A Generalized Notion of Time for Modeling Temporal Networks

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Abstract: Most approaches for modeling and analyzing temporal networks do not explicitly discuss the underlying notion of time. In this paper, we therefore introduce a generalized notion of time for temporal networks. Our approach also allows for considering non-deterministic time and incomplete data, two issues that are often found when analyzing data-sets extracted from online social networks, for example. In order to demonstrate the consequences of our generalized notion of time, we also discuss the implications for the computation of (shortest) temporal paths in temporal networks.

1   INTRODUCTION [1]

Temporal networks are a means for modeling and analyzing the temporal dimension of complex (networked) systems. However, most of the literature on temporal networks either does not explicitly discuss the underlying notion of time or uses a rather restrictive conception of time. In this paper, we discuss a generalized notion of time allowing for possible deviations from a linear flow of time. Thereby, our approach allows for considering non-determinism and incomplete data when analyzing temporal networks (two issues that often appear when dealing with real-world data, such as data extracted from online social networks).

We use a variant of the multilayer network concept to construct temporal networks alongside our generalized temporal abstraction. In particular, we consider a temporal network, as a (temporal) sequence of networks, a notion that can easily be expressed via the well-known multilayer network concept. This approach allows to clearly separate time from other (temporally varying) attributes that are attached to the edges or vertices in a network. This strict separation, however, requires to consider multivalued path length, the implications of which will be discussed as well.

For the purposes this paper, we use the notion

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[1]We provide an extended version of this paper on our Web page. The extended version includes additional text and examples that we had to cut from the paper due to the page restrictions for the proceedings version.

of multilayer networks as specified in Definition 1.1, which understands multilayer networks as a family of graphs connected with inter-graph edges.

Definition 1.1. *A weighted multilayer network is a triple* $\mathcal{M} := \langle \mathcal{G}, \mathcal{R}, \omega \rangle$ *such that for some arbitrary set of labels* I

- $\mathcal{G} := (G_\alpha)_{\alpha \in I}$ *is a family of weighted graphs* $G_\alpha := \langle V_\alpha, E_\alpha, \omega_\alpha \rangle$ *such that* $\forall \alpha, \beta \in I\ \alpha \neq \beta$ *we have* $V_\alpha \cap V_\beta = \varnothing$;
- $\mathcal{R} := (R_{\alpha\beta})_{\alpha,\beta \in I}$ *is a family of relations, such that* $\forall \alpha, \beta \in I\ \alpha \neq \beta$ *we have* $R_{\alpha\beta} \subseteq V(G_\alpha) \times V(G_\beta)$;
- $\omega : I \times I \times E(\mathcal{M}) \to \mathbb{R}$ *is the function*

$$\omega_{\alpha\beta} := \begin{cases} \omega_\alpha(e) & \alpha = \beta \\ \mathbb{R} & otw. \end{cases}$$

We define $\mathcal{V}(\mathcal{G}) := \{V(G) \mid G \in \mathcal{G}\}$ *and* $V(\mathcal{G}) := \bigcup_{V \in \mathcal{V}(\mathcal{G})} V$. *As well as* $\mathcal{E}(\mathcal{G}) := \{E(G) \mid G \in \mathcal{G}\}$ *and* $E(\mathcal{G}) := \bigcup_{E \in \mathcal{E}(\mathcal{G})} E$. *Moreover, let* $E(\mathcal{R}) = \bigcup_{R \in \mathcal{R}} R$. *Lastly,* $V(\mathcal{M}) := V(\mathcal{G})$ *and* $E(\mathcal{M}) := E(R) \cup E(\mathcal{G})$.[2]

This is far from the only notion of multilayer networks though. For example, one other formalism defines tuples for representing the layer membership of different vertices, while yet another represents a multilayer network solely as a tensor to be understood as an analogue to the adjacency matrix of ordinary graphs. For further details, see, e.g., (De Domenico et al., 2013), (Boccaletti et al., 2014) and (Kivelä et al., 2014).

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[2]For $G := \langle V, E \rangle$ let $V(G) := V$ and $E(G) := E$

A multiplex is commonly understood as a multi-layer network where all layers share the same set of vertices, i.e. $\forall U, W \in V(\mathcal{G})\ U = W$ (Kivelä et al., 2014; Boccaletti et al., 2014). For the purposes of this paper, we relax the strong equality implied by those characterizations, as apart from some technical issues with respect to path properties, strong equality seems to be inappropriate for our purposes. For example, an object (vertex or edge) x at time t may not share the same properties as the same object (vertex or edge) x at time $t' \neq t$.

Definition 1.2. *Let $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \equiv \rangle$ be a multilayer network with equivalence such that $\forall \alpha \in I \forall v, w \in V(G_\alpha)\ v \equiv w \Rightarrow v = w$. Moreover, the equivalence classes generated by $\equiv$ are denoted as $\bar{v} := \{w \mid w \in V(\mathcal{G})\ v \equiv w\}$ and we write $\bar{v}_\alpha := v \in V_\alpha \cap \bar{v}$. Such a multilayer network with equivalence is called a multiplex iff $\forall v \in V(\mathcal{G})\ |\bar{v}| = |\mathcal{G}|$.*

Unfortunately, when modeling the progression of time, equivalence is necessary but not sufficient. For achieving the required expressiveness we have to impose an additional order onto our structure. That is, if a graph G_α precedes a graph G_β according some order $\leq$, then the vertex v_α should precede v_β.

Definition 1.3. *Let $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \equiv, \leq \rangle$ be a multilayer network with equivalence and order. If $\leq$ is a partial ordering on $\mathcal{G}$. Moreover, $G_\alpha \leq G_\beta \iff (\forall v \in G_\alpha \forall w \in G_\beta (v \equiv w \Rightarrow v \leq w) \wedge (v \not\equiv w \Rightarrow v \not\Vdash w))$.*

Having defined the fundamental language of multilayer networks used in this paper, we shall now discuss the semantics and structure of time underlying the definition of temporal networks as presented in Section 3.

2 GENERALIZED FLOW OF TIME

We abstract from the notion of a temporal graph to discuss some more general observations regarding objects in a temporal dimension. When studying the notion of time we distinguish between two classes of properties. Informally, the properties concerned with the density of time and the ones determining the structure of time. Many concepts mentioned in this section can be found in and/or build on the ideas presented by (Venema, 1998), (Markosian et al., 2016) and (Goranko and Galton, 2015) and are heavily based on Kripke semantics (Van Ditmarsch et al., 2007).

Definition 2.1. *Let $\Phi = \langle T, R, \iota \rangle$ be a structure representing the flow of time, where T is a set of points, R is a relation over T, i.e. $R \subset T \times T$ and $\iota : T \to \wp(\mathcal{L})$ is a function assigning each point in time a set of properties P described in some language $\mathcal{L}$. If the properties at a point in time are not relevant we write $\Phi = \langle T, R \rangle$.*

The function ι, corresponding to the evaluation function of Kripke models, is best understood as a function that assigns each point in time a description of the world. R simply puts the points in time into relation. For example, one can embed a graph into a point in time, by attaching some axiomatisation of the desired graph theory together with a characterization of the graph itself.

There are some interresting conceptions of how points in time may relate, e.g. cyclic or bidirectional notions of time. Hence, in its most general form R should be open to interpretation. For example, in (Taylor et al., 2017) time is conceived as an undirected, discrete flow. However, as we intend to discuss a class of specific notions of time, allowing an arbitrary relation R is cumbersome. Hence, we restrict ourself for all subsequent discussions to the set of structures building upon a directed notion of time.

Definition 2.2. *(Venema, 1998) Let $\Phi = \langle T, \leq \rangle$ be a structure representing the flow of time, where T is a set of points and $\leq$ is a partial order over T.*

For more on partially orders see (Matthews, 1994). Moreover, we write $x < y := x \leq y \wedge x \neq y$ as a shorthand.

By imposing further restriction onto $\leq$ we can develop certain notions of time. For example, a flow of time is *linear* if its underlying order is total, i.e. $\forall x, y \in T\ x \leq y \vee y \leq x$; it is *strictly linear* if $\forall x, y \in T\ x < y \vee y < x \vee x = y$; it is *backwards-branching* if for a point a representing the present there are two incomparable points in the past, i.e. $\exists x, y \in T\ x \leq a \wedge y \leq a \wedge x \not\Vdash y$; it is *forwards-branching* if for a point a representing the present there are two incomparable points in the future, i.e $\exists x, y \in T\ a \leq x \wedge a \leq y \wedge x \not\Vdash y$; it is *backwards-serial* if there is always another point in the past, i.e. $\forall x \in T \exists y \in T\ y < x$ and *forwards-serial* if there is always another point in the future, i.e. $\forall x \in T \exists y \in T\ x < y$ (Venema, 1998).

The concept of *linear time* is highly intuitive: Time flows within a straight line, there are no alternative time lines, no branching and no cycles, allowing us to work in a deterministic fashion. For example, one encounters this notion of time when dealing with normal time series. Moreover, a good part of the literature regarding temporal networks, is concerned with linear time. However, one can easily conceptualize scenarios, where we deal with some kind of uncertainty or non-determanism. Here, the notion of *possible worlds* can guide our reasoning. Kripke models, which provide the foundation for the semantics of

modal logic, heavily rely on the concept of possible worlds (Venema, 1998; Van Ditmarsch et al., 2007).

In our case, each branch represents a possible future, thereby expressing the non-determinism of the future. For example, having a linear flow of time in the past that branches into the future, expresses that we are certain what happend in the past, but we can not predict the future with certainty. How those futures are obtained precicely, may it be through statistical inference, by consulting experts with domain knowledge, or being a product of a simulation with randomness is currently not of our concern. Considering possible worlds is especially useful when dealing with discrete objects such as graphs. That is, rather than introducing fuzzy edges (Sunitha and Mathew, 2013) we can, at least for our purposes, consider alternative worlds where an edge exists and some in which it does not. Similarly, backwards-branching could introduce the notion of unreliable data into our models. For example, if there two contradicting measurements of the same phenomenon at the same instance, one models them as two incomparable elements within the flow of time. Moreover, regardless of forward- or backward-branching, we do allow for collapsing flows of time. That is, two branches could meet at some point in time. Recall the unreliable data example, as an example for the applicability of such a structure.

When discussing the density of time, three general notions are common: a flow of time can be *discrete*, *dense* or *continuous*, thus existing in analogy to $\mathbb{N}$, $\mathbb{Q}$ and $\mathbb{R}$ respectively. However, as most measurements of the real world are processed by inherently discrete machines and this paper focuses on the analysis of discrete sequences of discrete objects, we will limit ourself to a discrete conception of time. However, for analytical and predictive purposes the other two models of time, especially the continuous one, should be investigated further in the context of temporal graphs (Venema, 1998).

The restriction to discrete notions of time allows for a concrete conception of successors (we can formalize steps in time). In linear time, the successor predicate S corresponds to a bijective mapping s where $t + 1 := s(t) \in T$. However, as soon as branching is permitted, $S(a,b)$ can be satisfied by multiple elements.

Definition 2.3. *Let* $\Phi = \langle T, \preceq \rangle$ *be a flow of time, then* $t + 1 := s(t)$ *is called the (direct) successor function iff*

$$s : T \to \wp(T) \quad s(t) \mapsto \{t' \mid t' \in T\ S(t,t')\}$$

for the successor predicate $S(a,b) := a < b \wedge \forall x \in T\ x \preceq a \vee b \preceq x$. *Moreover,* $t - 1 = p(t) := \{t' \mid \forall t' \in T\ s(t') = t\}$ *is called the (direct) predecessor function*

By taking several steps we obtain a path in time.

Definition 2.4. *Let* $\Phi = \langle T, \preceq \rangle$ *be a flow of time, then*

- *a forward path in time* $a \rightsquigarrow^+ b$ *between a and b is a sequence* $a \rightsquigarrow^+ b = t_1 \rightsquigarrow^+ t_n := t_1, t_2, \ldots t_{n-1}, t_n$ *where* $\forall i \in \{1, \ldots n-1\}\ t_{i+1} \in s(t_i)$;
- *a backwards path in time* $a \rightsquigarrow^- b$ *is defined in analogue to* $a \rightsquigarrow^+ b$ *with respect to the predecessor function p.*
- *if* $a \preceq b$, *a path in time is* $a \rightsquigarrow b := a \rightsquigarrow^+ b$; *and if* $b \preceq a$, *a path in time is* $a \rightsquigarrow b := a \rightsquigarrow^- b$.

The size of a path in time is determined by the elements in the path and denoted as $|a \rightsquigarrow b|$ *and can be understood as the amount of steps in time between a and b in* $a \rightsquigarrow b$.

Notice, if $a \rightsquigarrow^- b$ there exists some $b \rightsquigarrow^+ a$ such that $a \rightsquigarrow^- b = b \rightsquigarrow^+ a$. Moreover, if $a \rightsquigarrow b$ and $b \rightsquigarrow c$ for $a \preceq b$ and $c \preceq b$, the concatenation of those two paths $a \rightsquigarrow b \rightsquigarrow c$ is no path in time, since it is neither a forwards, nor a backwards path in time. Hence, by preventing a path from changing directions there can not be a path between two branches. The concept of *size* can be generalized to obtain a notion of *length*.

Definition 2.5. *Let* $\Phi = \langle T, \preceq, \omega \rangle$ *be a flow of time with spacing, with the function*

$$\omega : \mathcal{S}(\Phi) \to \mathbb{R}^+, x \mapsto \omega(x)$$

Where $\mathcal{S}(\Phi) := \{(x,y) \mid \forall x, y \in T\ S(x,y)\}$ *contains all steps in time. The length of a path in time* $a \rightsquigarrow b$ *with respect to* ω *is thus* $|a \rightsquigarrow b|_\omega := \sum_{e \in \mathcal{S}(a \rightsquigarrow b)} \omega(e)$.

Intuitively, ω stretches time, i.e. it assigns a duration onto a step in time. However, while the length of a path in time is a useful concept, it *cannot* serve as a measure of distance.

Example 2.6. *Consider the flow of time* $\Phi := \langle T, \preceq, \omega \rangle$
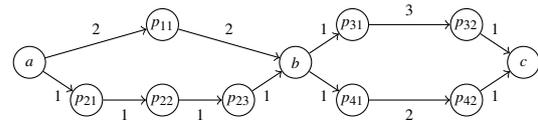
What is the distance δ *between* a, b *and between* b, c? *If constructed based on the notion of a path we can observe the following. For* $|\cdot|$ *we have* $\delta(a,b) = 2$ *or* $\delta(a,b) = 4$ *and for* $|\cdot|_\omega$ *we have* $\delta_\omega(a,b) = 4$ *or* $\delta_\omega(a,b) = 4$. *Analogously, we have* $\delta(b,c) = 3$ *or* $\delta(b,c) = 3$ *and* $\delta_\omega(b,c) = 5$ *or* $\delta_\omega(b,c) = 4$. [3]

To resolve this ambiguity we make use of a quasimetric with infinity (Schroeder, 2006). A fairly natural choice for modeling non-cyclic, directed flows of time. As one can easily travel into the future one instance at a time, but travelling into the past is, at least

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[3] $|\cdot|$ is shorthand for $|x \rightsquigarrow y|$ for some path $x \rightsquigarrow y$

for the common man, impossible. Moreover, by assigning an infinite distance to backtracking within our flow of time, the distance between two chains, also becomes infinite. One can easily check that this holds for chains with and without a join. More information about quasi-metrics can be found in (Matthews, 1994; Schroeder, 2006).

For the rest of this paper we use the following quasi-metric.

Definition 2.7. *Let* $\Phi = \langle T, \leq, \omega \rangle$ *be a flow of time with spacing. We call*

$$\delta_\omega(a,b) := \begin{cases} \min_{a \rightsquigarrow b}(|a \rightsquigarrow b|_\omega) & a \leq b \\ \infty & otw. \end{cases}$$

the distance between a and b (with respect to ω*). Moreover, we call* δ_1*, where* ω *is the constant function 1, the step distance between a and b.*

Proposition 2.8. $\Phi = \langle T, \leq \rangle$ *and* $\Phi = \langle T, \leq, \omega \rangle$ *are quasi-metric flows of time.*

Proof. Since δ_1 is a special case of δ_ω we only consider the latter. Firstly, $\delta_\omega(x,y) \geq 0$ holds as $\forall (x,y) \in \mathcal{S}(\Phi) \ \omega(x,y) \in \mathbb{R}^+$. Secondly, $\delta_\omega(x,y) = 0 = \delta(y,x) \iff x = y$ holds because $x = y \iff |x \rightsquigarrow y| = 0$, especially with respect to ω, and the empty sum is always 0. Finally, $\delta_\omega(x,z) \leq \delta_\omega(x,y) + \delta_\omega(y,z)$. If $z < x$ then $\delta_\omega(x,z) = \infty$. Since there are no circles, it must be that either $\delta_\omega(x,y) = \infty$ or $\delta_\omega(y,z) = \infty$. For $y < x$ or $z < y$ we have that in either case backtracking would be required, leading to a shift in direction, which by definition is not a valid path in time. If $x < y < z$, assume the contrary. Thus $\delta_\omega(x,z) > \delta_\omega(x,y) + \delta_\omega(y,z)$. Hence, $\exists x \rightsquigarrow y, y \rightsquigarrow z$, resulting in $x \rightsquigarrow y \rightsquigarrow z$ such that $|x \rightsquigarrow z|_\omega > |x \rightsquigarrow y \rightsquigarrow z|_\omega$. But $|x \rightsquigarrow z|_\omega$ is minimal, thus arriving at the desired contradiction. □

However, while this may allow us to speak about distance, in one form or another. It creates some semantic inconveniences or inconsistencies. Some of which are addressed in the following subsection.

2.1 Linear and Homogeneous Flows of Time

The notion of distance developed above may lead to unfortunate outcomes (see Example 2.9). Many of which can be reduced to the fact that, in their general form, flows in time can easily be used to express time as having varying density, i.e. the spacing between points may vary within a single chain or across branches. While sometimes useful, e.g. consider relativity of time, a characterization of time without such properties is desirable as well.

Example 2.9. *Consider the flow of time as in Example 2.6. We can find two paths from a to b and two paths from b to c. For the latter, our notion of step distance causes no issues, as all paths between b and c have the same size, i.e.* $|\cdot|$. *However, in the case a to b, two paths of different length, i.e* $|\cdot|_\omega$, *can be found. Hence, if one wants to consider all points at a certain step distance from b one obtains* $\delta_1(a,b) < \delta_1(a,p_{23})$ *but* $p_{23} < b$. *That is, the successor is closer than its predecessor. This can be resolved by adjusting the density of the flow of time across branches, e.g. by manipulating the spacing between two points.*

A flow of time, in which this issue cannot arise is a flow where every path between a join and a meet has the same length. Leading us to the definition of global-homogeneousness.

Definition 2.10. *Let* $\Phi = \langle T, \leq, \omega \rangle$ *be a quasi-metric flow of time. Then* Φ *is a globally-homogeneous flow of time iff* $\forall x, y, z \in T$

$$\delta_\omega(x,y) + \delta_\omega(y,z) = \infty \vee \delta_\omega(x,z) = \delta_\omega(x,y) + \delta_\omega(y,z)$$

Notice, that every linear flow of time satisfies this property. Lastly, we have to check whether global homogeneousness resolves the issue.

Proposition 2.11. *Let* $\Phi = \langle T, \leq, \omega \rangle$ *be a flow of time then* $\forall x, y, z \in T \ (x < y < z \implies \delta_\omega(x,y) < \delta_\omega(x,z))$ *if it is globally homogeneous*

Proof. If $\delta_\omega(x,y) + \delta_\omega(y,z) = \infty$, $\forall x \rightsquigarrow z \ (y \notin x \rightsquigarrow z)$. Hence, we only consider $x \leq y \leq z$. If either of $\leq$ are equal, we are done. Otherwise, we have $x < y < z$. Since, $\delta_\omega(x,z) = \delta_\omega(x,y) + \delta_\omega(y,z)$, thus $\delta_\omega(x,z) - \delta_\omega(y,z) = \delta_\omega(x,y)$ and since $0 < \delta_\omega(y,z)$, $\delta_\omega(x,y) < \delta_\omega(x,z)$ follows □

However, there is another notion of homogeneousness, namely local homogeneousness.

Example 2.12. *Consider the flow of time from Example 2.6. Interpret o as the present. We see that the past is globally-homogeneous, while the future is not. Even though, in the past there is a higher density of points in the bottom branch than in the top. While the inverse case holds in the future. So in some sense some strains of our time flow are denser than others.*

In Example 2.12 we regard "density" of time as the spacing of observations (Zumbach and Müller, 2001). Analogously to real world problems, consider a sequence of graphs $(G_t)_{t \in T}$ distributed unevenly along a time line, as a result from inconsistent measurements or a contraction of measurements to obtain certain properties. Hence, this is a problem that can

occur even in linear flows of time. Therefore, we introduce the concept of local-homogeneousness, to express that all points in T are evenly spaced with respect to ω. That is, we use a notion of homogeneity mentioned in (Zumbach and Müller, 2001).

Definition 2.13. $\Phi = \langle T, \leq, \omega \rangle$ *is a local-homogeneous flow of time iff* ω *is the constant function* x *with* $x \in \mathbb{R}^+$.

Thus we arrive at:

Definition 2.14. $\Phi = \langle T, \leq, \omega \rangle$ *is a homogeneous flow of time iff* Φ *is locally and globally homogeneous.*

It should be mentioned that those are not the only possible notions of homogeneity, consider for example (Venema, 1998). By using homogeneity we can now safely, navigate through a flow of time.

Definition 2.15. *Let* $\Phi = \langle T, \leq, \omega \rangle$ *be a globally-homogeneous flow of time, then for some point of origin* $a \in T$ *and some* $x \in \mathbb{R}$ *we have*

$$\Phi(a,x)_\omega = \begin{cases} \{y \mid \forall y \in T \; \delta_\omega(a,y) = |x|\} & 0 \leq x \\ \{y \mid \forall y \in T \; \delta_\omega(y,a) = |x|\} & 0 > x \end{cases}$$

However, more importantly, since global-homogeneity forces the distance between two points to be unambiguous, we can understand x to be a point on a linear time line and $\Phi(a,x)$ as a function that maps into a set of possible worlds. By restricting oneself to homogeneous flows of time, one ensures that all worlds across all branches are lined up. Allowing us to consider additional operations, such as the collapsing of possible worlds. Unfortunately, due to the page restrictions the various methods of doing so cannot be discussed here.

3 TEMPORAL NETWORKS

The notion of a temporal network, intuitively to be understood as a graph with an additional structure encoding the dimension of time, has been captured by multiple formalisms and is known under various names across different fields (Holme and Saramäki, 2012; Casteigts et al., 2012). Therefore, the general approach of multilayer networks was introduced to unify formalisms that extend the ordinary notion of a graph, this includes several formalisms concerned with capturing the notion of a temporal network. We are partially supporting this attempt.

In (Holme and Saramäki, 2012) an important distinction is made between *instance-based temporal graphs* and *interval-based temporal graphs*. The prior understands time as a sequence of instances, where, for example, an edge is labeled with a sequence of time stamps indicating the graph instances

that include this edge. Hence, all interactions between nodes can be modeled by intra-layer edges, and the inter-level edges corresponding to "identity" relations. While the latter allows for the modeling of interactions with a duration, i.e. an interaction starts at layer α and ends at layer β. We focus our attention on instance-based contact sequences.

Definition 3.1. *Let* $\mathcal{T} := \langle \mathcal{G}, \mathcal{R}, \equiv, \leq, \omega \rangle$ *be a weighted multilayer network with equivalence, we call it a instance-based temporal network iff*

- $\mathcal{G} := \{G_t \mid \forall t \in T\}$ *for some labeling* T.
- $\mathcal{R}$ *is a collection of all successor relations with respect to* $\leq$, *i.e.* $\mathcal{R} = \{R_{t_i t_j} \mid \forall G_{t_i}, G_{t_j} \in \mathcal{G} \; R_{t_i t_j} := \{(v,w) \mid \forall v \in G_{t_i} \forall w \in G_{t_j} \; S(G_{t_i}, G_{t_j}) \wedge v \equiv w\}\}$.
- ω *is defined such that it respects the intra-graph relation weights while assigning weights to every inter-graph relation, i.e.*

$$\omega_{t_i t_j}(e) := \begin{cases} \omega_{t_i}(e) & t_i = t_j \\ \mathbb{R}^+ & t_i \neq t_j \wedge G_{t_i} \leq G_{t_j} \\ \infty & otw. \end{cases}$$

Moreover, we observe that a temporal graph can be understood as a flow of time with additional structure. That is, consider the flow in time $\mathcal{T} := \langle \mathcal{G}, \leq, \iota, \omega \rangle$. Now for all $G_t \in \mathcal{G}$ we fix $\iota(G_t) = G_t$ where G_t is some weighted graph $G_t := \langle V_t, E_t, \omega_t \rangle$. By fixing the world at a certain point in time to being the same as its label ι becomes redundant. Let $\equiv$ be in Definition 1.2 and let $\leq$ extend to the vertices such that $\forall G_{t_i}, G_{t_j} \in \mathcal{G} \forall v \in G_{t_i} \forall w \in G_{t_j} \; (G_{t_i} \leq G_{t_j} \implies (\overline{v} = \overline{w} \iff v \leq w))$. Then $\mathcal{R}$ is just the set of successors with respect to our extended flow relation, where ω carries over.

3.1 Paths in Temporal Networks

The notion of a path in a static, non-weighted graph, is a fairly simple concept, with its length being defined as its size, i.e. its number of edges. In this case a shortest path can be computed efficiently and has an fairly unambiguous semantics (Tang et al., 2009; Wu et al., 2014). Unfortunately, this property is already lost when considering weighted graphs in general. As negative cycles increase the computational complexity of this problem, and weights in general require the distinction between similarity and dissimilarity measures. When analysing a network those two interpretations have to be distinguished carefully, as some measures only provide sensible results when ω is a similarity measure, while others require dissimilarity measures. Fortunately, it is possible to invert the semantic interpretation of the respective measure. One example of this would be

$\omega_s(x,y) = \frac{1}{1+\omega_d(x,y)}$, with ω_s and ω_d being some measure of similarity and dissimilarity respectively. Now consider a multilayer network, each layer having (possibly) different semantics. The same holds true for temporal networks, where we have a dissimilarity measure on inter-graph edges and another measure with different semantic on intra-graph eges. Hence, making the notion of what a shortest path may be even more difficult (Runkler, 2012; Goshtasby, 2012; Segarra and Ribeiro, 2016).

Within the context of temporal networks (Wu et al., 2014) introduce a set of minimum temporal paths, consisting of the *earliest-arrival path*, i.e. starting from a_0 find the path ending in the smallest $b \in \bar{b}$, the *latest-departure path*, i.e. what is the largest $a \in \bar{a}$, while still being able to reach $\bar{b}$; the *fastest path*, i.e. what is the shortest path between $\bar{a}$ and $\bar{b}$ minimising the difference between ending time and starting time and shortest path, thus the path that is the shortest with respect to traversal time. In (Tang et al., 2009) a temporal graph is conceptualised as a sequence of graphs. However, by limiting the amount of hops within each static graph, they manage to encode some notion of time into each static graph. Moreover, it is not uncommon to make the distinction between the size of a path and its duration explicit (Holme and Saramäki, 2012; Michail, 2016; Casteigts et al., 2012). Analogously to flows in time, this distinction roughly corresponds to $|\cdot|_1$ and $|\cdot|_\omega$. As already mentioned, when dealing with weighted temporal graphs an additional dimension is introduced. Namely, we distinguish not only between temporal steps and temporal distance, but also between intra-level steps and intra-level distance. One approach would be to use some norm to collapse those two dimensions. However, motivated by those examples and the issue of similarity and dissimilarity we instead use the concept of a path in time together its notion of distance, as well the notion of a path in a regular graph.

Definition 3.2. *Let* $\mathcal{T} := \langle \mathcal{G}, \mathcal{R}, \equiv, \preceq, \omega \rangle$ *be a instance-based temporal network and let* $v \in G_{t_i}$ *and* $w \in G_{t_j}$ *then the alternating sequence of regular paths and forward-paths in time* $\mathcal{T} \supseteq (v \leadsto_{\mathcal{T}} w) = (\bar{v}_{t_i} \leadsto_{\mathcal{T}} \bar{w}_{t_j}) := \bar{v}_{t_i} \leadsto_\Phi^+ \bar{v}_{t_k} \leadsto \bar{w}_{t_k} \cdots \leadsto_\Phi^+ \bar{u}_{t_j} \leadsto \bar{v}_{t_j}$ *is called the temporal path from* $\bar{v}_{t_i}$ *to* $\bar{w}_{t_j}$. *We write* $\lambda_\Phi(x \leadsto_{\mathcal{T}} y)$ *for the set of maximal paths in time and* $\lambda_{\mathcal{G}}(x \leadsto_{\mathcal{T}} y)$ *for the set of maximal intra-graph paths in* $x \leadsto_{\mathcal{T}} y$.[4]

By distinguishing between the types of paths in temporal graphs, we allow for a separation of measures, i.e. inter- and intra-length and size.

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[4]$a \leadsto_X b$ is shorthand for $a \leadsto b \subseteq X$.

Definition 3.3. *Let* $\mathcal{T} := \langle \mathcal{G}, \mathcal{R}, \equiv, \preceq, \omega \rangle$ *be an instance-based temporal network, let* $p := v \leadsto_{\mathcal{T}} w$ *be the temporal path from* v *to* w *with* $v, w \in V(\mathcal{T})$. *Then we define*

$$|p|_\omega := (|E(\lambda_\Phi(p))|_\omega, |E(\lambda_{\mathcal{G}}(p))|_\omega)$$

as the length of p with respect to ω *and* $|p|_1$ *as its size.*

By allowing for two dimensions with respect to path length and size we have to define a specific order on those values. A natural choice for this is the so called product order, i.e. $(x_i, y_i) \preceq (x_j, y_j) \iff x_i \leq x_j \wedge y_i \leq y_j$ (Dickson, 1913). However, one issue that immediately arises when using a product ordering, is the issue of partiality (see Example 3.4). Moreover, there can be various natural interpretations of shortest path problems. For example, the problem $\bar{a} \leadsto_{\mathcal{T}} \bar{b}$ addresses the desire to compute the set of overall shortest distances between two equivalence classes, while the problem $\bar{a}_{t_i} \leadsto_{\mathcal{T}} \bar{b}$ restricts the same question to a certain starting point and $\bar{a} \leadsto_{\mathcal{T}} \bar{b}_{t_j}$ to a specific arrival date. Lastly, $\bar{a}_{t_i} \leadsto_{\mathcal{T}} \bar{b}_{t_j}$ is de facto an ordinary shortest path problem.

Example 3.4. *The temporal graph* $\mathcal{T}$

where G_{t_1}, G_{t_2} *and* G_{t_3} *are respectively*
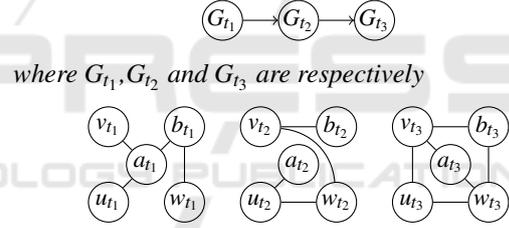
While each shortest path problem may lead to different results (see Table 1), the more significant observation is that from the shortest path between $\bar{x}$ *and* $\bar{y}$, $\bar{x} \leadsto \bar{z} \leadsto \bar{y}$ *it is not possible to conclude that* $\bar{x} \leadsto \bar{z}$ *is the shortest path between* $\bar{x}$ *and* $\bar{z}$. *Consider,* $\bar{a} \leadsto \bar{w}$ *and* $\bar{a} \leadsto \bar{b}$. *Thereby, prohibiting the save use of Dijkstra's algorithm for some of the specified problems.*

Table 1: Compute all possible shortest paths between $\bar{a}$ and $\bar{b}$ and all elements within those equivalence classes.

$|\cdot \leadsto \cdot|$	$\bar{b}_{t_1}$	$\bar{b}_{t_2}$	$\bar{b}_{t_3}$	$\bar{b}$
$\bar{a}_{t_1}$	$(0,1)$	$(1,1)$	$(2,1)$	$(0,1)$
$\bar{a}_{t_2}$	(∞,∞)	$(0,4)$	$(1,2)$	$(0,4),(1,2)$
$\bar{a}_{t_3}$	(∞,∞)	(∞,∞)	$(0,2)$	$(0,2)$
$\bar{a}$	$(0,1)$	$(1,1),(0,4)$	$(2,1),(0,2)$	$(0,1)$

While, in general, the multi-objective shortest path problem may need exponential runtime, we can do better due to its unique structure (Tarapata, 2007).

Proposition 3.5. *Let* $\mathcal{T} := \langle \mathcal{G}, \mathcal{R}, \equiv, \preceq, \omega \rangle$ *be a globally-homogeneous, instance-based temporal multiplex then for* $v, w \in V(\mathcal{T})$ *the length (and size) of all shortest path is comparable and thus the same.*

Proof. We know $\exists! G_{t_i}, G_{t_j} \in \mathcal{G}\ v \in G_{t_i} \wedge w \in G_{t_j}$. If there does not exits a path between v and w we are done. Otherwise, by global-homogeneity and due to $\preceq$ being directed, we obtain $\forall v \rightsquigarrow_{\mathcal{T}} w\ |v \rightsquigarrow_{\mathcal{T}} w|_\omega = (\delta_\omega(G_{t_i}, G_{t_j}), x)$. As they only differ in x, all of them are comparable and by minimality all shortest paths have the same length. $\qquad\square$

Proposition 3.6. *Let* $\mathcal{T} := \langle \mathcal{G}, \mathcal{R}, \equiv, \preceq, \omega \rangle$ *be a globally-homogeneous, instance-based temporal multiplex. The problem of finding the shortest path with respect to* $|\cdot|_\omega$ *over* $\mathcal{T}$ *from* v *to* w *for* $v, w \in V(\mathcal{T})$ *can be reduced in polynomial time to the problem of finding the shortest path between* v *and* w *in the weighted directed graph.*

Proof. Consider the construction $D := \langle V(\mathcal{T}), E(\mathcal{T}), \omega \rangle$. This transformation is linear. Show for $p_T := (v \rightsquigarrow_{\mathcal{T}} w) \subseteq \mathcal{T}$ we have $|v \rightsquigarrow_{\mathcal{T}} w|_\omega$ is minimal $\iff$ for $p_D := (v \rightsquigarrow_D w) \subseteq D$ we have $|v \rightsquigarrow_D w|_\omega$ is minimal. We observe $|E(\lambda_{\mathcal{G}}(p_T))|_\omega = \delta_\omega(v, w)$. Moreover, every path between v and w only differs in $|E(\lambda_\Phi(p_T))|_\omega$. Since, $|p_D|_\omega = \delta_\omega(v, w) + |E(\lambda_\Phi(p_T))|_\omega$, we obtain $|p_T|_\omega$ minimal must be equivalent with $|p_D|_\omega$ minimal. $\qquad\square$

Hence, finding the shortest path between two distinct vertices in a temporal graph can be solved by applying a variant of Dijkstra's algorithm and it is thus $O((n+m) \cdot log(n))$, where $|\mathcal{G}| \le n = |V(\mathcal{T})|$ and $m = |E(\mathcal{T})|$ (Barbehenn, 1998). By using this knowledge we can implement algorithms for computing the other shortest path problems. Even a naive implementation, i.e. one that computes all distances between $\bar{v}_t$ and every member of $\bar{w}$ is in $O((m+n) \cdot n \cdot log(n))$. Hence, searching for all minimal paths with respect to $|\cdot|_\omega$ is bounded by the same complexity, as finding minimal elements is at most $O(n^2)$. By applying the same algorithm for all $\bar{v}_t \in \bar{v}$ we obtain at most $O((m+n) \cdot n^2 \cdot log(n))$. This serves only as a rough estimate to show polynomial membership and to justify this approach form a computational complexity point of view.

4 DISCUSSION

We decided against using the classical notion of multiplex network as a basis for our approach and instead defined a tailored variant that explicitly considers our generalized notion of time. This allows us to classify a temporal path as a sub-network of the main temporal network, with the temporal path retaining the properties of being a temporal network itself. Moreover, as

discussed in (Taylor et al., 2017) restricting oneself to multiplex networks carries a tolerable loss of expressiveness, as one can add isolated "ghost nodes" to each graph G_t to obtain the (classical) multiplex property. Unfortunately, this implies that for certain measures the existence of such ghost nodes must be accounted for. While the relation modeling time is transitive, the corresponding inter-layer edges are neither transitive nor symmetric. Intuitively they reflect the movement of a vertex through time. Therefore, by accepting the multiplex property, node persistence is ensured, allowing for an easier semantic of a movement through time, i.e. a vertex can not pop in and out of existence as it pleases.

Future work may discuss possible ramifications with respect to common network measures. For example, by only considering the multivalued notion of distance on linear time, one can already detect different behaviours of some network measures. That is, consider the path-based centrality measures closeness and betweenness. As betweenness relies on counts of shortest paths its extension to a multivalued path length is straightforward. Even in linear time the various shortest path problems result in analogous betweenness measures, i.e. with and without restriction on the starting and end time. In contrast, while closeness centrality has a similar relationship with respect to time, any ranking of nodes will be a partial ordering. That is, since the length of a path is incorporated into this measure, its multivalued nature will be carried over. Lastly, while interesting, a discussion of network measures on non-linear models of time is, unfortunately, far beyond the scope of this paper.

5 CONCLUSION

Most approaches for the analysis of temporal networks do not explicitly discuss the underlying conception of time. Moreover, weighted temporal networks are still uncommon in the literature, and a direct discussion about how to reconcile the two semantic dimensions seems to be even more rare. In order to tackle those issues, this paper discussed time as a formal structure, thereby explicitly engaging with some of the underlying assumptions of time on which a temporal network may be operate.

Most prominently, we discussed some of the pitfalls that arise when dealing with non-deterministic time. Furthermore, our generalized abstraction of time promotes a clean separation of the semantics of time and the semantic interpretation of the network itself (i.e. the semantics of the vertices and edges in the corresponding network). To retain a clean separa-

tion of dimensions, we introduced the notion of multivalued (temporal) paths, a variant of paths that, on the one hand, enables a more in-depth understanding of temporal networks without sacrificing semantic integrity, while on the other hand introducing several new and interesting technical challenges, such as the computation of multivalued-centrality measures for temporal networks.

REFERENCES

Barbehenn, M. (1998). A note on the complexity of dijkstra's algorithm for graphs with weighted vertices. *IEEE transactions on computers*, 47(2):263.

Boccaletti, S., Bianconi, G., Criado, R., Del Genio, C. I., Gómez-Gardenes, J., Romance, M., Sendina-Nadal, I., Wang, Z., and Zanin, M. (2014). The structure and dynamics of multilayer networks. *Physics Reports*, 544(1):1–122.

Casteigts, A., Flocchini, P., Quattrociocchi, W., and Santoro, N. (2012). Time-varying graphs and dynamic networks. *International Journal of Parallel, Emergent and Distributed Systems*, 27(5):387–408.

De Domenico, M., Solé-Ribalta, A., Cozzo, E., Kivelä, M., Moreno, Y., Porter, M. A., Gómez, S., and Arenas, A. (2013). Mathematical formulation of multilayer networks. *Physical Review X*, 3(4):041022.

Dickson, L. E. (1913). Finiteness of the odd perfect and primitive abundant numbers with n distinct prime factors. *American Journal of Mathematics*, 35(4):413–422.

Goranko, V. and Galton, A. (2015). Temporal logic. In Zalta, E. N., editor, *The Stanford Encyclopedia of Philosophy*. Metaphysics Research Lab, Stanford University, winter 2015 edition.

Goshtasby, A. A. (2012). Similarity and dissimilarity measures. In *Image registration*, pages 7–66. Springer.

Holme, P. and Saramäki, J. (2012). Temporal networks. *Physics reports*, 519(3):97–125.

Kivelä, M., Arenas, A., Barthelemy, M., Gleeson, J. P., Moreno, Y., and Porter, M. A. (2014). Multilayer networks. *Journal of complex networks*, 2(3):203–271.

Markosian, N., Sullivan, M., and Emery, N. (2016). Time. In Zalta, E. N., editor, *The Stanford Encyclopedia of Philosophy*. Metaphysics Research Lab, Stanford University, fall 2016 edition.

Matthews, S. G. (1994). Partial metric topology. *Annals of the New York Academy of Sciences*, 728(1):183–197.

Michail, O. (2016). An introduction to temporal graphs: An algorithmic perspective. *Internet Mathematics*, 12(4):239–280.

Runkler, T. A. (2012). *Data Analytics*. Springer.

Schroeder, V. (2006). Quasi-metric and metric spaces. *Conformal Geometry and Dynamics of the American Mathematical Society*, 10(18):355–360.

Segarra, S. and Ribeiro, A. (2016). Stability and continuity of centrality measures in weighted graphs. *IEEE Transactions on Signal Processing*, 64(3):543–555.

Sunitha, M. and Mathew, S. (2013). Fuzzy graph theory: a survey. *Annals of Pure and Applied mathematics*, 4(1):92–110.

Tang, J., Musolesi, M., Mascolo, C., and Latora, V. (2009). Temporal distance metrics for social network analysis. In *Proceedings of the 2nd ACM workshop on Online social networks*, pages 31–36. ACM.

Tarapata, Z. (2007). Selected multicriteria shortest path problems: An analysis of complexity, models and adaptation of standard algorithms. *International Journal of Applied Mathematics and Computer Science*, 17(2):269–287.

Taylor, D., Myers, S. A., Clauset, A., Porter, M. A., and Mucha, P. J. (2017). Eigenvector-based centrality measures for temporal networks. *Multiscale Modeling & Simulation*, 15(1):537–574.

Van Ditmarsch, H., van Der Hoek, W., and Kooi, B. (2007). *Dynamic epistemic logic*, volume 337. Springer Science & Business Media.

Venema, Y. (1998). *Temporal logic*. Citeseer.

Wu, H., Cheng, J., Huang, S., Ke, Y., Lu, Y., and Xu, Y. (2014). Path problems in temporal graphs. *Proceedings of the VLDB Endowment*, 7(9):721–732.

Zumbach, G. and Müller, U. (2001). Operators on inhomogeneous time series. *International Journal of Theoretical and Applied Finance*, 4(01):147–177.