# Extended Possibilistic Fuzzification for Classification

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Abstract: In this paper, the extended possibilistic fuzzification for classification is proposed. Similar approach with the use of fuzzy–rough fuzzification (Nowicki and Starczewski, 2017; Nowicki, 2019) allows to obtain one of three decisions, i.e. ”yes”, ”no”, and ”I do not know”, The last label occurs when input information is imprecise, incomplete or in general uncertain, and consequently, determining the unequivocal decision is impossible. We extend three-way decision (Hu et al., 2017; Liu et al., 2016; Sun et al., 2017; Yao, 2010; Yao, 2011) into four-way decision by extending possibilistic fuzzification to the three–dimensional possibility and necessity measures of fuzzy events.

## 1 INTRODUCTION

Possibility distributions were introduced as an alternative to probability distributions. A possibility distribution on a set $X$ is a function $\varphi\colon X \to [0,1]$ such that $\sup_{x\in X} \varphi(x) = 1$. There are dual measures formed by a degree of possibility that some event is possible and a degree of necessity that ensures an event takes place. Generally, we can measure possibility and necessity degrees of a fuzzy event, whenever $A$ denotes a fuzzy set in $X$, the degrees of possibility and necessity of $A$ are be defined as follows (Zadeh, 1978)

$$\pi(A) = \sup_{x\in X} \min\left(\varphi(x), \mu_A(x)\right), \tag{1}$$

$$\nu(A) = \inf_{x\in X} \max\left(1-\varphi(x), \mu_A(x)\right). \tag{2}$$

Note that t-norms and t-conorms may be considered instead of min and max; however, such approach is closer related to a concept of a rough-fuzzy set. Exemplary calculations of possibility and necessity degrees are presented in Fig. 1.

The possibility is related to the difficulty to describe objects by means of suitable attributes. Two measures can independently classify events, as possible or certain, under possibility distribution describing imperfections of event's attributes. We need to obtain an a'priori knowledge about the imprecision of inputs in order to determine an proper shape of fuzzification. In many cases, knowledge about the nature of impressions is limited, thus a three–point estimation can be successfully applied in analogy to the probabilistic approaches of the triangular distribution in risk analysis, project management and business decision making. Obviously, when information about the fuzzification of an attribute is limited (e.g. its smallest and largest values), we apply interval de-

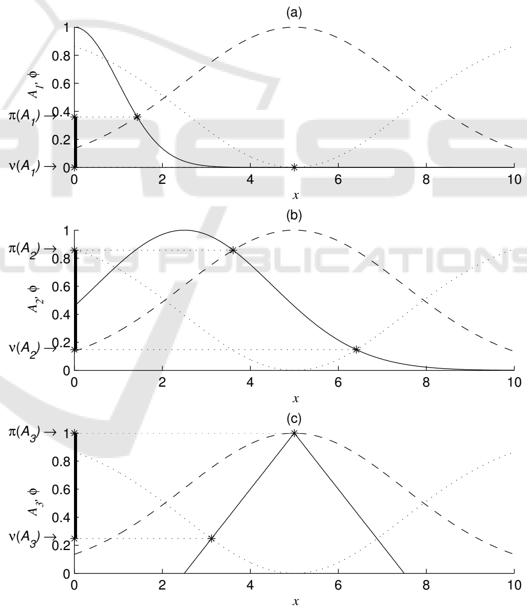


Figure 1: Calculation of possibility and necessity degrees of fuzzy sets: $\varphi$ — possibility distribution (*dashed lines*), $\mu_{A_i}$ — membership functions of fuzzy sets (*solid lines*), $\pi(A_i)$ and $\nu(A_i)$ — possibility and necessity, $i = 1,2,3$.

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scription of the membership uncertainty; however, if the most likely value of the attribute is also known, the fuzzification can be modelled by a triangular membership function in the truth interval $[0,1]$ which describes a type-2 fuzzy set (Najariyan et al., 2017; Han et al., 2016). The type-2 fuzzy subset is defined as a set $\mathbb{X}$ (called also as fuzzy-valued fuzzy set), denoted by $\tilde{A}$, which is a vague collection of elements characterized by membership function $\mu_{\tilde{A}}\colon \mathbb{X} \to \mathcal{F}([0,1])$, where $\mathcal{F}([0,1])$ is a set of all classical fuzzy sets in the unit interval $[0,1]$. Each $x \in \mathbb{X}$ is associated with a secondary membership function $f_x \in \mathcal{F}([0,1])$ i.e. a mapping $f_x\colon [0,1] \to [0,1]$. The fuzzy membership grade $\mu_{\tilde{A}}(x)$ is often called a fuzzy truth value, since its domain is the truth interval $[0,1]$. A type-1 membership function which is used in a type-2 set whose secondary membership grades are equal to the unity is called a principal membership function. The upper and lower bounds of a secondary membership function are respectively called upper and lower membership functions.

# 2 EXTENDED TRIANGULAR POSSIBILITY FUZZIFICATION

Uncertainty of input data should be modeled by a non-singleton fuzzification of system's inputs. In several classes of problems, we are able to assign triangular shapes of fuzzifying functions according to an a'priori knowledge about the uncertainty. Therefore, fuzzification of inputs can be considered in terms of possibility measures for input values $\mathbf{x}'$, while a membership function of the rule premise, $\mu_{A'}$, can be viewed as a possibility distribution. Consequently, the possibility of $A_k$ forms an upper bound of fuzzified inputs

$$\overline{\mu}_{A_k}\left(\mathbf{x}'\right) = \sup_{\mathbf{x}\in\mathbf{X}} T\left(\mu_{A'}\left(\mathbf{x},\mathbf{x}'\right), \mu_{A_k}\left(\mathbf{x}\right)\right), \tag{3}$$

the necessity of $A_k$ defines a lower bound of fuzzified inputs

$$\underline{\mu}_{A_k}\left(\mathbf{x}'\right) = \inf_{\mathbf{x}\in\mathbf{X}} S\left(N\left(\mu_{A'}\left(\mathbf{x},\mathbf{x}'\right)\right), \mu_{A_k}\left(\mathbf{x}\right)\right), \tag{4}$$

while the original membership function of $A_k$ is referred as an antecedent principal membership function.

Note that the possibility expression (3) is the same as the fuzzification in a traditional conjunction reasoning (Mouzouris and Mendel, 1997). On the contrary, the necessity expression (4) is the same as the fuzzification in an implication reasoning. With the use of both measures and the non-fuzzified principal membership function. We model more information about fuzzification.

Our method makes assumption that $\mu_{A'}(\mathbf{x},\mathbf{x}')$ varies in the whole spectrum of possible values of $\mathbf{x}'$ independently of $\mathbf{x}$. Thus, we are able to determine the upper limit of a t-norm according to (3), as well as the lower limit of an s-implication in (4). In Figure 2, the construction of possibility and necessity of antecedent (principal) $A_k$ is shown.

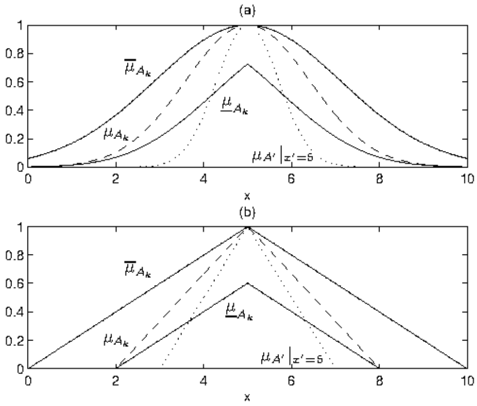


Figure 2: Extended possibilistic triangular fuzzifications of: (a) — Gaussian principal antecedent (*dashed line*), (b) — of triangular principal antecedent (*dashed line*); upper and lower membership functions (*solid lines*).

## 2.1 Triangular Fuzzification

Let two triangular membership functions be defined as the premise membership function, $\mu_{A'_n}(x_n) = \Big/\min\left(\frac{x_n-x'_n+\Delta_n}{\Delta_n}, \frac{x'_n+\Delta_n-x_n}{\Delta_n}\right)\Big/$, and the $k$-th antecedent membership function, expressed by $\mu_{A_{k,n}}(x_n) = \Big/\min\left(\frac{x_n-m_{k,n}+\delta_{k,n}}{\delta_{k,n}}, \frac{m_{k,n}+\gamma_{k,n}-x_n}{\gamma_{k,n}}\right)\Big/$. Moreover, let a t-conorm in (4) be the maximum, and the necessity antecedent function be defined by

$$\underline{\mu}_{A_k}(x'_n) = \inf_{x_n\in X_n}\left(\max\left(1-\mu_{A'_n}(x'_n,x_n), \mu_{A_{k,n}}(x_n)\right)\right). \tag{5}$$

For the left slope, $\max\left(\mu_{A'_n}(x_n,x'_n), \mu_{A_{k,n}}(x_n)\right)$ reaches its infimum at $x^*_n$ which satisfies

$$1 - \frac{x^*_n - x'_n + \Delta_n}{\Delta_n} = \frac{x^*_n - m_{k,n} + \delta_{k,n}}{\delta_{k,n}}. \tag{6}$$

Consequently,

$$x^*_n = \frac{\Delta_n m_{k,n} + \delta_{k,n} x'_n}{\Delta_n + \delta_{k,n}}. \tag{7}$$

Let us evaluate $\mu_{A_{k,n}}(x_n)$ for $x^*_n$ in both slopes

$$\mu_{A_{k,n}}(x^*_n) = \begin{cases} \frac{x'_n - m_{k,n} + \delta_{k,n}}{\Delta_n + \delta_{k,n}} & \text{if } x'_n \in \left[m_{k,n} - \Delta_n - \delta_{k,n}, \underline{m}_{k,n}\right] \\ \frac{m_{k,n} + \gamma_{k,n} - x'_n}{\Delta_n + \gamma_{k,n}} & \text{if } x'_n \in \left[\underline{m}_{k,n}, m_{k,n} + \Delta_n + \delta_{k,n}\right] \end{cases} \tag{8}$$

where $\underline{m}_{k,n}$ denotes a new center value.

It is profitable that the necessity being a lower bound of triangular fuzzification remains triangular,

$$\underline{\mu}_{A_k}(x_n') = \Big/ \min\left(\frac{x_n' - m_{k,n} + \delta_{k,n}}{\widetilde{\delta}_{k,n}}, \frac{m_{k,n} + \gamma_{k,n} - x_n'}{\widetilde{\gamma}_{k,n}}\right) \Big/ , \tag{9}$$

where $\widetilde{\delta}_{k,n} = \Delta_n + \delta_{k,n}$ and $\widetilde{\gamma}_{k,n} = \Delta_n + \gamma_{k,n}$. A new value of the center $\underline{m}_{k,n}$ can be obtained as $x_n'$ fullfilling the following

$$\begin{aligned} \frac{x_n'^* - m_{k,n} + \delta_{k,n}}{\Delta_n + \delta_{k,n}} &= \frac{m_{k,n} + \gamma_{k,n} - x_n'^*}{\Delta_n + \gamma_{k,n}} , \qquad (10) \\ \underline{m}_{k,n} &= x_n'^* = \frac{\Delta_n (\gamma_{k,n} - \delta_{k,n})}{2\Delta_n + \delta_{k,n} + \gamma_{k,n}} + m_{k,n} . \end{aligned}$$

By substituting $\underline{m}_{k,n}$ into (9)

$$\underline{h}_{k,n} = \frac{\gamma_{k,n} + \delta_{k,n}}{2\Delta_n + \delta_{k,n} + \gamma_{k,n}} . \tag{11}$$

Although the possibilistic measures implement non-singleton fuzzification using either fuzzy implications or fuzzy conjunctions, the reasoning schema is independent, and both implication and conjunction reasoning schemes can be here applied interchangeably.

# 3 GENERAL FL CLASSIFIER

Consider a type-2 fuzzy logic system with the uncertainty of the general form (Mendel, 2001; Starczewski, 2013; Nowicki, 2019). Such system can be adapted to classification tasks with the following form of rules:

$$\begin{aligned} &\mathrm{R}_k\colon \text{IF } v_1 \text{ is } A_{1,k} \text{ AND } v_2 \text{ is } A_{2,k} \text{ AND} \ldots \\ &\quad \text{THEN } x \in \omega_1(z_{1,k}), x \in \omega_2(z_{2,k}), \ldots \end{aligned} \tag{12}$$

where observations $v_i$ and objects $x$ are independent variables, $k = 1, \ldots, N$ is the number of $N$ rules, and $z_{j,k}$ is a membership degree of the object $x$ to the $j$–th class $\omega_j$ Memberships of objects are considered to be crisp rather than fuzzy, i.e.

$$z_{j,k} = \begin{cases} 1 & \text{if } x \in \omega_j \\ 0 & \text{if } x \notin \omega_j \end{cases} . \tag{13}$$

Each rule of a fuzzy system can be regarded as a certain two-place function $\mathrm{R}\colon [0,1]^2 \to [0,1]$. In the case of the conjunction-type fuzzy systems, function R is defined by any t-norm $\mathrm{R}(a,b) = \mathrm{T}(a,b)$, A logical approach use genuine fuzzy implications, i.e. strong implication $\mathrm{R}(a,b) = \mathrm{S}(\mathrm{N}(a), b)$, residual implications $\mathrm{R}(a,b) = \sup_{c\in[0,1]} \{c | \mathrm{T}(a,c) \le b\}$, quantum logic implications $\mathrm{R}(a,b) = \mathrm{S}(\mathrm{N}(a), \mathrm{T}(a,b))$. Traditional t-norm T, t-conorm S, negation N, have to be extended to operate on fuzzy values rather than numbers from $[0,1]$ (see eg. (Starczewski, 2013)).

The fuzzy reasoning process leads to the conclusion in the form of $y$ is $B'$, where $B'$ is aggregated from conclusions $B_k'$ for $k = 1, \ldots, N$ obtained as a result of fuzzy reasoning using separated rules $\mathrm{R}_k$. Compositions $B_k' = A' \circ \mathrm{R}(A_k, B_k)$ are fuzzy sets with the membership functions defined using $\sup - \mathrm{T}$ compositional rule of inference, i.e.

$$\mu_{B_k'}(y) = \sup_{\mathbf{x}\in\mathbf{X}} \mathrm{T}\left(\mu_{A'}(\mathbf{x}), \mathrm{R}\left(\mu_{A_k}(\mathbf{x}), \mu_{B_k}(y)\right)\right). \tag{14}$$

In the case of singleton fuzzification, equation (14) yields the following

$$\mu_{B_k'}(y) = \mathrm{R}\left(\mu_{A_k}(\mathbf{x}'), \mu_{B_k}(y)\right). \tag{15}$$

which allows for omitting a troublesome supremum.

In the case of conjunction reasoning, we aggregate $B' = \bigcup_{k=1}^{N} B_k'$, consequently

$$\mu_{B'}(y) = \mathop{\mathrm{S}}_{k=1}^{N} \mu_{B_k'}(y) \tag{16}$$

while in the case of genuine implications, aggregation is performed with the use of conjunctions $B' = \bigcap_{k=1}^{N} B_k'$, i.e.,

$$\mu_{B'}(y) = \mathop{\mathrm{T}}_{k=1}^{N} \mu_{B_k'}(y), \tag{17}$$

where all operations are on type-2 fuzzy sets.

## 3.1 Algebraic Operations

In (Starczewski, 2013), we have defined a regular t-norm on a set of triangular fuzzy truth numbers

$$\mu_{\mathrm{T}_{n=1}^{N} F_n}(u) = \max\left(0, \min\left(\lambda(u), \rho(u)\right)\right), \tag{18}$$

where

$$\lambda(u) = \begin{cases} \frac{u-l}{m-l} & \text{if } m > l \\ \text{singleton}(u-m) & \text{if } m = l, \end{cases} \tag{19}$$

$$\rho(u) = \begin{cases} \frac{r-u}{r-m} & \text{if } r > m \\ \text{singleton}(u-m) & \text{if } m = r, \end{cases} \tag{20}$$

and $l = \mathrm{T}_{n=1}^{N} l_n$, $m = \mathrm{T}_{n=1}^{N} m_n$, $r = \mathrm{T}_{n=1}^{N} r_n$. This formulation allows us to use ordinary t-norms for upper, principal and lower memberships independently. Moreover, we have proved the function given by (18) operating on triangular and normal fuzzy truth values is a t-norm on $\mathcal{L} = (\mathcal{F}_{\triangle}([0,1]), \sqsubseteq)$ (of type-2).

### 3.2 Triangular Centroid Type Redution

The first step transforming a type-2 fuzzy conclusion into a type-1 fuzzy set is called a type reduction. In classification, we perform only type reduction without the second step of final defuzzification. In (Starczewski, 2014), we have obtained exact type-type reduced sets for triangular type-2 fuzzy conclusions as a set of ordered discrete primary values $y_k$ and their secondary membership functions

$$f_k(u_k) = \Bigg/ \min\left(\frac{u_k - \underline{\mu}_k}{\widehat{\mu}_k - \underline{\mu}_k}, \frac{\overline{\mu}_k - u_k}{\overline{\mu}_k - \widehat{\mu}_k}\right) \Bigg/ \tag{21}$$

for $k = 1, \ldots, K$. The secondary membership functions are specified by upper, principal and lower membership grades, $\overline{\mu}_k > \widehat{\mu}_k > \underline{\mu}_k$, $k = 1, 2, \ldots, K$. Interval type reduction gives $[y_{\min}, y_{\max}]$ and $y_{\mathrm{pr}}$ is a centroid of the principal membership grades calculated by

$$y_{\mathrm{pr}} = \sum_{k=1}^{K} \frac{\widehat{\mu}_k y_k}{\widehat{\mu}_k}. \tag{22}$$

The exact centroid of the triangular type-2 fuzzy set is characterized by the following membership function:

$$\mu(y) = \begin{cases} \frac{y - y_{\mathrm{left}}(y)}{(1-q_{\mathrm{l}}(y))y + q_{\mathrm{l}}(y)y_{\mathrm{pr}} - y_{\mathrm{left}}(y)} & \text{if } y \in [y_{\min}, y_{\mathrm{pr}}] \\ \frac{y - y_{\mathrm{right}}(y)}{(1-q_{\mathrm{r}}(y))y + q_{\mathrm{r}}(y)y_{\mathrm{pr}} - y_{\mathrm{right}}(y)} & \text{if } y \in [y_{\mathrm{pr}}, y_{\max}] \end{cases}, \tag{23}$$

where the parameters are

$$q_{\mathrm{l}}(y) = \frac{\sum_{k=1}^{K} \widehat{\mu}_k}{\sum_{k=1}^{K} \overleftarrow{\mu}_k(y)}, \quad q_{\mathrm{r}}(y) = \frac{\sum_{k=1}^{K} \widehat{\mu}_k}{\sum_{k=1}^{K} \overrightarrow{\mu}_k(y)},$$

$$y_{\mathrm{left}}(y) = \frac{\sum_{k=1}^{K} \overleftarrow{\mu}_k(y)\, y_k}{\sum_{k=1}^{K} \overleftarrow{\mu}_k(y)}, \quad y_{\mathrm{right}}(y) = \frac{\sum_{k=1}^{K} \overrightarrow{\mu}_k(y)\, y_k}{\sum_{k=1}^{K} \overrightarrow{\mu}_k(y)},$$

with

$$\overleftarrow{\mu}_k(y) = \begin{cases} \overline{\mu}_k & \text{if } y_k \le y \\ \underline{\mu}_k & \text{otherwise} \end{cases},$$

$$\overrightarrow{\mu}_k(y) = \begin{cases} \overline{\mu}_k & \text{if } y_k \ge y \\ \underline{\mu}_k & \text{otherwise} \end{cases}.$$

### 3.3 Type Reduction in Classification

In classification, $y_k$ are either equal to 0 or to 1. Therefore, instead of the Karnik–Mendel iterative type reduction, we propose the following procedure. In the case of conjunction (Mamdani) type of fuzzy reasoning, the lower and upper membership grades are expressed as follows

$$\underline{z}_j = \frac{\sum\limits_{\substack{k=1 \\ k:\, \overline{z}_j^k = 1}}^{N} \mu_{A_{\mathrm{L}}^k}(\overline{\mathbf{v}})}{\sum\limits_{k=1}^{N} \mu_{A_{\mathrm{L}}^k}(\overline{\mathbf{v}})} \qquad \overline{z}_j = \frac{\sum\limits_{\substack{k=1 \\ k:\, \overline{z}_j^k = 1}}^{N} \mu_{A_{\mathrm{U}}^k}(\overline{\mathbf{v}})}{\sum\limits_{k=1}^{N} \mu_{A_{\mathrm{U}}^k}(\overline{\mathbf{v}})}, \tag{24}$$

where $A_{\mathrm{L}}^k$ and $A_{\mathrm{U}}^k$ are expressed as follows

$$A_{\mathrm{L}}^k = \begin{cases} A_*^k & \text{if } \overline{z}_j^k = 1 \\ A^{k*} & \text{if } \overline{z}_j^k = 0 \end{cases} \qquad A_{\mathrm{U}}^k = \begin{cases} A^{k*} & \text{if } \overline{z}_j^k = 1 \\ A_*^k & \text{if } \overline{z}_j^k = 0 \end{cases}. \tag{25}$$

Whenever the classifier is built with the use of logical-type reasoning, we can use the following analogy

$$\underline{z}_j = \frac{\sum\limits_{\substack{k=1 \\ k:\, \overline{z}_j^k = 1}}^{N} \sum\limits_{\substack{r=1 \\ r:\, \overline{z}_j^r = 0}}^{N} \mathrm{N}\left(\mu_{A_{\mathrm{L}}^r}(\overline{\mathbf{v}})\right)}{\sum\limits_{k=1}^{N} \sum\limits_{\substack{r=1 \\ r:\, \overline{z}_j^r \neq \overline{z}_j^k}}^{N} \mathrm{N}\left(\mu_{A_{\mathrm{L}}^r}(\overline{\mathbf{v}})\right)}, \tag{26}$$

$$\overline{z}_j = \frac{\sum\limits_{\substack{k=1 \\ k:\, \overline{z}_j^k = 1}}^{N} \sum\limits_{\substack{r=1 \\ r:\, \overline{z}_j^r = 0}}^{N} \mathrm{N}\left(\mu_{A_{\mathrm{U}}^r}(\overline{\mathbf{v}})\right)}{\sum\limits_{k=1}^{N} \sum\limits_{\substack{r=1 \\ r:\, \overline{z}_j^r \neq \overline{z}_j^k}}^{N} \mathrm{N}\left(\mu_{A_{\mathrm{U}}^r}(\overline{\mathbf{v}})\right)}, \tag{27}$$

where $A_{\mathrm{L}}^k$ and $A_{\mathrm{U}}^k$ are defined as previously, by equations (25), and N is any fuzzy negation $\mathrm{N}(x) = 1 - x$.

### 3.4 Interpretation of Type-reduced Sets

A proper interpretation of obtained is a complex problem for the extended possibilistic fuzzy classification. If $\underline{z}_j$ is a lower membership grade of an object $x$ to a class $\omega_j$ and $\overline{z}_j$ is its upper membership grade in the form of equations (24) respectively, then we suggest to fix a threshold value, e.g. 0.5 and perform a crisp decision in the following way:

$$\begin{cases} x \in \omega_j & \text{if } \underline{z}_j \ge \frac{1}{2} \text{ and } \overline{z}_j > \frac{1}{2} \\ x \notin \omega_j & \text{if } \underline{z}_j < \frac{1}{2} \text{ and } \overline{z}_j \le \frac{1}{2} \\ \text{likely possible class.} & \text{if } \underline{z}_j < \frac{1}{2} \text{ and } \widehat{z}_j^* \ge \frac{1}{2} \\ \text{likely impossible class.} & \text{otherwise.} \end{cases} \tag{28}$$

## 4 SIMULATION RESULTS

The following scheme of experiments is provided:

1. An ordinary (type-1) fuzzy system on exact data in a laboratory environment is trained. This system becomes a framework for a possibilistic system. In the performed simulations the standard Back Propagation learning method was used.
2. Real-time systems usually operate on noisy signals, and the nature of measurement noise might

Table 1: Accuracy for classification (in %) of Iris data with additional Gaussian noise to all inputs $\sigma_i = 0.1\Delta x_i$, where $\Delta x_i$ is a range of $x_i$, $i = 1,2,3,4$; σ — standard deviation of p.

| System | $\bar{\sigma}_i$ | Incorrect lrn./test | Unclassified incorrect suggestion lrn./test | Unclassified correct suggestion lrn./test | Correct lrn./test |
|---|---|---|---|---|---|
| Logical-type, no noise | | | | | |
| singleton | — | 18.7/24.7 | — | — | 81.3/75.3 |
| Logical-type, noised inputs | | | | | |
| singleton | — | 37.8/36.5 | — | — | 62.2/63.5 |
| fuzzy - rough | 0,01 | 31,8/33,9 | 6,0/2,5 | 8,1/10,7 | 54,1/52,9 |
| | 0,02 | 27,9/27,0 | 9,9/9,5 | 16,7/19,8 | 45,4/43,7 |
| | 0,03 | 23,3/23,3 | 14,5/13,1 | 24,0/26,5 | 38,2/37,1 |
| | 0,10 | 12,1/11,8 | 25,7/24,7 | 54,6/56,2 | 7,6/7,3 |
| | 0,20 | 7,8/8,4 | 30,0/28,1 | 62,0/63,1 | 0,2/0,4 |
| | 0,30 | 6,1/6,5 | 31,8/30,0 | 62,2/63,5 | 0,0/0,0 |
| | 0,40 | 4,3/4,5 | 33,5/32,0 | 62,2/63,5 | 0,0/0,0 |
| | 0,50 | 3,8/2,9 | 34,0/33,6 | 62,2/63,5 | 0,0/0,0 |
| | 0,60 | 2,7/3,1 | 35,1/33,4 | 62,2/63,5 | 0,0/0,0 |
| | 1,00 | 0,5/0,5 | 37,3/35,9 | 62,2/63,5 | 0,0/0,0 |
| non - singleton | 0.01 | 37.3/37.5 | — | — | 62.7/62.5 |
| | 0.02 | 37.2/36.3 | — | — | 62.8/63.7 |
| | 0.03 | 37.8/40.1 | — | — | 62.2/59.9 |
| | 0.10 | 40.5/41.3 | — | — | 59.5/58.7 |
| | 0.20 | 45.2/45.9 | — | — | 54.8/54.1 |
| | 0.30 | 46.9/49.4 | — | — | 53.1/50.6 |
| | 0.40 | 49.3/49.3 | — | — | 50.7/50.7 |
| | 0.50 | 51.8/52.1 | — | — | 48.2/47.9 |
| | 0.60 | 52.9/53.3 | — | — | 47.1/46.7 |
| | 1.00 | 58.8/59.3 | — | — | 41.2/40.7 |
| Conjunction-type, no noise | | | | | |
| singleton | — | 0.4/7.3 | — | — | 99.6/92.7 |
| Conjunction-type, noised inputs | | | | | |
| singleton | — | 14.3/16.1 | — | — | 85.7/83.9 |
| fuzzy - rough | 0,01 | 9,9/10,4 | 4,4/5,7 | 5,0/2,3 | 80,7/81,5 |
| | 0,02 | 6,5/7,5 | 7,8/8,6 | 10,5/9,3 | 75,2/74,6 |
| | 0,03 | 4,3/4,0 | 10,0/12,1 | 17,1/15,5 | 68,6/68,3 |
| | 0,10 | 0,1/0,2 | 14,2/15,9 | 56,8/55,6 | 28,9/28,3 |
| | 0,20 | 0,0/0,0 | 14,3/16,1 | 82,4/80,2 | 3,3/3,7 |
| | 0,30 | 0,0/0,0 | 14,3/16,1 | 85,7/83,9 | 0,0/0,0 |
| | 0,40 | 0,0/0,0 | 14,3/16,1 | 85,7/83,9 | 0,0/0,0 |
| | 0,50 | 0,0/0,0 | 14,3/16,1 | 85,7/83,9 | 0,0/0,0 |
| | 0,60 | 0,0/0,0 | 14,3/16,1 | 85,7/83,9 | 0,0/0,0 |
| | 1,00 | 0,0/0,0 | 14,3/16,1 | 85,7/83,9 | 0,0/0,0 |
| non - singleton | 0.01 | 14.5/13.9 | — | — | 85.5/86.1 |
| | 0.02 | 14.0/14.9 | — | — | 86.0/85.1 |
| | 0.03 | 14.1/13.9 | — | — | 85.9/86.1 |
| | 0.10 | 14.5/15.6 | — | — | 85.5/84.4 |
| | 0.20 | 20.3/18.9 | — | — | 79.7/81.1 |
| | 0.30 | 34.6/33.7 | — | — | 65.4/66.3 |
| | 0.40 | 51.9/50.3 | — | — | 48.1/49.7 |
| | 0.50 | 63.6/63.8 | — | — | 36.4/36.2 |
| | 0.60 | 68.3/69.0 | — | — | 31.7/31.0 |
| | 1.00 | 70.3/70.5 | — | — | 29.7/29.5 |

be known. Following this, a white Gaussian noise with a standard deviation value $\sigma_i$ corresponding to the $i$–th input was added.

3. The additional noise should match non-singleton fuzzification. Consequently, non-singleton fuzzification and possibilistic fuzzification using Gaussian membership functions with standard deviation values $\bar{\sigma}_i$ were performed.

We have decided to present a multiple output fuzzy rough set system, in which each class was trained against all other classes. All membership functions were of the Gaussian type. The Cartesian product was realized by the algebraic product t-norm. Both

Table 2: Accuracy for classification (in %) of Wisconsin Breast Cancer data with additional Gaussian noise to all inputs $\sigma_i = 0.1\Delta x_i$, where $\Delta x_i$ is a range of $x_i$, $i = 1,2,3,4$.

| System | $\bar{\sigma}_i$ | Incorrect lrn./test | Unclassified incorrect suggestion lrn./test | Unclassified correct suggestion lrn./test | Correct lrn./test |
|---|---|---|---|---|---|
| Logical-type, no noise | | | | | |
| singleton | — | 2.9/5.0 | — | — | 97.1/95.0 |
| Logical-type, noised inputs | | | | | |
| singleton | — | 23.7/25.1 | — | — | 76.3/74.9 |
| fuzzy - rough | 0,01 | 18,5/18,8 | 5,3/6,3 | 6,0/5,6 | 70,3/69,3 |
| | 0,02 | 13,2/13,7 | 10,5/11,4 | 13,4/12,0 | 62,9/62,9 |
| | 0,03 | 8,7/8,9 | 15,0/16,2 | 20,5/18,9 | 55,8/56,0 |
| | 0,10 | 0,2/0,4 | 23,5/24,7 | 51,5/50,7 | 24,8/24,2 |
| | 0,20 | 0,0/0,1 | 23,7/25,0 | 64,3/63,9 | 11,9/11,0 |
| | 0,30 | 0,0/0,0 | 23,7/25,1 | 71,6/70,3 | 4,7/4,6 |
| | 0,40 | 0,0/0,0 | 23,7/25,1 | 74,5/72,9 | 1,8/2,0 |
| | 0,50 | 0,0/0,0 | 23,7/25,1 | 75,9/74,3 | 0,4/0,6 |
| | 0,60 | 0,0/0,0 | 23,7/25,1 | 76,3/74,9 | 0,0/0,0 |
| | 1,00 | 0,0/0,0 | 23,7/25,1 | 76,3/74,9 | 0,0/0,0 |
| non - singleton | 0.01 | 24.0/24.8 | — | — | 76.0/75.2 |
| | 0.02 | 23.8/24.0 | — | — | 76.2/76.0 |
| | 0.03 | 23.9/23.7 | — | — | 76.1/76.3 |
| | 0.10 | 24.1/24.7 | — | — | 75.9/75.3 |
| | 0.20 | 24.9/24.9 | — | — | 75.1/75.1 |
| | 0.30 | 25.9/27.1 | — | — | 74.1/72.9 |
| | 0.40 | 27.0/27.1 | — | — | 73.0/72.9 |
| | 0.50 | 27.9/28.2 | — | — | 72.1/71.8 |
| | 0.60 | 29.3/29.6 | — | — | 70.7/70.4 |
| | 1.00 | 34.8/34.5 | — | — | 65.2/65.5 |
| Conjunction-type, no noise | | | | | |
| singleton | — | 2.6/4.8 | — | — | 97.4/95.2 |
| Conjunction-type, noised inputs | | | | | |
| singleton | — | 19.6/20.0 | — | — | 80.4/80.0 |
| fuzzy - rough | 0,01 | 14,0/15,2 | 5,5/4,8 | 6,2/6,9 | 74,3/73,0 |
| | 0,02 | 9,0/9,6 | 10,6/10,4 | 13,8/14,2 | 66,6/65,8 |
| | 0,03 | 5,4/5,9 | 14,1/14,1 | 22,4/22,7 | 58,0/57,3 |
| | 0,10 | 0,1/0,4 | 19,4/19,6 | 56,7/56,8 | 23,7/23,1 |
| | 0,20 | 0,0/0,1 | 19,6/19,9 | 69,3/69,6 | 11,2/10,4 |
| | 0,30 | 0,0/0,0 | 19,6/20,0 | 76,2/76,0 | 4,2/4,0 |
| | 0,40 | 0,0/0,0 | 19,6/20,0 | 78,9/78,3 | 1,6/1,7 |
| | 0,50 | 0,0/0,0 | 19,6/20,0 | 80,1/79,4 | 0,3/0,6 |
| | 0,60 | 0,0/0,0 | 19,6/20,0 | 80,4/80,0 | 0,0/0,0 |
| | 1,00 | 0,0/0,0 | 19,6/20,0 | 80,4/80,0 | 0,0/0,0 |
| non - singleton | 0.01 | 18.9/19.4 | — | — | 81.1/80.6 |
| | 0.02 | 17.3/18.1 | — | — | 82.7/81.9 |
| | 0.03 | 15.3/15.7 | — | — | 84.7/84.3 |
| | 0.10 | 5.8/6.8 | — | — | 94.2/93.2 |
| | 0.20 | 3.8/4.5 | — | — | 96.2/95.5 |
| | 0.30 | 3.9/4.2 | — | — | 96.1/95.8 |
| | 0.40 | 3.9/4.1 | — | — | 96.1/95.9 |
| | 0.50 | 4.1/4.2 | — | — | 95.9/95.8 |
| | 0.60 | 4.3/4.2 | — | — | 95.7/95.8 |
| | 1.00 | 12.5/12.6 | — | — | 87.5/87.4 |



logical-type and conjunction-type fuzzy systems were compared in their singleton, non-singleton and possibilistic realizations. The tests were carried out using 10-fold cross validation. Tables 1-3 present a direct comparison of average results for six classifiers presented in the paper, i.e. the fuzzy classifier with singleton fuzzification, the classifier with classic non-singleton fuzzification and the classifier with proposed extended possibilistic fuzzification, while all of them have been realized in two versions with two different implication methods. The classifiers with non-singleton and possibilistic fuzzification have been examined for various levels of assumed uncertainty of input data. These levels are relative with respect to

Table 3: Accuracy for classification (in %) of Pima Indians Diabetes data with additional Gaussian noise to all inputs $\sigma_i = 0.1\Delta x_i$, where $\Delta x_i$ is a range of $x_i$, $i = 1,2,3,4$.

| System | $\bar{\sigma}_i$ | Incorrect lrn./test | Unclassified incorrect suggestion lrn./test | Unclassified correct suggestion lrn./test | Correct lrn./test |
|---|---|---|---|---|---|
| Logical-type, no noise | | | | | |
| singleton | — | 11.5/29.7 | — | — | 88.5/70.3 |
| Logical-type, noised inputs | | | | | |
| singleton | — | 32.3/33.1 | — | — | 67.7/66.9 |
| fuzzy - rough | 0,01 | 16,6/18,0 | 15,7/15,1 | 19,4/20,2 | 48,3/46,8 |
| | 0,02 | 7,4/7,9 | 24,9/25,2 | 38,2/37,7 | 29,6/29,3 |
| | 0,03 | 3,2/3,5 | 29,1/29,5 | 51,2/50,7 | 16,6/16,3 |
| | 0,10 | 0,0/0,0 | 32,3/33,1 | 67,6/66,8 | 0,1/0,1 |
| | 0,20 | 0,0/0,0 | 32,3/33,1 | 67,7/66,9 | 0,0/0,0 |
| | 0,30 | 0,0/0,0 | 32,3/33,1 | 67,7/66,9 | 0,0/0,0 |
| | 0,40 | 0,0/0,0 | 32,3/33,1 | 67,7/66,9 | 0,0/0,0 |
| | 0,50 | 0,0/0,0 | 32,3/33,1 | 67,7/66,9 | 0,0/0,0 |
| | 0,60 | 0,0/0,0 | 32,3/33,1 | 67,7/66,9 | 0,0/0,0 |
| | 1,00 | 0,0/0,0 | 32,3/33,1 | 67,7/66,9 | 0,0/0,0 |
| non - singleton | 0.01 | 32.3/33.2 | — | — | 67.7/66.8 |
| | 0.02 | 32.1/33.0 | — | — | 67.9/67.0 |
| | 0.03 | 32.3/33.8 | — | — | 67.7/66.2 |
| | 0.10 | 32.1/31.9 | — | — | 67.9/68.1 |
| | 0.20 | 32.2/32.7 | — | — | 67.8/67.3 |
| | 0.30 | 32.0/33.1 | — | — | 68.0/66.9 |
| | 0.40 | 31.8/33.1 | — | — | 68.2/66.9 |
| | 0.50 | 31.8/33.3 | — | — | 68.2/66.7 |
| | 0.60 | 31.7/32.2 | — | — | 68.3/67.8 |
| | 1.00 | 31.7/33.1 | — | — | 68.3/66.9 |
| Conjunction-type, no noise | | | | | |
| singleton | — | 11.5/28.6 | — | — | 88.5/71.4 |
| Conjunction-type, noised inputs | | | | | |
| singleton | — | 32.8/33.3 | — | — | 67.2/66.7 |
| fuzzy - rough | 0,01 | 15,4/16,9 | 17,4/16,4 | 22,0/22,9 | 45,2/43,8 |
| | 0,02 | 6,0/6,5 | 26,8/26,8 | 42,1/42,3 | 25,1/24,3 |
| | 0,03 | 2,2/2,6 | 30,6/30,8 | 54,3/54,1 | 12,9/12,6 |
| | 0,10 | 0,0/0,0 | 32,8/33,3 | 67,1/66,6 | 0,1/0,1 |
| | 0,20 | 0,0/0,0 | 32,8/33,3 | 67,2/66,7 | 0,0/0,0 |
| | 0,30 | 0,0/0,0 | 32,8/33,3 | 67,2/66,7 | 0,0/0,0 |
| | 0,40 | 0,0/0,0 | 32,8/33,3 | 67,2/66,7 | 0,0/0,0 |
| | 0,50 | 0,0/0,0 | 32,8/33,3 | 67,2/66,7 | 0,0/0,0 |
| | 0,60 | 0,0/0,0 | 32,8/33,3 | 67,2/66,7 | 0,0/0,0 |
| | 1,00 | 0,0/0,0 | 32,8/33,3 | 67,2/66,7 | 0,0/0,0 |
| non - singleton | 0.01 | 32.6/33.6 | — | — | 67.4/66.4 |
| | 0.02 | 32.3/32.8 | — | — | 67.7/67.2 |
| | 0.03 | 31.7/32.7 | — | — | 68.3/67.3 |
| | 0.10 | 28.3/29.9 | — | — | 71.7/70.1 |
| | 0.20 | 28.8/29.6 | — | — | 71.2/70.4 |
| | 0.30 | 30.4/30.6 | — | — | 69.6/69.4 |
| | 0.40 | 32.0/32.4 | — | — | 68.0/67.6 |
| | 0.50 | 33.3/33.4 | — | — | 66.7/66.6 |
| | 0.60 | 33.8/33.5 | — | — | 66.2/66.5 |
| | 1.00 | 34.8/34.6 | — | — | 65.2/65.4 |

the input domains and are expressed by parameters $\overline{\sigma}_i$ (spreads) taking values from 0 to 1. When the value is equal to 0, the both classifiers are identical to the corresponding singleton classifiers. The value of the spread close to 1 means that uncertainty covers the whole range, i.e., the actual input value can be any value in the range regardless of the actually measured one. In such situation, a correct classification cannot be expected. Besides, in the case of classic non-singleton fuzzification, similar results in the whole range of spread can be observed in Tables 1-3. The numbers of correct classifications achieve the barely perceptible maximum. Moreover, for individual classifiers, the maximum is reached at different values of spread. This situation confirms that classic non-singleton fuzzification does not incorporate uncertainty in input data. In contrary, the proposed classifier with possibilistic fuzzification actually takes uncertainty into account. Some samples could be unclassified if the level of uncertainty is such high that it does not allow for an explicit classification. When the uncertainty covers the whole range ($\sigma$ equal to 1), all samples are classified to the boundary region of classes, in other words, are unclassified. The described behavior of the classifier is desirable in situations of the high level of uncertainty. The same properties are observed for both examined methods of inference.

## 5 CONCLUSIONS

In the presented paper the non-singleton fuzzification have beens used to handle the imprecision of input measurements or noisy input data. The simulated classification examples have demonstrated that possibilistic fuzzy systems (based on implications or conjunctions) can produce no false classification performing only certain or possible assignments. It seems promising in such areas as medical diagnosis that possibilistic fuzzy systems give uncertain answers rather than wrong answers. Without difficulty, not classified cases can be redirected to a new more particular investigation. Our future goal is to optimize the percentage of correct classifications providing that incorrect classification rate is equal zero.

The derived class of possibilistic fuzzy systems is the rationally proper approach to uncertain classification, while the classical non-singleton fuzzy systems do not incorporate properly uncertainty of input data, particularly, even in cases of complete uncertainty they give. Actually, they ignore the fact of uncertainty in data. The possibilistic fuzzy systems work properly when there is some redundancy in input data. Using such classifiers as the valuable parts of ensemble systems is a subject of future investigations.

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