1}, v_{i+1,j}): 1 \leq l \leq m\}
\]

\[
= d_G''(v_{i,1}, v_{i+2}) = i + 1,
\]

\[
d_G'(y, S_1) = \min\{d_G''(v_{i,j}, v_{i+1,1}): 1 \leq l \leq m\}
\]

\[
= d_G''(v_{i,j}, v_{i+1}) = j - 1,
\]

\[
d_G'(z, S_3) = \min\{d_G''(v_{i,k}, v_{i,1}): 1 \leq l \leq m\}
\]

\[
= d_G''(v_{i,k}, v_{i+1}) = k - 1.
\]

Therefore, we obtain that
Let \( G' = G \cup E_1 \cup E_2 \) and \( G \subseteq G'' \subseteq G' \). Then, \( pd(G'') = 3 \).

**Proof.** Since \( G'' \) is not a path, \( pd(G'') \geq 3 \). To show the upper bound of partition dimension of \( G \), define a partition \( \Pi = \{ S_1, S_2, S_3 \} \) of \( G'' \) where \( S_1 = \{ v_{i,1}: 1 \leq i \leq m \}, S_2 = \{ v_{i,j}: 1 \leq i \leq m, 2 \leq j \leq i+1 \} \), and \( S_3 \) contains the rest vertices of \( G \). By the definition of partition \( \Pi \), for a vertex \( v_{i,j} \in V(G) \) in \( S_a \) where \( i \in [1, m] \), \( j \in [1, n_i] \) and \( a \in [1, 3] \), we have the representation of \( v_{i,j} \) with respect to the partition \( \Pi \) as follows.

\[
\begin{aligned}
 r(v_{i,j}|\Pi) &= \\
&= \begin{cases} 
(0,1,i+1), & \text{if } j = 1, \\
(j-1,0,i+2-j), & \text{if } j \in [2, i+1], \\
(j-1,j-i-1), & \text{if } j \in [i+2,n_i]. 
\end{cases}
\end{aligned}
\]

Let us show that \( \Pi \) is a resolving partition of \( G'' \). We consider any two vertices \( x, y \in V(G'') \). If \( x \) and \( y \) are in the different partition class, then clearly that they have distinct representation. Now we suppose that \( x, y \in S_a \) for some \( a \in [1, 3] \). If \( x = v_{p,q} \) and \( y = v_{r,s} \) where \( 1 \leq p \leq m \) and \( 2 \leq q < r \leq p + 1 \) or \( p + 2 \leq q < r \leq n_p \), then \( d_{G''}(x, S_a) = q - 1 \). \( x \) and \( y \) have distinct representation. Now we suppose that \( x = v_{p,q} \) and \( y = v_{r,s} \) in \( S_a \) where \( 1 \leq p < q \leq m, 2 \leq q \leq p + 1 \) and \( 2 \leq s \leq r + 1 \), then \( d_{G''}(x, S_a) = q - 1 = s - 1 = d_{G''}(y, S_a) \). Therefore, \( r(x|\Pi) \neq r(y|\Pi) \).

Now, assume that \( x = v_{p,q} \) and \( y = v_{r,s} \) in \( S_a \) for some \( a \in [1, 3] \) and \( p, r \in [1, m] \) where \( p \neq q \). For two vertices \( x = v_{p,q} \) and \( y = v_{r,s} \) in \( S_1 \) where \( 1 \leq p \leq m \), then \( d_{G''}(x, S_1) = p + 1 \). For two vertices \( x = v_{p,q} \) and \( y = v_{r,s} \) in \( S_2 \) where \( 1 \leq p < r \leq m, 2 \leq q \leq p + 1 \) and \( 2 \leq s \leq r + 1 \), then \( d_{G''}(x, S_2) = p + 1 = 1 = d_{G''}(y, S_2) \). For two vertices \( x = v_{p,q} \) and \( y = v_{r,s} \) in \( S_3 \) where \( 1 \leq p < r \leq m, p + 2 \leq q \leq n_p \) and \( r + 2 \leq s \leq n_r \), if \( d_{G''}(x, S_3) = k - 1, \) then \( d_{G''}(y, S_3) = k - 1 = d_{G''}(y, S_3) \). Otherwise, \( d_{G''}(x, S_3) \neq d_{G''}(y, S_3) \). For two vertices \( x = v_{p,q} \) and \( y = v_{r,s} \) in \( S_3 \) where \( 1 \leq p < r \leq m, p + 2 \leq q \leq n_p \) and \( r + 2 \leq s \leq n_r \), if \( d_{G''}(x, S_3) = k \), then \( d_{G''}(y, S_3) = k - 1 = d_{G''}(y, S_3) \). Otherwise, \( d_{G''}(x, S_3) \neq d_{G''}(y, S_3) \).

The four graphs in Figure 4 give an illustration of the graphs provided for Theorem 3.3. These graphs are (a) \( G = P_8 \cup P_5 \cup P_9 \cup P_7, (b) G' = G \cup E_1 \cup E_2, (c) G'' = G \cup E_1 \cup E_2, \) and (d) \( G'' \subseteq G' \). Note that from Theorem 3.3, \( pd(G) = pd(G') = pd(G'') = pd(G''') = 3 \).

In the next result, we also construct graphs from disjoint union of paths \( G = \bigcup_{i=1}^{m} P_{n_i} \), so that their partition dimensions are equal to 3 as well.

**Theorem 3.4.** Let \( G'' = G \cup E_1 \cup E_2 \) and \( G \subseteq G'' \subseteq G' \). Then, \( pd(G') = 3 \).

**Proof.** Let \( H = G'' - F' \). This is easy to see that \( pd(H) \geq 3 \). To show that \( pd(H) \leq 3 \), define a partition \( \Pi'' = \{ S'_1, S'_2, S'_3 \} \) of \( H \) where \( S'_1 = \{ v_{i,1}: 1 \leq i \leq m \}, S'_2 = \{ v_{i,j}: 1 \leq i \leq m, 2 \leq j \leq i+1 \} \), and \( S'_3 = \{ v_{i,j}: 1 \leq i \leq m, i+2 \leq j \leq n_i \} \). (c) \( G' = G \cup E_1 \cup E_2 \), and (d) \( G'' \subseteq G' \). Note that from Theorem 3.3, \( pd(G) = pd(G') = pd(G'') = pd(G''') = 3 \).

By the definition of a partition \( \Pi'' \) of \( V(H) \), for three vertices \( x, y, z \in V(H) \) where \( x = v_{i,1} \in S'_1, y = v_{i,j} \in S'_2 \) and \( z = v_{i,k} \in S'_3 \) for some \( i \in [2, m], j \in [2, i+1] \) and \( k \in [i+2, n_i] \), we have

\[
d_{H}(x, S'_2) = \min\{d_H(v_{i,1}, v_{i,l}): 1 \leq l \leq m \} = d_H(v_{i,1}, v_{i+2}) = 1.
\]
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