Infinite Trees with Finite Dimensions

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Keywords: Infinite Graph, Tree, Metric Dimension, Partition Dimension, Locating-Chromatic Number.

Abstract: The properties of graph we consider are metric dimension, partition dimension, and locating-chromatic number. Infinite graphs can have either infinite or finite dimension. Some necessary conditions for an infinite graph with finite metric dimension has been studied in 2012. Infinite graphs with finite metric dimension will also have finite partition dimension and locating-chromatic number. In this paper we find a relation between the partition dimension (locating chromatic number) of an infinite tree with the metric dimensions of its special subtree. We also show that it is possible for an infinite trees with infinite metric dimension to have finite partition dimension (locating-chromatic number).

1 INTRODUCTION

A graph G is an infinite graph if the number of vertices is infinite. Throughout this paper, we will only consider connected graphs. The *distance* from vertex u to vertex v ($d_G(u,v)$) is the number of edges in a shortest path from u to v. The distance from $v \in V$ to $S \subseteq V$ ($d_G(u,S)$) is the minimum distance from u to all vertices in S. If the context is clear, we simply use $d(u,v)$ and $d(u,S)$.

Let w be a vertex of graph G, we say that w *resolves* vertex u and vertex v in G if $d(w,u) \neq d(w,v)$. A set of vertices S is called a *resolving set* of G if any two different vertices (u,v) is resolved by some vertices in S. Note that verifying S is a resolving set is achieved by checking all vertices outside of S. The *metric dimension* of G ($dim(G)$) is the minimum cardinality of a resolving set. The *coordinate* of v (with respect to S), denoted by $r_S(v)$, is the vector of distances from v to vertices in S.

Similarly, a set of vertices S *resolves* u and v if $d(u,S) \neq d(v,S)$. Let $\Pi = \{\pi_1, \pi_2, \cdots, \pi_k\}$ be a partition of V, Π is a *resolving partition* if and different vertices u and v is resolved by a partition class in Π. To verify Π is a resolving partition, we only need to check vertices in the same partition class. The *partition dimension* of G ($pd(G)$) is the minimum number of partition classes in a resolving partition. The *representation* of v ($r(v|\Pi)$), is the vector of distances from v to the partition classes in the ordered partition Π.

A map $c : V \to \{1, 2, \cdots, k\}$ is a *k-coloring* if any two adjacent vertices u and v receive different colors. A coloring c is a *locating k-coloring* (or simply *locating coloring*) if the partition Π_c of V induced by c, is a resolving partition. The *locating-chromatic number* of G ($\chi_L(G)$), is the smallest integer k such that G has a locating k-coloring. The *color code* of v is $r_c(v) = r(v|\Pi_c)$.

Note that any locating coloring will induce a resolving partition, therefore we have $pd(G) \leq \chi_L(G)$. Throughout this paper, the word dimension corresponds to either metric dimension, partition dimension, or locating chromatic number.

A well known relation between the metric dimension of a graph and its partition dimension is given in the following theorem.

Theorem 1.1. *(G. Chartrand and Zhang, 2000) For any graph G,*

$$pd(G) \leq \dim(G) + 1.$$

The relation between metric dimension and locating-chromatic number of a graph is given as follows.

Theorem 1.2. *(G. Chartrand and Zhang, 2002) Let G be a graph with chromatic number $\chi(G)$ (the smallest positive integer k such that G have a k-coloring). Then,*

$$\chi_L(G) \leq \dim(G) + \chi(G).$$

Hafidh, Y. and Baskoro, E.
Infinite Trees with Finite Dimensions.
DOI: 10.5220/0009876300002775
In *Proceedings of the 1st International MIPAnet Conference on Science and Mathematics (IMC-SciMath 2019)*, pages 11-13
ISBN: 978-989-758-556-2

2 INFINITE GRAPH WITH FINITE DIMENSIONS

2.1 Metric dimension

Cácares and Puertas (2012) showed that any infinite graph with bounded degree always has finite metric dimension.

Theorem 2.1. *(J. Cáceres and Puertas, 2012) If G is an infinite graph with maximum degree Δ and $m(\geq 1)$ vertices of degree at least 3, then $dim(G)$ is finite. Moreover $dim(G) \leq m\Delta$.*

The converse of this theorem is not true in general but it is true for any tree.

Theorem 2.2. *(J. Cáceres and Puertas, 2012) If an infinite tree has finite metric dimension, then the set of vertices of degree at least 3 is finite and has bounded degree.*

These two theorems give a characterization for infinite trees to have finite metric dimension.

Corollary 2.1. *Let T be an infinite tree, then $dim(T)$ is finite if and only if T has bounded degree and the number of vertices of degree at least 3 is finite.*

Let T be a tree other than a path, a *branch* is a vertex with degree more than two. An *end path* is a path connecting a leaf (vertex of degree one) to its nearest branch. A *major branch* is a branch with at least one end path. The metric dimension of a tree is given in the following theorem.

Theorem 2.3. *(Slater, 1975) Let T be a finite tree with $b \geq 1$ major branches $v_1, \cdots, v_b$ and l leaves. If k_i is the number of end paths from v_i, then $dim(T) = l - b = \sum_{i=1}^{b}(k_i - 1)$.*

The previous theorem $\left(dim(T) = \sum_{i=1}^{b}(k_i - 1)\right)$ is also true for any infinite graph, but we need to generalize the definition of an end path to also be an infinite ray from a branch, (J. Cáceres and Puertas, 2012).

2.2 Partition dimension and locating-chromatic number

In this section we find a relation between the partition dimension and locating-chromatic number of an infinite tree with the metric dimensions of its special subtree. This show that it is possible for an infinite trees with infinite metric dimension to have finite partition dimension (locating-chromatic number).

Let T be any infinite tree, define $[T]$ as the tree obtained by the following way : for every major branch v, every end path starting from v is contracted into one vertex, see figure 1. Note that $[T]$ is a subtree

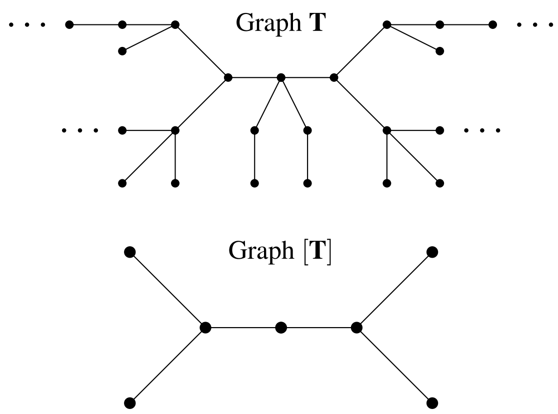

Figure 1: Graph T and $[T]$

of T.

Theorem 2.4. *If T be an infinite tree with bounded degree. If the maximum number of end path from a branch is κ, then*

$$pd(T) \leq dim([T]) + \kappa + 1.$$

Proof. We will construct a resolving partition for T with cardinality $dim([T]) + \kappa + 1$. Consider the following algorithm:

1. Let $W = \{w_1, w_2, \cdots, w_{dim([T])}\}$ be a resolving set for subtree $[T]$.
2. For every major vertex v_i of T $(i = 1, 2, \cdots)$, let $P_{i1}, P_{i2}, \cdots$ be the end paths from v_i.
3. Define $\Pi = \{\pi_1, \pi_2, \cdots, \pi_{dim([T])+\kappa+1}\}$ with:
   - $\pi_i = \{w_i\}$ for $i = 1, 2, \cdots, dim([T])$;
   - $\pi_{i+dim([T])} = \cup_j (V(P_{ji}) - \{v_j\})$ for $i = 1, 2, \cdots, \kappa$; and
   - $\pi_{dim([T])+\kappa+1} = V(T) - \left(\pi_1 \cup \cdots \cup \pi_{dim([T])+\kappa}\right)$.

Let u and v be any two different vertices in the same partition. If $u, v \in \pi_{dim([T])+\kappa+1}$, then $u, v \in V([T])$, which means that w_i will resolve u and v for some i, and so $\pi_i = \{w_i\}$ also resolve u and v.

If u and v in π_i with $dim([T]) + 1 \leq i \leq dim([T]) + \kappa$, let u' and v' be the nearest branch form u and v, respectively. If $d(u, u') = d(v, v')$, then the vertex w_i in W that distinguishes u' and v' will also distinguish u and v, this implies that π_i will distinguishes u and v. If $d(u, u') < d(v, v')$ then the partition containing u' will distinguish u and v, a similar argument can be applied if $d(u, u') > d(v, v')$.

So every pair of vertices is resolved by some partition class, therefore Π is a resolving partition, and $pd(T) \leq dim([T]) + \kappa + 1$. □

Theorem 2.5. *Let T be an infinite tree with bounded degree. If the maximum number of end path from a branch is κ, then*

$$\chi_L(T) \leq dim([T]) + \kappa + 2.$$

Proof. First, we will construct a resolving coloring for T with cardinality $dim([T]) + \kappa + 2$. Consider the following algorithm:

1. Let $W = \{w_1, w_2, \cdots, w_{dim([T])}\}$ be a resolving set for subtree $[T]$.
2. For every major vertex v_i of T $(i = 1, 2, \cdots)$, let $P_{i1}, P_{i2}, \cdots$ be the end paths from v_i.
3. Define the following coloring on $V(T)$:
   - Fix a vertex v in T, for every vertex in $V(T)$, if the vertex has odd distance to v, color that vertex with $dim([T]) + \kappa + 1$, otherwise color the vertex with $dim([T]) + \kappa + 2$;
   - Recolor w_i with i for $i = 1, 2, \cdots, dim([T])$;
   - For every vertex in P_{ij}, if the distance to its nearest branch is odd, recolor the vertex with $dim([T]) + j$.

Now we prove that c is a resolving coloring. Let u and v be two different vertices and assume that $r_c(u) = r_c(v)$. Since W is a resolving set for $[T]$, if $r_W(u) = r_W(v)$, then u and v must be in two different end paths from the same major branch, lets say u in P_{ij} and v in P_{ik}. But that means the color j and k will distinguish u and v.

Therefore c is a resolving coloring, and $\chi_L(T) \leq dim([T]) + \kappa + 2$. □

The previous two theorems, together with corollary 2.1 give a sufficient condition for infinite trees to have finite partition dimension and locating chromatic number.

Corollary 2.2. *Let T be a tree with bounded degree and $[T]$ only have a finite number of branches, then $pd(T)$ and $\chi_L(T)$ are finite.*

Note that Theorem 2.4 and Theorem 2.5 also work for finite graphs.

Corollary 2.3. *Let T be a tree (finite or infinite). If the maximum number of end path from a branch is γ, then*

$$pd(T) \leq dim([T]) + \gamma + 1$$

and

$$\chi_L(T) \leq dim([T]) + \gamma + 2.$$

Remark 2.1. *For an infinite tree T with $pd(T)$ or $\chi_L(T)$ finite, it is not necessary that $dim(T)$ is finite. For example, if T is an infinite comb (a tree obtained by attaching one pendant to each vertex in an infinite path), then $dim(T)$ is infinite (J. Cáceres and Puertas, 2012). Since $[T]$ is an infinite path with metric dimension two, we have $pd(T) \leq 4$ and $\chi_L(T) \leq 5$ by Theorem 2.4 and Theorem 2.5.*

ACKNOWLEDGMENTS

This research has been supported by World Class Research Grant, Ministry of Research, Technology and Higher Education, Indonesia.

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