Correlation Coefficient of Modal Level Operators: An Application to Medical Diagnosis

Alex Bertei, Renata H. S. Reiser and Luciana Foss
Federal University of Pelotas (UFPEL), Centre for Technological Development (CDTEC), Laboratory of Ubiquitous and Parallel Systems (LUPS), Street Gomes Carneiro, Pelotas, Brazil

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Abstract: This paper studies the correlation coefficient (A-CC) related to the Atanassov's intuitionistic fuzzy sets (A-IFS) which are obtained as image of modal operators. Extended results from the action of A-CC over necessity and possibility modal operators are considered, determining the A-CC of A-IFS obtained as image of the $!A$ and $?A$ modal level operators and discussing the main conditions under which the main properties related to such fuzzy sets are preserved by conjugate and complement operations. In addition, a simulation based on the proposal methodology using modal level operators is applied to a medical diagnosis analysis.

1 INTRODUCTION

The Atanassov's intuitionistic fuzzy logic (A-IFLs) comprises a generalization of multi-valued fuzzy logic by taking into account the membership and non-membership degrees of the information from Atanassov's intuitionistic fuzzy sets (A-IFSs) as discussed by Bustince and Burillo (Bustince and Burillo, 1995) approach and also providing the hesitation margin of the index of intuitionist (A-IFIx), as reported by Szmidt and Kacprzyk approach (Szmidt and Kacprzyk, 2012). Many other approaches lead to a great numbers of studies:

1. Relating similarity measure of A-IFSs to analyse the consensus of an expert preference into a group decision making (González-Arteaga et al., 2016);
2. Dealing with similarity measure to indicate the similar degree between two A-IFSs (Szmidt and Kacprzyk, 2007); and
3. Analysing the entropy of A-IFSs and describing its fuzziness degree (Szmidt and Kacprzyk, 2001).

They are closely connected with the correlation coefficient (A-CC) between two A-IFSs, which is able to express the whole expert systems in fuzzy reasoning, mainly those applied to decision-making processes such as clustering analysis (Meng et al., 2016), digital image processing, medical diagnosis and also including pattern recognition (Huang and Guo, 2017).

A-CC should provide an expression given by real parameters from -1, as the most negative (decreasing) linear relationship to 1, as the most positive (increasing) linear relationship. So, the closer an A-CC is to either -1 or 1, the stronger the correlation between these A-IFS is.

This article mainly focusses on intuitionistic fuzzy modal (A-IFM) operators and their algebraic properties have been studied by different authors. Extending the results presented in (Bertei and Reiser, 2018), this article studies A-CC to modal level operators based on their analytical expressions, which can be applied to fuzzy data analysis for classification in prediction and diagnosis in decision making. By interpreting an A-IFS as the image of modal level operators, as necessity and possibility on $U = [0,1]$, is possible to obtain a simple A-CC even when A-IFM are obtained by the action of duality and conjugate operators.

This paper is organized as follows: Section2 considers the related work presenting brief comparisons performing A-CC in A-IFS. Section 3 states the foundations on A-IFL reporting main concepts of modal level operations, including the action of automorphisms and negation operators in order to obtain conjugate and complement of A-IFS. Section 4 brings the main concepts of correlation coefficient from A-IFL. New results in Section 5 show that dual and conjugate operators are preserved by modal α-level operators. In Section 6, the study includes the main results based on A-CC obtained by modal operators. In the Section 7 is presents an application for the medical diagnosis. Finally, conclusions and further work are discussed in Section 8.

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2 RELATED WORKS

In the following, Table 1 presents a brief description of the main papers, summarizing aggregation operators and applied research area. The main aggregators used to construct the correlation coefficient are identified, as well as examples of applicability, which the great majority uses this coefficient in problems of fuzzy multiple criteria decision making (MCDM).

In (Szmidt et al., 2012), an extension of previous work (Szmidt and Kacprzyk, 2010) on A-CC is presented, measuring how strong an A-IFS relationship can be, indicating the positively or negatively correlated fuzzy set.

The A-CC analysis obtained as image of intuitionistic fuzzy t-norms and t-conorms is accomplished in (Reiser et al., 2013). They consider the action of automorphisms and the class of strong fuzzy negations. A-CC related to conjugate and dual constructions of these fuzzy connectives are studied.

Arithmetic operations on trapezoidal fuzzy intuitionistic fuzzy sets (TzFIFSs) are discussed in (Robinson and Amirtharaj, 2014) the multiattribute decision making (MADM) model proposed, using A-CC of TzFIFS for ranking the alternatives together with weighted averaging (WA) and weighted geometric (WG) operators.

In (Singh, 2015), the authors propose A-CC for picture fuzzy sets, which are extensions of A-IFS including situations when facing human opinions involving more answers (yes, abstain, no, and refusal).

In (Bertei et al., 2016) a correlation between A-IFSs obtained as image of strong negations is presented considering the action of strong fuzzy negations verifying the conditions under which the A-CC in A-IFS and their corresponding conjugate constructions are obtained. Moreover, algebraic expressions of A-CC are discussed by considering representable intuitionistic automorphisms.

The membership and non-membership degrees of A-IFS are considered in (Liu et al., 2016), providing a new approach to measuring the A-CC degree between the IFSs infinite sets. The method not only reflects the symbol attribute of an A-CC degree, but also preserves the integrity of related A-IFS.

In (Zhao and Xu, 2016), a new measure was applied to an algorithm for MADM, using the A-CC and its desirable axiomatic properties to define the intuitionistic fuzzy ideal solution (IFIS) and the intuitionistic fuzzy negative ideal solution (IFNIS). It is extended to the interval-valued approach (A-IvIFS).

A novel weighted A-CC formulation proposed in (Garg, 2016) measures the relationship on the Pythagorean fuzzy sets (PFS), which are one of the most successful methods in terms of comprehensively representing uncertain and vague information.

An MCDM problem is studied in (Solanki et al., 2016). This proposal refines TOPSIS using A-CC, characterizing an intuitionistic fuzzy decision matrix considering criteria as incompleteness and imprecision in the evaluation process. Intuitionistic fuzzy weighted averaging (IFWA) operator aggregate each DMs opinions for evaluating the relevance of alternatives. Then positive-ideal and negative-ideal solutions are calculated, using A-CC a relative closeness coefficient of the alternatives is obtained.

The theory of neutrosophic sets presented by (Smarandache, 1999) is a powerful technique to handle incomplete, indeterminate and inconsistent information in the real world. A-CC between Dynamic single-valued neutrosophic multiset (DSVNM) and a weighted coefficient between DSVNMs are presented to measure the correlation degrees between DSVNMs, and their properties are investigated by (Ye, 2017).

Huang (Huang and Guo, 2017) introduced an improved A-CC of the IFSs, discussing its properties in the IFS theory and the generalization of the coefficient of IvIFS is also introduced.

Choquet integral is used in (Qu et al., 2017) presenting a new extension of A-CC. When the weight information about criteria represented by IFSs is incomplete, a fuzzy measure model for the optimal measures on the criteria set is established, which can be used to determine the criteria fuzzy measure.

Main results in (Bertei and Reiser, 2018) extend the studies in (Bertei et al., 2016) analyzing A-CC obtained as the image of modal level operators. The actions of the necessity and the possibility, are considered to verify under which conditions an A-CC preserves main properties related to A-IFS.

3 PRELIMINARY

Firstly, a brief account on A-IFS is stated.

Consider a non-empty and finite universe $\mathscr{U} = \{x_1,\ldots,x_n\}$ an the unitary interval $[0,1] = U$. According with (Bustince and Burillo, 1995), an Atanassosv's intutionistic fuzzy set A (A-IFS) based on $\mathscr{U}$ is expressed as

$$A = \{(x, \mu_A(x), \nu_A(x)) \colon x \in \mathscr{U}\} \tag{1}$$

whenever the membership and non-membership functions $\mu_A, \nu_A : \mathscr{U} \to U$ are related by the inequality $\mu_A(x_i) + \nu_A(x_i) \leq 1$, for all $i \in \mathbb{N}_n = \{1,2,\ldots,n\}$.

Additionally, a function $\pi : A \to U$ given as

$$\pi_A(x_i) = 1 - \mu_A(x_i) - \nu_A(x_i) \tag{2}$$

Table 1: Comparisons performing correlation coefficient in A-IFL.

Paper	Title	Aggregation	Applied Field
(Szmidt et al., 2012)	Correlation between intuitionistic fuzzy sets: Some conceptual and numerical extensions	Arithmetic mean	Data analysis and decision making
(Reiser et al., 2013)	Correlations from conjugate and dual intuitionistic fuzzy triangular norms and conorms	Sum	Decision making and similarity measure
(Robinson and Amirtharaj, 2014)	MADM Problems with Correlation Coefficient of Trapezoidal Fuzzy Intuitionistic Fuzzy Sets	TzFIFWA, TzFIFWG	Ranking alternatives
(Singh, 2015)	Correlation coefficients for picture fuzzy sets	Sum	Clustering algorithm for picture fuzzy sets
(Bertei et al., 2016)	Correlation coefficient analysis based on fuzzy negations and representable automorphisms	Arithmetic mean	Fuzzy data analysis and decision making
(Liu et al., 2016)	A new correlation measure of the intuitionistic fuzzy sets	Variance and covariance	Medical diagnosis
(Zhao and Xu, 2016)	Intuitionistic fuzzy multi-attribute decision making with ideal-point-based method and correlation measure	Quadratic mean	Decision making
(Garg, 2016)	A Novel Correlation Coefficients between Pythagorean Fuzzy Sets and Its Applications to Decision-Making Processes	Weighted mean	Decision making
(Solanki et al., 2016)	A correlation based Intuitionistic fuzzy TOPSIS method on supplier selection problem	IFWA	Decision making
(Ye, 2017)	Correlation Coefficient between Dynamic Single Valued Neutrosophic Multisets and Its Multiple Attribute Decision-Making Method	Weighted mean	Decision making
(Huang and Guo, 2017)	An Improved Correlation Coefficient of Intuitionistic Fuzzy Sets	Quasi-arithmetic mean	Medical diagnosis and clustering
(Qu et al., 2017)	Choquet integral correlation coefficient of intuitionistic fuzzy sets and its applications	Choquet integral	Decision making
(Bertei and Reiser, 2018)	Correlation Coefficient Analysis Performed On Duality And Conjugate Modal-Level Operators	Arithmetic mean	Fuzzy data analysis and decision making

is called the intuitionistic fuzzy index (IFIx) or hesitance degree of an A-IFS A. The set of all above related A-IFSs is denoted by $\mathscr{C}(A)$.

Let $\tilde{U} = \{\tilde{x}_i = (x_{i1}, x_{i2}) \in U^2 : x_{i1} + x_{i2} \leq 1\}$ be the set of all intuitionistic fuzzy values such that $\tilde{x}_i$ is a pair of membership and non-membership degrees of an element $x_i \in \mathscr{U}$, i.e. $(x_{i1}, x_{i2}) = (\mu_A(x_i), \nu_A(x_i))$. And, the related IFIx is given as $\pi_A(x_i) = x_{i3} = 1 - x_{i1} - x_{i2}$, for all $i \in \mathbb{N}_n = \{1, 2, \ldots, n\}$.

The projections $l_{\tilde{U}^n}, r_{\tilde{U}^n} : \tilde{U}^n \to U^n$ are given by:

$$l_{\tilde{U}^n}(\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n) = (x_{11}, x_{21}, \ldots, x_{n1}) \quad (3)$$

$$r_{\tilde{U}^n}(\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n) = (x_{12}, x_{22}, \ldots, x_{n2}) \quad (4)$$

The order relation $\leq_{\tilde{U}}$ on $\tilde{U}$ is defined as: $\tilde{x} \leq_{\tilde{U}} \tilde{y} \Leftrightarrow x_1 \leq y_1$ and $x_2 \geq y_2$. Moreover, $\tilde{0} = (0,1) \leq_{\tilde{U}} \tilde{x} \leq_{\tilde{U}} (1,0) = \tilde{1}$, for all $\tilde{x} \in \tilde{U}$.

3.1 Intuitionistic Fuzzy Negations

Intuitionistic fuzzy negations and intuitionistic automorphisms are studied in the following. See more details in (Bustince et al., 2003).

An intuitionistic fuzzy negation (A-IFNs) $N_I : \tilde{U} \to \tilde{U}$ is a function such that, for all $\tilde{x}$, $\tilde{y}$ in $\tilde{U}$:

N_I1 $N_I(\tilde{0}) = N_I(0,1) = \tilde{1}$ and $N_I(\tilde{1}) = N_I(1,0) = \tilde{0}$;

N_I2 If $\tilde{x} \geq_{\tilde{U}} \tilde{y}$ then $N_I(\tilde{x}) \leq_{\tilde{U}} N_I(\tilde{y})$, $\forall \tilde{x}, \tilde{y} \in \tilde{U}$.

In (Bustince et al., 2000), if an IFN N_I also satisfies the involutive property

N_I3 $N_I(N_I(\tilde{x})) = \tilde{x}$, $\forall \tilde{x} \in \tilde{U}$,

N_I is called a strong A-IFN.

According with (Deschrijver et al., 2004, Theorem 3.6), N_I is a strong A-IFNs iff there exists a strong fuzzy negation N on U such that:

$$N_I(x_1,x_2) = (N(N_S(x_2)), N_S(N(x_1))). \quad (5)$$

Thus, N_I is an example of N-representable IFN. Moreover, if $N = N_S$, equation (5) can be given as

$$N_{SI}(\tilde{x}) = N_{S_I}(x_1,x_2) = (x_2,x_1). \quad (6)$$

By (Bustince et al., 2004), the complement of an IFS A w.r.t. N_I in (5) is given as

$$A_{N_I} = \{(x, N_I(\mu_A(x), \nu_A(x))) : x \in \mathscr{U}\}. \quad (7)$$

When $N = N_S$ in Eq.(5), then the complement of an IFS A with respect to N_{SI} is expressed as

$$\overline{A} = \{(x, \nu_A(x), \mu_A(x)) : x \in \mathscr{U}\}. \quad (8)$$

Let N_I be an IFN. The function $f_{N_I} : \tilde{U}^n \to \tilde{U}$ is the N_I-dual operator of $f : \tilde{U}^n \to \tilde{U}$ given as follows:

$$f_{N_I}(\tilde{x}_1, \ldots, \tilde{x}_n) = N_I(f(N_I(x_1), \ldots, N_I(\tilde{x}_n))). \quad (9)$$

For further information, see (Atanassov and Gargov, 1989; Atanassov, 1986; Atanassov, 1999).

3.2 Modal Operators

Following (Atanassov, 1983), two operators are considered over the IFSs, transforming an IFS into a fuzzy set (FS). These two operators are similar to the logical operators of necessity ($\Box$) and possibility ($\Diamond$) and their properties resemble those of Modal Logic.

Adverbial locutions as "very or absolutely" and "more or less" are interpreted as the linguistic modifiers necessity and possibility, modifying the evaluation of the linguistic Boolean truth values: "true" and "false" (Atanassov, 1986).

In deduction process, the analytic representation of such expressions plays an important role, and the A-CC analysis is able to identify the close correlation related A-IFSs (Dombi, 2013).

Relevant properties are reported below:

Definition 1. *(Atanassov, 1999, Def. 1.41) Let A be an A-IFS. The related $\Box$A-IFS and $\Diamond$A-IFS obtained by the necessity and possibility modal operators are, respectively, given as follows:*

$$\Box A = \{\langle x, \mu_A(x), 1-\mu_A(x)\rangle | x \in \mathscr{U}\}; \quad (10)$$

$$\Diamond A = \{\langle x, 1-\nu_A(x), \nu_A(x)\rangle | x \in \mathscr{U}\}. \quad (11)$$

Obviously, by Def. 1, if A is an ordinary fuzzy set then $\Box A = A = \Diamond A$.

Proposition 2. *(Atanassov, 1999, Prop. 1.42) For every A-IFS, the following properties are verified:*

$$\overline{\Box\overline{A}} = \Diamond A, \qquad \overline{\Diamond\overline{A}} = \Box A, \quad (12)$$

$$\Box\Box A = \Box A, \qquad \Box\Diamond A = \Diamond A, \quad (13)$$

$$\Diamond\Box A = \Box A, \qquad \Diamond\Diamond A = \Diamond A. \quad (14)$$

3.3 Modal α-level Operators

Initially, the operators $K_\alpha, L_\alpha : \tilde{U} \to \tilde{U}$ are defined as:

$$K_\alpha(x_1,x_2) = \left(\max\left(\frac{1}{2},x_1\right), \min\left(\frac{1}{2},x_2\right)\right); \quad (15)$$

$$L_\alpha(x_1,x_2) = \left(\min\left(\frac{1}{2},x_1\right), \max\left(\frac{1}{2},x_2\right)\right). \quad (16)$$

Further, related IFS are given in the following.

Definition 3. *(Atanassov, 1999, Def. 1.99) Let A be an A-IFS. For $\alpha \in U$, K_αA-IFS and L_αA-IFS are respectively given as follows:*

$$K_\alpha A = \{(x, \max(\alpha, x_1), \min(\alpha, x_2)) : x \in \mathscr{U}\}; \quad (17)$$

$$L_\alpha A = \{(x, \min(\alpha, x_1), \max(\alpha, x_2)) : x \in \mathscr{U}\}. \quad (18)$$

The complementary relation $K_\alpha A = \overline{L_\alpha A}$ is obtained from the pair (K_α, L_α) of N_{SI}-dual operators.

When $\alpha = \frac{1}{2}$, we use the notation $! \equiv K_{\frac{1}{2}}$ and $? \equiv L_{\frac{1}{2}}$ and related $!A$-IFS and $?A$-IFS are given below:

Definition 4. *(Atanassov, 1999, Def.1.96) Let A be an A-IFS. The related two modal level operators !A-IFS and ?A-IFS are respectively given as follows*

$$!A = \left\{\left(x, \max\left(\frac{1}{2},x_1\right), \min\left(\frac{1}{2},x_2\right)\right) : x \in \mathscr{U}\right\}; \quad (19)$$

$$?A = \left\{\left(x, \min\left(\frac{1}{2},x_1\right), \max\left(\frac{1}{2},x_2\right)\right) : x \in \mathscr{U}\right\}. \quad (20)$$

Theorem 5. *(Atanassov, 1999, Theorema 1.97) Let A and B be IFS, the following holds:*

$$!A = \overline{?\overline{A}}, \qquad !?A = ?!A, \quad (21)$$

$$!(A\cap B) = !A \cap !B, \qquad !(A\cup B) = !A \cup !B, \quad (22)$$

$$?(A\cap B) = ?A \cap ?B, \qquad ?(A\cup B) = ?A \cup ?B. \quad (23)$$

Theorem 6. *(Atanassov, 1999, Theorema 1.98)For every IFSs A, the following properties are verified:*

$$\Box !A = !\Box A, \qquad \Box ?A = ?\Box A, \quad (24)$$

$$\Diamond !A = !\Diamond A, \qquad \Diamond ?A = ?\Diamond A. \quad (25)$$

3.4 Conjugate Operators

In (Reiser and Bedregal, 2017, Def.1), a function $\Phi : \tilde{U} \to \tilde{U}$ is an intuitionistic fuzzy automorphism (A-IFA) on $\tilde{U}$ if Φ is a bijective and non-decreasing function, $\tilde{x} \leq_{\tilde{U}} \tilde{y} \Leftrightarrow \Phi(\tilde{x}) \leq_{\tilde{U}} \Phi(\tilde{y})$.

$Aut(\tilde{U})$ denotes the set of all A-IFA, extending the notion of a fuzzy automorphism $\phi : U \to U$ in $Aut(U)$.

And, the action of $\Phi : \tilde{U} \to \tilde{U}$ on $f_I : \tilde{U}^n \to \tilde{U}$ is a function $f_I^\Phi : \tilde{U} \to \tilde{U}$ called intuitionistic conjugate (A-IFA) of f_I and defined as follows:

$$f_I^\Phi(\tilde{x}_1, \ldots, \tilde{x}_n) = \Phi^{-1}(f_I(\Phi(\tilde{x}_1), \ldots, \Phi(\tilde{x}_n))). \quad (26)$$

Now, the ϕ-representability of an A-IFA is reported:

Proposition 7. *(Reiser and Bedregal, 2017, Prop. 5) Let $\phi \in Aut(U)$ and $\phi \in Aut(U)$. The a ϕ-representable A-IFA $\Phi \in Aut(\tilde{U})$ is defined as follows:*

$$\Phi(x_1,x_2) = (\phi(x_1), 1-\phi(1-x_2)). \quad (27)$$

Proposition 8. *(Bertei and Reiser, 2018, Proposition IV.16) Consider a Φ-representable automorphism given by Eq.(27) and $\Box$A-IFS and $\Diamond$A-IFS given by Eqs. (10) and (11), respectively. Then, for all $\tilde{x} = (x_1,x_2) \in \tilde{U}$ the following holds:*

$$(\Box)^{\Phi}(\tilde{x}) = \Box(x_1, 1-x_1); \quad (28)$$

$$(\Diamond)^{\Phi}(\tilde{x}) = \Diamond(1-x_2, x_2). \quad (29)$$

4 CORRELATION FROM A-IFL

Using denotation related to Eqs. (2), (3)a and (3)b:

$$(\mu_A(x_1), \mu_A(x_2), \ldots, \mu_A(x_n)) = (x_{11}, x_{21}, \ldots, x_{n1}) = \mathbf{x}_{i1};$$
$$(\nu_A(x_1), \nu_A(x_2), \ldots, \nu_A(x_n)) = (x_{12}, x_{22}, \ldots, x_{n2}) = \mathbf{x}_{i2};$$
$$(\pi_A(x_1), \pi_A(x_2), \ldots, \pi_A(x_n)) = (x_{13}, x_{23}, \ldots, x_{n3}) = \mathbf{x}_{i3}.$$

and the two corresponding classes of the quasi-arithmetic means are reported below:
(i) the arithmetic mean related to an A-IFS A, given as follow:
$m(\mathbf{x}_{i1}) = \frac{1}{n}\sum_{i=1}^{n} x_{i1}$; $m(\mathbf{x}_{i2}) = \frac{1}{n}\sum_{i=1}^{n} x_{i2}$; $m(\mathbf{x}_{i3}) = \frac{1}{n}\sum_{i=1}^{n} x_{i3}$.
(ii) the quadratic mean, performed over the difference between each intuitionistic fuzzy value of an A-IFS A and the corresponding arithmetic mean of all its values are described in the following:

$$m_2(\mathbf{x}_{i1}) = \sqrt{\sum_{i=1}^{n}\left(x_{i1} - \frac{1}{n}\sum_{j=1}^{n} x_{j1}\right)^2};$$
$$m_2(\mathbf{x}_{i2}) = \sqrt{\sum_{i=1}^{n}\left(x_{i2} - \frac{1}{n}\sum_{i=j}^{n} x_{j2}\right)^2};$$
$$m_2(\mathbf{x}_{i3}) = \sqrt{\sum_{i=1}^{n}\left(x_{i3} - \frac{1}{n}\sum_{j=1}^{n} x_{j3}\right)^2}.$$

Thus, the quotient between product values obtained by taking two sums performed over such classes of quasi-arithmetic means extending the coefficient correlation definition to the Atanassov-intuitionistic fuzzy approach.

Definition 9. *(Szmidt and Kacprzyk, 2012) The A-CC between A and B in $\mathscr{C}(A)$ is given as follows:*

$$\mathbf{C}(A,B) = \frac{1}{3}(C_1(A,B) + C_2(A,B) + C_3(A,B)) \quad (30)$$

wherever the following holds:

$$C_1(A,B) = \frac{\sum_{i=1}^{n}\left(x_{i1} - \frac{1}{n}\sum_{j=1}^{n} x_{j1}\right)\left(y_{i1} - \frac{1}{n}\sum_{j=1}^{n} y_{j1}\right)}{\sqrt{\sum_{i=1}^{n}\left(x_{i1} - \frac{1}{n}\sum_{j=1}^{n} x_{j1}\right)^2 \sum_{i=1}^{n}\left(y_{i1} - \frac{1}{n}\sum_{j=1}^{n} y_{j1}\right)^2}}$$

$$C_2(A,B) = \frac{\sum_{i=1}^{n}\left(x_{i2} - \frac{1}{n}\sum_{j=1}^{n} x_{j2}\right)\left(y_{i2} - \frac{1}{n}\sum_{j=1}^{n} y_{j2}\right)}{\sqrt{\sum_{i=1}^{n}\left(x_{i2} - \frac{1}{n}\sum_{j=1}^{n} x_{j2}\right)^2 \sum_{i=1}^{n}\left(y_{i2} - \frac{1}{n}\sum_{j=1}^{n} y_{j2}\right)^2}}$$

$$C_3(A,B) = \frac{\sum_{i=1}^{n}\left(x_{i3} - \frac{1}{n}\sum_{j=1}^{n} x_{j3}\right)\left(y_{i3} - \frac{1}{n}\sum_{j=1}^{n} y_{j3}\right)}{\sqrt{\sum_{i=1}^{n}\left(x_{i3} - \frac{1}{n}\sum_{j=1}^{n} x_{j3}\right)^2 \sum_{i=1}^{n}\left(y_{i3} - \frac{1}{n}\sum_{j=1}^{n} y_{j3}\right)^2}}$$

In (Szmidt and Kacprzyk, 2012), the correlation coefficient $\mathbf{C}(A,B)$ in Eq. (30) considers both factors:
(i) the amount of information expressed by the membership and non-membership degrees expressed by $C_1(A,B)$ and $C_2(A,B)$, respectively; and
(ii) the reliability of information expressed by the hesitation margins in $C_3(A,B)$.

Additionally, for fuzzy data, these expressions just make sense for A-IFS variables whose values vary and avoid zero in the denominator. Moreover, $\mathbf{C}(A,B)$ fulfils the following properties:

(i) $\mathbf{C}(A,B) = \mathbf{C}(A,B)$;

(ii) If $A = B$ then $\mathbf{C}(A,B) = 1$;

(iii) $-1 \leq \mathbf{C}(A,B) \leq 1$.

Proposition 10. *(Bertei et al., 2016, Prop.1) Let N be a strong A-IFNs, A and B be A-IFSs and $\overline{A}$ and $\overline{B}$ be their corresponding complements. The following holds:*

$$C_1(A,\overline{B}) = C_2(\overline{A},B); \quad (31)$$
$$C_2(A,\overline{B}) = C_1(\overline{A},B); \quad (32)$$
$$C_3(A,\overline{B}) = C_3(\overline{A},B). \quad (33)$$

Corollary 11. *(Bertei et al., 2016, Corollary.1) Let N be a strong A-IFNs, A and B are A-IFSs and $\overline{A}$ and $\overline{B}$ be their corresponding complements. The following holds:*

$$\mathbf{C}(A,\overline{B}) = \mathbf{C}(\overline{A},B). \quad (34)$$

5 RESULTS ON CONJUGATE MODAL LEVEL OPERATORS

In this section, dual and conjugate operators are preserved by modal α-level operators.

Proposition 12. *Consider a Φ-representable automorphism given by Eq. (27) and K_α-IFS and L_α-IFS given by Eqs. (17) and (18), respectively. For all $\tilde{x} = (x_1, x_2) \in \tilde{U}$, the following holds:*

$$(K_\alpha)^\Phi(\tilde{x}) = (\phi^{-1}(\max(\alpha, \phi(x_1))), 1-\phi^{-1}(1-\min(\alpha, 1-\phi(1-x_2)))); \quad (35)$$

$$(L_\alpha)^\Phi(\tilde{x}) = (\phi^{-1}(\min(\alpha, \phi(x_1))), 1-\phi^{-1}(1-\max(\alpha, 1-\phi(1-x_2)))). \quad (36)$$

Proof. For all $\tilde{x} = (x_1, x_2) \in \tilde{U}$, we have that

$(K_\alpha)^\Phi(\tilde{x}) =$

$= \Phi^{-1}(K_\alpha(\Phi(x_1, x_2)))$ by Eq.(26)

$= \Phi^{-1}(\max(\alpha, \phi(x_1)), \min(\alpha, 1-\phi(1-x_2)))$ by Eq.(27)

$= (\phi^{-1}(\max(\alpha, \phi(x_1))), 1-\phi^{-1}(1-\min(\alpha, 1-\phi(1-x_2))))$ by Eq.(27)

Analogously, Eq.(36) can be proved. Therefore, Proposition 12 is verified. □

Corollary 13. *Consider a Φ-representable automorphism given by Eq. (27) and !A-IFS and ?A-IFS given by Eqs. (19) and (20), respectively. For all $\tilde{x} = (x_1, x_2) \in \tilde{U}$, the following holds:*

$$(!)^\Phi(\tilde{x}) = \left(\phi^{-1}\left(\max\left(\frac{1}{2}, \phi(x_1)\right)\right), 1-\phi^{-1}\left(1-\min\left(\frac{1}{2}, 1-\phi(1-x_2)\right)\right)\right);$$

$$(?)^\Phi(\tilde{x}) = \left(\phi^{-1}\left(\min\left(\frac{1}{2}, \phi(x_1)\right)\right), 1-\phi^{-1}\left(1-\max\left(\frac{1}{2}, 1-\phi(1-x_2)\right)\right)\right).$$

Proposition 14. *Consider $\Phi \in Aut(\tilde{U})$ and K_α-IFS and L_α-IFS given by Eqs. (17) and (18), respectively. For all $\tilde{x} = (x_1, x_2) \in \tilde{U}$, the following holds:*

$$K_\alpha^\Phi(\tilde{x}) = N_{SI}^\Phi\left(L_\alpha^\Phi(\tilde{x})\right) \quad \text{and} \quad L_\alpha^\Phi(\tilde{x}) = N_{SI}^\Phi\left(K_\alpha^\Phi(\tilde{x})\right).$$

Proof. For all $\tilde{x} \in \tilde{U}$, the results below are verified:

$N_{SI}^\Phi\left(L_\alpha^\Phi(\tilde{x})\right) = N_{SI}^\Phi(\Phi^{-1}(L_\alpha(\Phi(x_1, x_2))))$ by Eq.(26)

$= \Phi^{-1}(N_{SI}(L_\alpha(\Phi(x_1, x_2))))$ by Eq.(26)

$= \Phi^{-1}(N_{SI}(L_\alpha(\phi(x_1), 1-\phi(1-x_2))))$ by Eq.(27)

$= \Phi^{-1}(N_{SI}((\min(\alpha, \phi(x_1)), \max(\alpha, 1-\phi(1-x_2))))$ by Eq.(36)

$= \Phi^{-1}(\max(\alpha, 1-\phi(1-x_2)), \min(\alpha, \phi(x_1))))$ by Eq.(6)

$= (\phi^{-1}(\max(\alpha, \phi(x_1)), 1-\phi^{-1}(1-\min(\alpha, 1-\phi(1-x_2))))$ by Eq.(27)

$= K_\alpha^\Phi(\tilde{x})$, by Eq.(35).

Since N_{SI} is a strong IFN, the other equation can be straightforward proved. So, Prop. 14 is verified. □

6 RESULTS ON A-CC AND MODAL LEVEL OPERATORS

This section studies main results of A-CC related to A-IFS, $!A$-IFS, $?A$-IFS, $\Box A$-IFS and $\Diamond A$-IFS. For that, consider $i \in \mathbb{N}_n$, $k \in \mathbb{N}_3$ and the notations below:

$$\alpha_{ik} = min\left(\frac{1}{2}, x_{ik}\right), \qquad \beta_{ik} = max\left(\frac{1}{2}, x_{ik}\right).$$

Proposition 15. *The A-CC between A-IFS A and ?A-IFS is given as*

$$\mathbf{C}(A, ?A) = \frac{1}{3}\left(C_1(A, ?A) + C_2(A, ?A) + C_3(A, ?A)\right) \quad (37)$$

whenever the following holds:

$$C_1(A, ?A) = \frac{\sum_{i=1}^{n}\left(x_{i1} - \frac{1}{n}\sum_{j=1}^{n} x_{j1}\right)\left(\alpha_{i1} - \frac{1}{n}\sum_{j=1}^{n} \alpha_{j1}\right)}{\sqrt{\sum_{i=1}^{n}\left(x_{i1} - \frac{1}{n}\sum_{j=1}^{n} x_{j1}\right)^2 \sum_{i=1}^{n}\left(\alpha_{i1} - \frac{1}{n}\sum_{j=1}^{n} \alpha_{j1}\right)^2}}$$

$$C_2(A, ?A) = \frac{\sum_{i=1}^{n}\left(x_{i2} - \frac{1}{n}\sum_{j=1}^{n} x_{j2}\right)\left(\beta_{i2} - \frac{1}{n}\sum_{j=1}^{n} \beta_{j2}\right)}{\sqrt{\sum_{i=1}^{n}\left(x_{i2} - \frac{1}{n}\sum_{j=1}^{n} x_{j2}\right)^2 \sum_{i=1}^{n}\left(\beta_{i2} - \frac{1}{n}\sum_{j=1}^{n} \beta_{j2}\right)^2}}$$

$$C_3(A, ?A) = \frac{\sum_{i=1}^{n}\left(x_{i3} - \frac{1}{n}\sum_{j=1}^{n} x_{j3}\right)\left(\alpha_{i1} + \beta_{i2} - \frac{1}{n}\sum_{j=1}^{n} \alpha_{j1} + \beta_{j2}\right)}{\sqrt{\sum_{i=1}^{n}\left(x_{i3} - \frac{1}{n}\sum_{j=1}^{n} x_{j3}\right)^2 \sum_{i=1}^{n}\left(\alpha_{i1} + \beta_{i2} - \frac{1}{n}\sum_{j=1}^{n} \alpha_{j1} + \beta_{j2}\right)^2}}$$

Proof. Let A-IFS A and $?A$-IFS given by Eqs.(1) and (20), respectively. $C_1(A, ?A)$ and $C_2(A, ?A)$ follow from (11) and (30). And, the related resultant margin to C_3 is given as follows:

$C_3(A, ?A) =$

$$= \frac{\sum_{i=1}^{n}\left(x_{i3} - \frac{1}{n}\sum_{j=1}^{n} x_{j3}\right)\left(1 - \alpha_{i1} - \beta_{i2} - \frac{1}{n}\sum_{j=1}^{n} 1 - \alpha_{j1} - \beta_{j2}\right)}{\sqrt{\sum_{i=1}^{n}\left(x_{i3} - \frac{1}{n}\sum_{j=1}^{n} x_{j3}\right)^2 \sum_{i=1}^{n}\left(1 - \alpha_{i1} - \beta_{i2} - \frac{1}{n}\sum_{j=1}^{n} 1 - \alpha_{j1} - \beta_{j2}\right)^2}}$$

$$= \frac{\sum_{i=1}^{n}\left(x_{i3} - \frac{1}{n}\sum_{j=1}^{n} x_{j3}\right)\left(\alpha_{i1} + \beta_{i2} - \frac{1}{n}\sum_{j=1}^{n} \alpha_{j1} + \beta_{j2}\right)}{\sqrt{\sum_{i=1}^{n}\left(x_{i3} - \frac{1}{n}\sum_{j=1}^{n} x_{j3}\right)^2 \sum_{i=1}^{n}\left(\alpha_{i1} + \beta_{i2} - \frac{1}{n}\sum_{j=1}^{n} \alpha_{j1} + \beta_{j2}\right)^2}}$$

Therefore, Proposition 15 is verified. □

Proposition 16. *The A-CC between A-IFS and $\overline{?A}$-IFS is given as follows:*

$$\mathbf{C}(A,\overline{?A}) = \frac{1}{3}\left(C_1(A,\overline{?A})+C_2(A,\overline{?A})+C_3(A,\overline{?A})\right) \quad (38)$$

whenever the following holds

$$C_1(A,\overline{?A}) = \frac{\sum_{i=1}^{n}\left(x_{i1}-\frac{1}{n}\sum_{j=1}^{n}x_{j1}\right)\left(\beta_{i2}-\frac{1}{n}\sum_{j=1}^{n}\beta_{j2}\right)}{\sqrt{\sum_{i=1}^{n}\left(x_{i1}-\frac{1}{n}\sum_{j=1}^{n}x_{j1}\right)^2\sum_{i=1}^{n}\left(\beta_{i2}-\frac{1}{n}\sum_{j=1}^{n}\beta_{j2}\right)^2}}$$

$$C_2(A,\overline{?A}) = \frac{\sum_{i=1}^{n}\left(x_{i2}-\frac{1}{n}\sum_{j=1}^{n}x_{j2}\right)\left(\alpha_{i1}-\frac{1}{n}\sum_{j=1}^{n}\alpha_{j1}\right)}{\sqrt{\sum_{i=1}^{n}\left(x_{i2}-\frac{1}{n}\sum_{j=1}^{n}x_{j2}\right)^2\sum_{i=1}^{n}\left(\alpha_{i1}-\frac{1}{n}\sum_{j=1}^{n}\alpha_{j1}\right)^2}}$$

$$C_3(A,\overline{?A}) = \frac{\sum_{i=1}^{n}\left(x_{i3}-\frac{1}{n}\sum_{j=1}^{n}x_{j3}\right)\left(\beta_{i2}+\alpha_{i1}-\frac{1}{n}\sum_{j=1}^{n}\beta_{j2}+\alpha_{j1}\right)}{\sqrt{\sum_{i=1}^{n}\left(x_{i3}-\frac{1}{n}\sum_{j=1}^{n}x_{j3}\right)^2\sum_{i=1}^{n}\left(\beta_{i2}+\alpha_{i1}-\frac{1}{n}\sum_{j=1}^{n}\beta_{j2}+\alpha_{j1}\right)^2}}$$

Proof. $C_1(A,\overline{?A})$ and $C_2(A,\overline{?A})$ follows from Eqs. (1), (8), (20) and (30). $C_3(A,\overline{?(A)})$ is given as follows:

$$C_3(A,\overline{?(A)}) =$$

$$= \frac{\sum_{i=1}^{n}\left(x_{i3}-\frac{1}{n}\sum_{j=1}^{n}x_{j3}\right)\left(1-\beta_{i2}-\alpha_{i1}-\frac{1}{n}\sum_{j=1}^{n}1-\beta_{j2}-\alpha_{j1}\right)}{\sqrt{\sum_{i=1}^{n}\left(x_{i3}-\frac{1}{n}\sum_{j=1}^{n}x_{j3}\right)^2\sum_{i=1}^{n}\left(1-\beta_{i2}-\alpha_{i1}-\frac{1}{n}\sum_{j=1}^{n}1-\beta_{j2}-\alpha_{j1}\right)^2}}$$

$$= \frac{\sum_{i=1}^{n}\left(x_{i3}-\frac{1}{n}\sum_{j=1}^{n}x_{j3}\right)\left(\beta_{i2}+\alpha_{i1}-\frac{1}{n}\sum_{j=1}^{n}\beta_{j2}+\alpha_{j1}\right)}{\sqrt{\sum_{i=1}^{n}\left(x_{i3}-\frac{1}{n}\sum_{j=1}^{n}x_{j3}\right)^2\sum_{i=1}^{n}\left(\beta_{i2}+\alpha_{i1}-\frac{1}{n}\sum_{j=1}^{n}\beta_{j2}+\alpha_{j1}\right)^2}}$$

Thus, Proposition 16 is also verified. □

Proposition 17. *Let ?A-IFS and !A-IFS given by Eqs.(20) and (19), respectively. The following holds:*

$$\mathbf{C}(A,!\overline{A}) = \mathbf{C}(A,\overline{?A}) \quad (39)$$

Proof. By Eq.(30) we have that:

$$C_1(A,!\overline{A}) = \frac{\sum_{i=1}^{n}\left(x_{i1}-\frac{1}{n}\sum_{j=1}^{n}x_{j1}\right)\left(\beta_{i2}-\frac{1}{n}\sum_{j=1}^{n}\beta_{j2}\right)}{\sqrt{\sum_{i=1}^{n}\left(x_{i1}-\frac{1}{n}\sum_{j=1}^{n}x_{j1}\right)^2\sum_{i=1}^{n}\left(\beta_{i2}-\frac{1}{n}\sum_{j=1}^{n}\beta_{j2}\right)^2}}$$

$$= C_1(A,\overline{?A})$$

$$C_2(A,!\overline{A}) = \frac{\sum_{i=1}^{n}\left(x_{i2}-\frac{1}{n}\sum_{j=1}^{n}x_{j2}\right)\left(\alpha_{i1}-\frac{1}{n}\sum_{j=1}^{n}\alpha_{j1}\right)}{\sqrt{\sum_{i=1}^{n}\left(x_{i2}-\frac{1}{n}\sum_{j=1}^{n}x_{j2}\right)^2\sum_{i=1}^{n}\left(\alpha_{i1}-\frac{1}{n}\sum_{j=1}^{n}\alpha_{j1}\right)^2}}$$

$$= C_2(A,\overline{?A})$$

$$C_3(A,!\overline{A}) =$$

$$= \frac{\sum_{i=1}^{n}\left(x_{i3}-\frac{1}{n}\sum_{j=1}^{n}x_{j3}\right)\left(1-\beta_{i2}-\alpha_{i1}-\frac{1}{n}\sum_{j=1}^{n}1-\beta_{j2}-\alpha_{j1}\right)}{\sqrt{\sum_{i=1}^{n}\left(x_{i3}-\frac{1}{n}\sum_{j=1}^{n}x_{j3}\right)^2\sum_{i=1}^{n}\left(1-\beta_{i2}-\alpha_{i1}-\frac{1}{n}\sum_{j=1}^{n}1-\beta_{j2}-\alpha_{j1}\right)^2}}$$

$$= C_3(A,\overline{?A})$$

Therefore, Proposition (17) is verified. □

Corollary 18. *Let $A-IFS$ A, $?A-IFS$ and $!A-IFS$ given as Eqs. (1), (20) and (19), respectively. Based on their NS-dual constructions, the following holds:*

$$\mathbf{C}(A,!\overline{A}) \overset{Eq.(34)}{=} \mathbf{C}(\overline{A},!A) \overset{Eq.(21)a}{=} \mathbf{C}(\overline{A},\overline{?\overline{A}}). \quad (40)$$

Proof. It follows from Propositions (10) and (17) also considering results from above propositions. □

Proposition 19. *Let A-IFS A, ?A-IFS and $\Diamond$A-IFS given by Eqs. (1), (20) and (11) respectively. The following holds:*

$$\mathbf{C}(A,\Diamond ?\overline{A}) = \frac{1}{3}\left(C_1(A,\Diamond ?\overline{A})+C_2(A,\Diamond ?\overline{A})\right), \quad (41)$$

whenever the following holds

$$C_1(A,\Diamond ?\overline{A}) = (-1)\frac{\sum_{i=1}^{n}\left(x_{i1}-\frac{1}{n}\sum_{j=1}^{n}x_{j1}\right)\left(\beta_{i1}-\frac{1}{n}\sum_{j=1}^{n}\beta_{j1}\right)}{\sqrt{\sum_{i=1}^{n}\left(x_{i1}-\frac{1}{n}\sum_{j=1}^{n}x_{j1}\right)^2\sum_{i=1}^{n}\left(\beta_{i1}-\frac{1}{n}\sum_{j=1}^{n}\beta_{j1}\right)^2}}$$

$$C_2(A,\Diamond ?\overline{A}) = \frac{\sum_{i=1}^{n}\left(x_{i2}-\frac{1}{n}\sum_{j=1}^{n}x_{j2}\right)\left(\beta_{i1}-\frac{1}{n}\sum_{j=1}^{n}\beta_{j1}\right)}{\sqrt{\sum_{i=1}^{n}\left(x_{i2}-\frac{1}{n}\sum_{j=1}^{n}x_{j2}\right)^2\sum_{i=1}^{n}\left(\beta_{i1}-\frac{1}{n}\sum_{j=1}^{n}\beta_{j1}\right)^2}}$$

Proof. Straightforward. □

Corollary 20. *Let $A-IFS$ A, $?A-IFS$ and $\Diamond A-IFS$ given as Eqs. (1), (20) and (11), respectively. Then the following holds:*

$$\mathbf{C}(A,\Diamond ?\overline{A}) \overset{Eq.(13)b}{=} \mathbf{C}(A,\Box\Diamond ?\overline{A}) \overset{Eq.(34)}{=} \mathbf{C}(\overline{A},\Box\Diamond ?A);$$

$$\mathbf{C}(A,\Diamond ?\overline{A}) \overset{Eq.(34)}{=} \mathbf{C}(\overline{A},\Diamond ?A) \overset{Eq.(14)b}{=} \mathbf{C}(\overline{A},\Diamond\Diamond ?A).$$

Proof. It results from Propositions 2, 10 and 19. □

Proposition 21. *Let A be an $A-IFS$. The correlation between IFS A and $\Box!A$-IFS is given as*

$$\mathbf{C}(A,\Box !A) = -\mathbf{C}\left(A,\Diamond ?\overline{A}\right), \tag{42}$$

Proof. Straightforward. □

Corollary 22. *Let $A-IFS$ A, $!A-IFS$ and $\Diamond A-IFS$ given as Eqs. (1), (19), and (11) respectively. Then the following holds:*

$$\mathbf{C}(A,\Box !A) \overset{Eq.(21)}{=} \mathbf{C}(A,\overline{\Box ?\overline{A}}) \overset{Eq.(34)}{=} \mathbf{C}(\overline{A},\Box ?\overline{A}). \tag{43}$$

$$\mathbf{C}(A,\Box !A) \overset{Eq.(13)a}{=} \mathbf{C}(A,\Box\Box !A) \overset{Eq.(14)a}{=} \mathbf{C}(A,\Diamond\Box\Box !A). \tag{44}$$

Proof. It results from Propositions 2, 10 and 21. □

Proposition 23. *Let $?A$-IFS, $!A$-IFS, $\Diamond A$-IFS and $\Box A$-IFS given by Eqs. (20), (19), (11) and (10) respectively. The following holds:*

$$\mathbf{C}(\Box ?A,\Diamond !A) = \frac{2}{3}\left(C_1(\Box ?A,\Diamond !A)\right), \tag{45}$$

whenever the following holds

$$C_1(\Box ?A,\Diamond !A) = (-1)\frac{\sum_{i=1}^{n}\left(\alpha_{i1}-\frac{1}{n}\sum_{j=1}^{n}\alpha_{j1}\right)\left(\alpha_{i2}-\frac{1}{n}\sum_{j=1}^{n}\alpha_{j2}\right)}{\sqrt{\sum_{i=1}^{n}\left(\alpha_{i1}-\frac{1}{n}\sum_{j=1}^{n}\alpha_{j1}\right)^2\sum_{i=1}^{n}\left(\alpha_{i2}-\frac{1}{n}\sum_{j=1}^{n}\alpha_{j2}\right)^2}}$$

Proof. By Equations. (20), (19), (11), (10) and (30) we have the following results:

$$C_1(\Box ?A,\Diamond !A) =$$

$$= \frac{\sum_{i=1}^{n}\left(\alpha_{i1}-\frac{1}{n}\sum_{j=1}^{n}\alpha_{j1}\right)\left(1-\alpha_{i2}-\frac{1}{n}\sum_{j=1}^{n}1-\alpha_{j2}\right)}{\sqrt{\sum_{i=1}^{n}\left(\alpha_{i1}-\frac{1}{n}\sum_{j=1}^{n}\alpha_{j1}\right)^2\sum_{i=1}^{n}\left(1-\alpha_{i2}-\frac{1}{n}\sum_{j=1}^{n}1-\alpha_{j2}\right)^2}}$$

$$= (-1)\frac{\sum_{i=1}^{n}\left(\alpha_{i1}-\frac{1}{n}\sum_{j=1}^{n}\alpha_{j1}\right)\left(\alpha_{i2}-\frac{1}{n}\sum_{j=1}^{n}\alpha_{j2}\right)}{\sqrt{\sum_{i=1}^{n}\left(\alpha_{i1}-\frac{1}{n}\sum_{j=1}^{n}\alpha_{j1}\right)^2\sum_{i=1}^{n}\left(\alpha_{i2}-\frac{1}{n}\sum_{j=1}^{n}\alpha_{j2}\right)^2}}$$

$$= C_2(\Box ?A,\Diamond !A)$$

Since $C_3(\Box ?A,\Diamond !A) = 0$, Prop. 23 is verified. □

Corollary 24. *Let $?A-IFS$, $!A-IFS$, $\Diamond A-IFS$ and $\Box A-IFS$ given as Eqs. (20), (19), (11) and (10), respectively. Then the following holds:*

$$\mathbf{C}(\Box ?A,\Diamond !A) \overset{Eq.(21)}{=} \mathbf{C}(\Box ?A,\overline{\Diamond ?\overline{A}}) \overset{Eq.(34)}{=} \mathbf{C}(\overline{\Box ?A},\Diamond ?\overline{A}).$$

$$\mathbf{C}(\Box ?A,\Diamond !A) \overset{Eq.(13)a}{=} \mathbf{C}(\Box\Box ?A,\Diamond !A).$$

Proof. It follows from Propositions 10, 2 and 23. □

Proposition 25. *Let $?A$-IFS, $!A$-IFS, $\Diamond A$-IFS and $\Box A$-IFS given by Eqs. (20), (19), (11) and (10) respectively. The following holds:*

$$\mathbf{C}(\Box ?A,\overline{\Diamond !A}) = -\frac{2}{3}\left(C_1(\Box ?A,\Diamond !A)\right),$$

Proof. It follows from Proposition 23. □

Corollary 26. *Let $?A-IFS$, $!A-IFS$, $\Diamond A-IFS$ and $\Box A-IFS$ given as Eqs. (20), (19), (11) and (10), respectively. Then the following holds:*

$$\mathbf{C}(\Box ?A,\overline{\Diamond !A}) \overset{Eq.(34)}{=} \mathbf{C}(\overline{\Box ?A},\Diamond !A) \overset{Eq.(21)}{=} \mathbf{C}(\overline{\Box ?A},\overline{\Diamond ?\overline{A}}).$$

Proof. It follows from Propositions 2, 10 and 25. □

Proposition 27. *For an A-IFS A, we have that:*

$$\mathbf{C}(\overline{\Box ?A},\overline{\Diamond !A}) = \frac{2}{3}\left(C_1(\Box ?A,\Diamond !A)\right). \tag{46}$$

Proof. It follows from Prop. 23 and Corollary 6. □

An application considering the previous theoretical results is presented in the following.

7 MADM - MEDICAL DIAGNOSIS

Previous analytical expressions of modal operator A-CC are applied in developing a method to medical diagnosis (MADM-MD) which is adapted from (Xu, 2006)) related to a medical knowledge base, providing a proper diagnosis D = {*Viral fever (VF) , Malaria (Ma) , Typhoid (Ty) , Stomach problem (SP) , Chest problem (CP)*} for a patient with the given symptoms S = {*temperature (T), headache (H), stomach pain (SPa), cough (C), chest pain (CPa)*} described in terms of A-IFSs. Two methodologies are applied:
(i) the former uses necessity and possibility modal operators in MADM-MD; and
(ii) the latter extends the method in order to apply the modal type operators ?A and !A.

The possibility modal-operator (◇) in the Eq. (11) and necessity modal-operator □ in Eq. (10) together with the operator (□?) are applied to values from Table 1 (T1) in (Xu, 2006), resulting in values of Table 2. In addition, each symptom is described by its related membership and non-membership degrees.

The necessity modal operator (□) and possibility modal-operator (◇) along with modal-level operators (!) and (◇!) are applied to values of Table 2 (T2) in (Xu, 2006), and symptom results are described in Table 3. The set of patients is P = {*Al, Bob, Joe,*

$Ted\}$. Furthermore, we need to seek a diagnosis for each patient p_i, for $i = 1,2,3,4$.

Table 2: Symptoms characteristic for the diagnoses.

	Op	*VF*	*Ma*	*Ty*	*SP*	*CP*
T	$\Diamond T1$	(1,0)	(1,0)	(0.7,0.3)	(0.3,0.7)	(0.2,0.8)
	$\Box ?T1$	(0.4,0.6)	(0.5,0.5)	(0.3,0.7)	(0.1,0.9)	(0.1,0.9)
H	$\Diamond T1$	(0.5,0.5)	(0.4,0.6)	(0.9,0.1)	(0.6,0.4)	(0.2,0.8)
	$\Box ?T1$	(0.3,0.7)	(0.2,0.8)	(0.5,0.5)	(0.2,0.8)	(0,1)
SPa	$\Diamond T1$	(0.3,0.7)	(0.1,0.9)	(0.3,0.7)	(1,0)	(0.2,0.8)
	$\Box ?T1$	(0.1,0.9)	(0,1)	(0.2,0.8)	(0.5,0.5)	(0.2,0.8)
C	$\Diamond T1$	(0.7,0.3)	(1,0)	(0.4,0.6)	(0.3,0.7)	(0.2,0.8)
	$\Box ?T1$	(0.4,0.6)	(0.5,0.5)	(0.2,0.8)	(0.2,0.8)	(0.2,0.8)
CPa	$\Diamond T1$	(0.3,0.7)	(0.2,0.8)	(0.1,0.9)	(0.3,0.7)	(0.9,0.1)
	$\Box ?T1$	(0.1,0.9)	(0.1,0.9)	(0.1,0.9)	(0.2,0.8)	(0.5,0.5)

Table 3: Symptoms characteristic for the patient.

	Op	*T*	*H*	*SPa*	*C*	*CPa*
Al	$\Box T2$	(0.8,0.2)	(0.6,0.4)	(0.2,0.8)	(0.6,0.4)	(0.9,0.1)
	$\Diamond !T2$	(0.9,0.1)	(0.9,0.1)	(0.5,0.5)	(0.9,0.1)	(0.5,0.5)
Bob	$\Box T2$	(0,1)	(0.4,0.6)	(0.6,0.4)	(0.1,0.9)	(0.1,0.9)
	$\Diamond !T2$	(0.5,0.5)	(0.6,0.4)	(0.9,0.1)	(0.5,0.5)	(0.5,0.5)
Joe	$\Box T2$	(0.8,0.2)	(0.8,0.2)	(0,1)	(0.2,0.8)	(0,1)
	$\Diamond !T2$	(0.9,0.1)	(0.9,0.1)	(0.5,0.5)	(0.5,0.5)	(0.5,0.5)
Ted	$\Box T2$	(0.6,0.4)	(0.5,0.5)	(0.3,0.7)	(0.7,0.3)	(0.3,0.7)
	$\Diamond !T2$	(0.9,0.1)	(0.6,0.4)	(0.6,0.4)	(0.8,0.2)	(0.6,0.4)

We calculate the A-CC in Eq. (30) between Tables 2 and 3 deriving a diagnosis for each patient p_i, for i = 1,2,3,4. In the first step, the method is performed applying the A-CC between the operators $\Diamond T1$ and $\Box T2$. And, in the second one, the method uses the operators $\Box ?T1$ and $\Diamond !T2$ deriving the related A-CC. All the results for the considered patients are listed in Table 4.

Table 4: Resulting A-CC of symptoms for each patient.

	A-CC	*VF*	*Ma*	*Ty*	*SP*	*CP*
Al	$(\Diamond T1, \Box T2)$	**0,610**	0,568	0,538	-0,255	-0,452
	$(\Box ?T1, \Diamond !T2)$	**0,642**	0,555	0,441	-0,441	-0,448
Bob	$(\Diamond T1, \Box T2)$	-0,430	-0,472	0,029	**0,646**	-0,208
	$(\Box ?T1, \Diamond !T2)$	-0,381	-0,460	0	**0,634**	-0,103
Joe	$(\Diamond T1, \Box T2)$	0,455	0,326	**0,632**	-0,158	-0,327
	$(\Box ?T1, \Diamond !T2)$	0,361	0,238	**0,562**	-0,361	-0,488
Ted	$(\Diamond T1, \Box T2)$	0,553	**0,632**	0,344	-0,363	-0,375
	$(\Box ?T1, \Diamond !T2)$	0,544	**0,614**	0	-0,389	-0,189

Based on the arguments in Table 4, for both methods a proper diagnosis coincides as follows: Al suffers from Viral fever, Bob from a stomach problem, Joe from Typhoid, and Ted from Malaria. Additionally, one can observe that in (Xu, 2006)), the diagnosis is the same in two patients (Bob and Joe) and it is reverse in other two (Al and ted). Despite the A-CC expression used in (Xu, 2006, Definition 3.1)) is able to preserve the property on which any two IFSs equals one iff these two IFSs are the same, it does not consider the action of the IFIx π providing the measure of hesitance degree. The distinct methodologies justified the difference in the results.

8 CONCLUSION

In this paper, the analytical expressions of A-CC were considered to pairs of modal α-level operators $K_{\frac{1}{2}}$ and $L_{\frac{1}{2}}$, in particular, for $!A$ and $?A$, also including their conjugate operators. Moreover, we present an application of the A-CC with the modal operators of necessity, possibility, and related modal-level operators $?A$ and $!A$.

Further work intends to extend these studies of A-IFSs to other fuzzy connectives frequently applied to making decision based on fuzzy systems.

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