# Spectral Algorithm for Line Graphs to Find Overlapping Communities in Social Networks

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Abstract: A great deal of community detection communities is based on the maximization of the measure known as modularity. There is a dearth of literature on overlapping community detection algorithms, in spite of the importance of the applications and the overwhelming number of community detection algorithms yet proposed. To this end, one of the suggestions in the literature consists of partitioning the set of edges into communities, also known as link partitions, by applying community detection algorithms to line graphs. In line with this, in this paper, overlapping vertex communities are obtained from link partitions by a method that selects the communities of the edges that represent the highest modularity gain. We also introduce a spectral algorithm to find link partitions from line graphs. We show that the modularity of communities in line graphs is equivalent to the adaptation of modularity of communities in the original graphs, when considering the non-backtracking matrix instead of the adjacency matrix in its formula. The results of the experiments carried out with overlapping community detection algorithms showed that the proposed method is competitive with state-of-the-art algorithms.



## 1 INTRODUCTION

Complex networks can model real systems such as social, metabolic and citation networks. These networks can be partitioned into communities of densely connected vertices for a better understanding of the characteristics of the system. In the context of social networks, the identification of communities is of great interest in applications such as targeting market campaigns, prediction of interactions amongst users and detecting non-human users.

The traditional approach to detect communities in networks is to find the best vertex partition according to a given measure that assesses its quality. Examples of measures are the modularity (Newman and Girvan, 2004), the statistical significance (Lancichinetti et al., 2011) and the map equation (Rosvall and Bergstrom, 2007).

The number of edges within the groups minus the number of edges expected in a random graph with the same degrees as the graph under the consideration is virtually the definition of modularity. The higher the modularity of a partition, the better the partition according to this measure.

Because the modularity maximization problem is NP-Hard (Brandes et al., 2008), its exact optimization is unrealistic in scenarios where the networks have a large number of vertices. Therefore, the literature has been virtually focused on developing approximation methods and heuristics. Spectral decomposition methods in particular have presented good results in (Nascimento and De Carvalho, 2011) and in (Newman, 2013b). Krzakala et al. (2013) introduced the non-backtracking matrix which indicates whether or not a pair of edges is adjacent. Based on computational experiments, the authors suggested that the spectrum of such matrix is more meaningful than the spectrum of the adjacency matrix to detect communities in networks.

A characteristic of social networks is the existence of overlapping communities meaning that vertices belong to more than one community (Xie et al., 2013). A user of Facebook, for example, might join different groups. Nonetheless, many works are strict on selecting only one community for each vertex, i.e., disjoint communities (Xie et al., 2013). Among the methods capable of finding either disjoint or overlapping communities, the ones presented in (Lancichinetti et al., 2011) and in (Xie et al., 2011) are worth mentioning.



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The partitioning of edges into communities to define link partitions has also been studied in the literature as an approach to identify overlapping vertex communities (Evans and Lambiotte, 2009). Evans and Lambiotte (2009) suggested that link partitions can be obtained by (vertex) community detection algorithms applied to their respective line graphs. Zhou et al. (2017), however, observed that the resulting partitions may present communities with an undesirable large number of overlappings.

In this paper, we adapted the modularity measure to assess the adjacency relations between edges instead of vertices. Moreover, we demonstrated that this adaptation is equivalent to the classical modularity in line graphs. As another contribution, we introduce a strategy to identify disjoint and overlapping vertex communities from link partitions and a spectral divisive community detection algorithm, here called *SpecDiv*. *SpecDiv* is a hybridization of the Leading Eigenvalue algorithm proposed by Newman (2006) with a local search heuristic. The proposed strategy constructs the vertex communities according to the link partitions found by *SpecDiv* and based on the gain of modularity.

The computational experiments showed that this strategy was competitive with state-of-the-art overlapping community detection algorithms. Furthermore, we obtain link partitions by applying classical community detection algorithms to line graphs. The experiments carried out with real and sparse LFR artificial networks showed that disjoint partitions obtained by the proposed strategy using line graphs are closer to the expected partitions than those found by traditional community detection methods. In both cases, *SpecDiv* is competitive with the reference algorithms. These results suggest that link partitions can also be advantageous for existing vertex community detection algorithms. As part of the experiments, we also illustrate a practical application of the overlapping community detection in a Twitter network from the literature.

# 2 RELATED WORKS

This section presents the community detection methods related to the introduced algorithm. First, we briefly discuss the disjoint community detection methods. After, we examine the overlapping community detection methods and, in special, link community methods.

## 2.1 Disjoint Community Detection in Networks

This section briefly describes disjoint community detection methods used as reference in the experiments performed and presented in this paper. They are: the Louvain method (Blondel et al., 2008), Infomap (Rosvall and Bergstrom, 2007), Label Propagation Algorithm (Raghavan et al., 2007) and Walktrap (Pons and Latapy, 2005). We also discuss the Leading Eigenvalue method of Newman (2006) since our algorithm is based on it and highlight a few other spectral-based methods. As an extensive literature review on this subject is out of the scope of this paper, we refer the reader to the recent survey of Fortunato and Hric (2016). We chose these methods due to their good results to a wide variety of networks and consolidated use as reference algorithms in the literature (Yang et al., 2016).

There are two overlapping community detection methods discussed in the next section which are also capable of finding disjoint communities: the Order Statistics Local Optimization Method (OSLOM) (Lancichinetti et al., 2011) and the Speak Listener Propagation Algorithm (SLPA) (Xie et al., 2011).

To find the vertex partitions, the Louvain and the Leading Eigenvalue methods aim at maximizing the modularity measure. The Louvain method is a greedy strategy that merges communities of vertices that increase the modularity value. The Leading Eigenvalue method is based on the observation that the signs of components of the eigenvector associated with the largest eigenvalue of the modularity matrix can be used to partition the set of vertices into two communities. The method then hierarchically divides each community into two new communities.

Infomap aims at optimizing the map equation measure, which is inspired by the duality between the community detection problem and random walks paths in digraphs. Walktrap also relies on random walks to form vertex partitions by inferring the probability of random walkers leaving communities to reach vertices from other communities. The Label Propagation Algorithm (LP) is a strategy which propagates labels through the network to define the label of a vertex based on the communities to which its neighbors belong to.

Several other spectral-based methods for finding disjoint communities in networks exists besides the Leading Eigenvalue method. Zhang and Newman (2015), for example, designed a heuristic based on the spectral decomposition of the modularity maximization matrix. As previously mentioned, Krzakala et al. (2013) introduced the spectral decomposition of the

non-backtracking matrix. Both Newman (2013a) and Singh and Humphries (2015) studied variations of the non-backtracking matrix to improve its representativeness regarding networks.

Closely related to the spectral decomposition methods are the methods based on statistical inference (Newman, 2013b; Ali and Couillet, 2017), which Newman (2013b) showed to be equivalent to modularity maximization decomposition methods under specific parameter settings. These methods detect communities by fitting the stochastic block models to the networks using maximum likelihood methods.

## 2.2 Overlapping Community Detection in Networks

As mentioned earlier, a reduced number of algorithms to detect overlapping communities can be found in the literature. Among them, OSLOM and SLPA perform such task with satisfactory outcomes when applied to LFR networks as pointed out in the literature review presented by Xie et al. (2013).

SLPA is an extension of LP to detect overlapping communities in networks. SLPA keeps the information about the community labels propagated to each vertex along the iterations. A post-processing strategy is responsible for defining the most frequent community labels of the vertices as their communities, based on a threshold parameter. OSLOM is a heuristic method that employs a fitness function to assess the statistical significance of the communities compared to a null model, i.e., a network with no community structure.

Evans and Lambiotte (2009) and Ahn et al. (2010) suggested detecting communities of links instead of vertices to deal with overlapping communities. In this strategy, each vertex belongs to the communities of its incident edges. Evans and Lambiotte (2009) recommended the detection of link partitions through the partitioning of the vertices of the line graphs. Ahn et al. (2010), on the other hand, studied the hierarchical partitioning of links in graphs by measuring the similarity of edges using a measure known as Jaccard index.

Although Xie et al. (2013) have shown through experiments that OSLOM and SLPA outperformed the link partitioning method of Ahn et al. (2010), approaching the problem by line graphs enables the use of any vertex community detection algorithm to find overlapping communities. In this sense, methods that aim at finding communities of links have been increasingly studied in the last years. Some examples of algorithms that find overlapping communities based on line graphs are the genetic algorithms proposed by Pizzuti (2009), Shi et al. (2013) and Li and Liu (2018).

Spectral methods have also been used to detect overlapping communities from link partitions. The pioneer spectral method for this task was the one introduced by Zhang et al. (2007). The algorithm proposed by the authors is based on the spectral decomposition of an adaptation of the modularity measure to deal with the soft assignment of vertices to communities. The overlapping communities are achieved by analyzing how much each vertex belongs to each community. Jiang and McQuay (2012) also explored the soft assignment of nodes to communities through a spectral decomposition method.

Gui et al. (2018) designed a spectral method that used the eigenvectors associated with the two largest eigenvalues of the Laplacian matrix of a line graph to project the graph into points in a plane. These points were then used to calculate the similarity between the vertices in order to hierarchically merge them into communities.

As opposed to disjoint community detection, for which several spectral clustering methods have been studied (Nascimento and De Carvalho, 2011), the literature still lacks spectral heuristics that maximize the modularity measure in order to find overlapping communities. Even though Nicosia et al. (2009), for example, introduced an adaptation of the modularity measure to assess overlapping vertex communities, they used a genetic algorithm to maximize it rather than a spectral-based method.

# 3 MODULARITY MAXIMIZATION

Let $G = (V, E)$ be an undirected and simple graph such that $V$ and $E$ are its set of vertices and edges, respectively.

The set of vertices is represented by integer scalars, such that $V = \{1, 2, \ldots, |V|\}$. An edge is an unordered tuple $(i, j)$ that has end points in two vertices $i, j \in V$. The adjacency matrix $A = [a_{ij}] \in \mathbb{R}^{|V|\times|V|}$ represents the pairwise relation between vertices of $G$. An element $a_{ij}$ is 1 if edge $(i, j) \in E$, and 0, otherwise. The total number of the edges is denoted by $m = \sum_{i\in V}\sum_{j\in V}\frac{a_{ij}}{2}$. The degree of vertex $i$ is given by $d_i = \sum_{j\in V} a_{ij}$.

Let the communities of a graph be defined by integer scalars called community labels. A vertex partition into communities is defined as $\mathcal{P} = \{c_1, c_2, ..., c_{|V|}\}$, where $c_i$ is the label of the community that vertex $i$ belong to. Two vertices are in the same community when they share the same label.

Considering the community labels of vertices $i$ and $j$, i.e. $c_i$ and $c_j$, let $\delta_{c_i c_j}$ be 1 if $c_i = c_j$ and 0, otherwise. The modularity matrix is given by $B = [b_{ij}] \in \mathbb{R}^{|V|\times|V|}$, where each element is calculated according to Equation (1).

$$b_{ij} = a_{ij} - \frac{d_i d_j}{2m} \tag{1}$$

The modularity of a partition $\mathcal{P}$ is given $Q(\mathcal{P})$ according to Equation (2).

$$Q(\mathcal{P}) = \frac{1}{2m} \sum_{i\in V} \sum_{j\in V} b_{ij}\delta_{c_i c_j} \tag{2}$$

According to Equation (2), higher values of modularity are obtained when the number of edges within the communities is greater than the expected number of edges of a random graph with the same degree sequence as $V$.

## 3.1 Non-backtracking Matrix

An edge $e_1 = (i, j)$ is said to be adjacent to another edge $e_2 = (k, r)$ if and only if they share exactly one end point. The set of edges adjacent to $e_1 = (i, j)$ is thereby given by $\mathcal{N}(i, j) = \{(k, r) \in E$ such that $(i, j)$ and $(k, r)$ are adjacent in $G\}$. Moreover, the degree of an edge $e_1 = (i, j)$ is the number of its adjacent edges and is given by $d'_{(i,j)} = |\mathcal{N}(i, j)|$.

Because we only deal with undirected graphs, we adapted the non-backtracking matrix introduced by Krzakala et al. (2013) to ignore edge directions. Let $O = [o_{(i,j),(k,r)}] \in \mathbb{R}^{|E|\times|E|}$ be the non-backtracking matrix, where $(i, j), (k, r) \in E$ and each element is given by Equation (3).

$$o_{(i,j),(k,r)} = \begin{cases} 1, \text{ if } (i, j) \text{ is adjacent to } (k, r) \\ 0, \text{ otherwise} \end{cases} \tag{3}$$

Therefore, $o_{(i,j),(k,r)}$ is 1 if edges $(i, j)$ and $(k, r)$ share exactly one end vertex, i.e., they are adjacent. The non-backtracking matrix can be understood as an adjacency matrix between edges. The analogous definition of $m$ to the non-backtracking matrix is $y = \sum_{(i,j)\in E} \sum_{(k,r)\in E} \frac{o_{(i,j),(k,r)}}{2}$. Furthermore, the non-backtracking modularity matrix is given by $H = [h_{(i,j),(k,r)}] \in \mathbb{R}^{|E|\times|E|}$, where each element is calculated according to Equation (4).

$$h_{(i,j),(k,r)} = o_{(i,j),(k,r)} - \frac{d'_{(i,j)} d'_{(k,r)}}{2y} \tag{4}$$

We introduce an adaptation of the modularity measure to consider the non-backtracking matrix instead of the adjacency matrix in Equation (5).

$$Qo(\mathcal{P}) = \frac{1}{2y} \sum_{(i,j)\in E} \sum_{(k,r)\in E} h_{(i,j),(k,r)}\delta_{c_i c_j}\delta_{c_k c_r} \tag{5}$$

By replacing the adjacency matrix by the non-backtracking matrix, Equation (5) evaluates the modularity over the edges of $G$, instead of $E$: it calculates the number of edges within communities minus the expected number edges in a random graph whose edges have the same degree sequence as $E$.

## 3.2 Line Graphs

Let $G' = (V', E')$ be the line graph of $G$. Its set of vertices is $V' = E$, i.e., each edge of $G$ is a vertex of $G'$. The set of edges of $G'$ is given by $E' = \{(e_1, e_2)$, such that $e_1 \in E$ and $e_2 \in E$ are adjacent in $G\}$. The number of edges of $G'$ is denoted by $m'$ and coincides with $y$, i.e., $m' = y$.

The adjacency matrix of $G'$ is $A' = [a'_{(i,j),(k,r)}] \in \mathbb{R}^{|E|\times|E|}$ where $a'_{(i,j),(k,r)}$ receives value 1 if and only if edges $(i, j), (k, r) \in E$ are adjacent in $G$. Thereby, $a'_{(i,j),(k,r)} = o_{(i,j),(k,r)}$ and, consequently, $A' = O$. The degree of a vertex $e'_1 = (i, j) \in V'$ is defined by $d'_{e'_1} = \sum_{e'_2\in V'} a'_{e'_1 e'_2} = d'_{(i,j)}$.

A vertex partition of $G'$ represents a link partition of $G$. A link partition is here denoted by $\mathcal{P}' = \{c'_1, c'_2, ..., c'_{|E|}\}$, where $c'_e$ is the label of the community to which $e \in E$ belongs to.

Let the modularity of a partition $\mathcal{P}'$ of line graph $G'$ be defined by Equation (6).

$$\begin{aligned} Q'(\mathcal{P}') &= \frac{1}{2m'} \sum_{e_1\in V'} \sum_{e_2\in V'} \left( a'_{e_1 e_2} - \frac{d'_{e_1} d'_{e_2}}{2m'} \right) \delta_{c'_{e_1} c'_{e_2}} \\ &= Q'(\mathcal{P}') = \frac{1}{2y} \sum_{(i,j)\in E} \sum_{(k,r)\in E} h_{(i,j),(k,r)}\delta_{c'_{(i,j)} c'_{(k,r)}} \end{aligned} \tag{6}$$

On the one hand, $Qo(\mathcal{P})$ (Equation (5)) assesses a vertex partition and, on the other, $Q'(\mathcal{P}')$ (Equation (6)) evaluates a link partition.

Consider a pair of edges $(i, j) \in E$ and $(k, r) \in E$. The product $\delta_{c_i c_j}\delta_{c_k c_r}$ is equal to 1 if and only if $i$, $j$, $k$ and $r$ are in the same community in vertex partition $\mathcal{P}$ of $G$. In these cases, edges $(i, j)$ and $(k, r)$ will be in the same community in the related link partition $\mathcal{P}'$. Consequently, $Qo(\mathcal{P})$ reduces to $Q'(\mathcal{P}')$.

Nevertheless, $Q'(\mathcal{P}')$ does not reduce directly into $Qo(\mathcal{P})$. Two edges $(i, j), (k, r) \in E$ might be in the same community in a link partition $\mathcal{P}'$. However, one or more of their end points might be end point of other edges that belong to different communities in $\mathcal{P}'$. A vertex can be in any of the communities of its incident edges in a vertex partition $\mathcal{P}$. This observation induces a natural overlapping of communities in $\mathcal{P}$.

Therefore, maximizing the non-backtracking modularity in a graph $G$ is equivalent to maximize the classical modularity in its line graph $G'$, provided that the former accounts for overlapping communities.

# 4 SPECTRAL DECOMPOSITION

This section presents the spectral decomposition of the modularity measure of a partition $\mathcal{P}_F$ in an arbitrary graph $F = (V_F, E_F)$ that can be either $G$ or $G'$. If $F$ is a line graph, then the modularity is equivalent to the non-backtracking modularity in $G$. The decomposition of the modularity is based on the study performed by Newman (2006). Assume that $m_F \in \mathbb{R}$ and $B_F = [bf_{ij}] \in \mathbb{R}^{|V_F|\times|V_F|}$ are the number of edges and the modularity matrix of $G_F$, respectively.

First, consider a bipartition of the vertices of $F$. Let $s = [s_i] \in \mathbb{R}^{|V_F|}$ be a vector where each element $s_i$ is either 1 or $-1$, depending on which of the communities vertex $i$ belongs to. The modularity can be expressed in a matricial form in function of $s$, as presented in Equation (7).

$$Q(s) = \frac{1}{2m_F} s^T B_F s \tag{7}$$

Let $\lambda_1, \lambda_2, \ldots, \lambda_{|V_F|}$ be the eigenvalues of $B_F$ decreasingly sorted by value, i.e., $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{|V_F|}$. Moreover, let $\Lambda \in \mathbb{R}^{|V_F|\times|V_F|}$ be a diagonal matrix such that the diagonal element in row $j$ and column $j$ is given by the $j-th$ largest eigenvalue, $\lambda_j$. The eigenvectors of $B_F$ are arranged in matrix $U = [u_{ij}] \in \mathbb{R}^{|V_F|\times|V_F|}$ such that each column $j$ is the eigenvector $u_j$ associated with the eigenvalue $\lambda_j$.

Matrix $B_F$ is symmetric because $F$ is undirected and thus admits an eigen-decomposition. The vector $s$ can be written as a linear combination of the eigenvectors: $s = \sum_{i\in V_F} \alpha_i u_i$, where $\alpha_i = u_i^T s \in \mathbb{R}$. Equation (7) can then be rewritten as shown in Equation (8).

$$\begin{aligned} Q(s) &= \frac{1}{2m_F} \sum_{i\in V_F} \alpha_i u_i^T B_F \sum_{j\in V_F} \alpha_j u_j \\ &= \frac{1}{2m_F} \sum_{i\in V_F} (u_i^T s)^2 \lambda_i \end{aligned} \tag{8}$$

Newman (2006) suggested approximating Equation (8) using the largest eigenvalue $\lambda_1$. The decision problem is thus to decide the values of each element $s_i$, $i \in V_F$, that result in the largest value in Equation (8). Because an element $s_i$ can be 1 or $-1$, each term $(u_i^T s)^2 \lambda_i, i \in V_F$, is maximized by setting $s_i$ as 1 if $u_{i1} > 0$ and as $-1$ if $u_{i1} < 0$. In case $u_{i1} = 0$, the choice of $s_i$ does not change the value of the term. Therefore, the resulting partition is $\mathcal{P}_F = \{c_1, c_2, \ldots, c_{|V_F|}\}$, where $c_i = 1$ when $s_i = -1$ and $c_i = 2$ when $s_i = 1$, for $i \in V_F$.

## 4.1 Hierarchical Division of Communities

In order to hierarchically divide the pair of communities found through the spectral decomposition explained in the earlier section, Newman (2006) suggested creating subgraphs induced by the vertices of each community. This approach, however, requires the recalculation of the modularity matrix of the induced graphs to consider the degree sequence of the original graph.

In this paper, instead of analyzing the modularity matrix of the induced graphs, we define the modularity matrix of each community by selecting only the rows and columns of the vertices that belong to the community.

To show that the introduced strategy is correct, let $Z$ be a set containing the indices of the vertices that belong to an arbitrary community labeled $l_a$. Equation (9) isolates the contribution of the vertices in $Z$ to the modularity measure.

$$\begin{aligned} Q(\mathcal{P}_F) = &\frac{1}{2m_F} \sum_{i\in Z} \sum_{j\in Z} bf_{ij} \delta_{c_i c_j} \\ &+ \frac{1}{2m_F} \sum_{i,j:\{i,j\}\in(V_F\times V_F)-(Z\times Z)} bf_{ij} \delta_{c_i c_j} \end{aligned} \tag{9}$$

where $(V_F \times V_F) - (Z \times Z)$ has as elements all pairs of vertices $i$, $j$ such that both $j$ and $i$ does not belong to $Z$.

The first term of Equation (9) is the contribution of community labeled $l_a$ to the modularity. According to this Equation, maximizing the first term does not change the value of the second term in Equation (9).

Let $D$ be a mapping between the vertices from $Z$ and elements of $\{1, 2, \ldots, |Z|\}$, i.e.: $D = \{(i, \upsilon)$ such that $i \in Z$ and $\upsilon$ is selected without replacement from $\{1, 2, .., |Z|\}\}$. Let $B_Z = [bf_{\upsilon\rho}] \in \mathbb{R}^{|Z|\times|Z|}$ be the modularity matrix whose element $b_{\upsilon\rho}$ has the value of $b_{ij}$, $i : (i, \upsilon) \in D$ and $j : (j, \rho) \in D$.

Also, let $z = [z_\upsilon] \in \mathbb{R}^{|Z|}$ be the vector whose elements $z_\upsilon$ receive value 1 or $-1$ depending on which community vertex $i : (i, \upsilon) \in D$ belong to. Thereby, the modularity of the community labeled $l_a$ is given by $Q_Z(z) = \frac{1}{2m_F} z B_Z z$.

Because the spectral decomposition of $Q_Z(z)$ is analogous to the one performed on $Q(s)$, the signs of the eigenvectors associated with the leading eigenvalue of $B_Z$ can be used to choose the values of $z$ that maximize $Q_Z(z)$. After choosing $z$ using the leading

eigenvalue approach, two new community labels are selected for the vertices in $Z$.

# 5 VERTEX COMMUNITY DETECTION IN LINE GRAPHS

The first step of the introduced strategy is to find a link partition by applying a vertex community detection algorithm to the corresponding line graph.

In addition to the introduced strategy to obtain vertex partitions from link partitions, this paper also introduces a divisive spectral community detection algorithm.

## 5.1 Divisive Algorithm

The spectral divisive heuristic introduced in this paper is strongly based on the leading eigenvalue algorithm proposed by Newman (2006).

Algorithm 1 presents the heuristic, called *SpecDiv*. It takes as input a graph $F = (V_F, E_F)$, which can be either the original graph $G$ or its corresponding line graph $G'$, the maximum number of communities to be created, *max_labels*, and the minimum number of vertices required to further divide a community, *min_size*. The algorithm returns a vertex partition for the input graph $F$. Hence, in case $F$ is the line graph, the algorithm returns a link partition of the original graph.

In line 1 of Algorithm 1 is constructed the modularity matrix $B_F$ according to Equation (1). Recall that if $F$ is a line graph, then the modularity matrix of $F$ is equivalent to the non-backtracking matrix of the original graph. In line 3, a partition $\mathcal{P}_F$ is created wherein all vertices belong to the community labeled 0. In line 4 the list $\mathcal{L}$ which stores the labels of the communities that can be further divided into two communities is defined. Line 7 sets the modularity matrix of the initial iteration, $B_F^{(1)}$, as $B_F$.

From lines 8 to 29, *SpecDiv* repeatedly breaks in two each community in $\mathcal{L}$ until either there is no community left in $\mathcal{L}$ or a predefined maximum number of communities is reached.

In lines 9 and 10, the set $Z$ of indices of vertices in the community labeled $lr$, chosen randomly from $\mathcal{L}$, is defined. In case the number of vertices indicated by $Z$ is greater than *min_size*, in line 12, the pairwise mappings $(i, \upsilon)$ between vertices $i \in Z$ and sequential numbers $\upsilon \in \{1, 2, \ldots, |Z|\}$ are assigned to set $D$. This set is used to map the vertices of $Z$ to indices that correspond to the appropriate components of $B_F$ and $u$. In line 13, $B_F^{(it)}$ is constructed by selecting only the rows and columns of $B_F$ that correspond to vertices in $Z$, according to Section 4.1.

In line 14, the selected element $lr$ from the list $\mathcal{L}$ is removed. In line 15, the largest eigenvalue, $\lambda$ and the respective eigenvector, $u_1$, of $B_F^{(it)}$ are calculated. We used the *implicitly restarted Arnoldi method* from

Algorithm 1: *SpecDiv*.

**Input :** $F = (V_F, E_F)$, *max_labels*, *min_size*
**Output:** $\mathcal{P}_F*$

1 $B_F :=$ compute the modularity matrix of $F$ according to Equation (1)
2 $lb := 0$
3 $\mathcal{P}_F := \{c_1 = 0, c_2 = 0, \ldots, c_{|V_F|} = 0\}$
4 $\mathcal{L} := \{0\}$
5 $\mathcal{P}_F* := \mathcal{P}_F$
6 $it := 1$
7 $B_F^{(1)} := B_F$
8 **while** $\mathcal{L} \neq \emptyset$ *and number of communities of* $\mathcal{P}_F$ *is less than max_labels* **do**
9 $lr :=$ randomly chose a community label in $\mathcal{L}$
10 $Z := \{i$ such that $c_i = lr$ for $i = 1, \ldots, |V_F|\}$
11 **if** $|Z| >$ *min_size* **then**
12 $D := \{(i, \upsilon)$ such that $i \in Z$ and $\upsilon$ is selected without replacement from $\{1, 2, .., |Z|\}\}$
13 $B_F^{(it)} :=$ Select only the rows and columns of $B_F$ which to vertices in $Z$ according to Section 4.1
14 $\mathcal{L} := \mathcal{L} \backslash \{lr\}$
15 $\lambda, u_1 :=$ Compute the largest eigenvalue and associated eigenvector of $B_F^{(it)}$
16 $\mathcal{U}^+ := \{i \in Z$ such that $u_{\upsilon 1} \geq 0$, $\upsilon : (i, \upsilon) \in D\}$
17 $\mathcal{U}^- := \{i \in Z$ such that $u_{\upsilon 1} < 0$, $\upsilon : (i, \upsilon) \in D\}$
18 $c_i = lb$, for $i \in \mathcal{U}^+$
19 $c_i = lb + 1$, for $i \in \mathcal{U}^-$
20 $Q(\mathcal{P}_F) :=$ Calculate modularity according to Equation (2).
21 **repeat**
22 $\mathcal{P}_F, \Delta Q :=$ move each $i \in Z$ to the community labeled $lb$ or $lb+1$ that results in the largest modularity gain
23 **until** $\Delta Q \leq 0$;
24 **if** $Q(\mathcal{P}_F) > Q(\mathcal{P}_F*)$ **then** $\mathcal{P}_F* = \mathcal{P}_F$;
25 **if** *communities labeled lb and* $lb+1$ *are not empty* **then** $\mathcal{L} := \mathcal{L} \cup \{lb\} \cup \{lb+1\}$ ;
26 $lb := lb + 2$
27 $it := it + 1$
28 **repeat**
29 $\mathcal{P}_F*, \Delta Q* :=$ move each $i \in V_F$ to the community that results in the largest modularity gain
30 **until** $\Delta Q* \leq 0$;
31 **return** $\mathcal{P}_F*$

ARPACK library (Lehoucq et al., 1998) for this step [1]. In lines 16 and 17, $\mathcal{U}^+$ and $\mathcal{U}^-$ are the sets of vertices from $Z$ whose corresponding signs of the components of $u_1$ larger than or equal to zero are assigned to $\mathcal{U}^+$, whereas those lower than 0 are assigned to $\mathcal{U}^-$. In lines 18 and 19, vertices from $\mathcal{U}^+$ and $\mathcal{U}^-$ receive two new community labels $lb$ and $lb+1$ in partition $\mathcal{P}_F$, respectively. In case one of the sets is empty, the community is not divided. Otherwise, the modularity of partition $\mathcal{P}_F$ is calculated according to Equation (2) in line 21.

After that or when $|Z| \leq 8$, a local search strategy in line 23 moves each vertex in $Z$ to the community labeled $lb$ or $lb+1$ that results the largest modularity gain. To avoid biased results, we select the vertices to be moved in a random order. The local search is repeated until there is no movement to improve the modularity gain.

In line 25, the algorithm updates the best partition found along the iterations, $\mathcal{P}_F*$, to $\mathcal{P}_F$ if $Q(\mathcal{P}_F)$ is higher than $Q(\mathcal{P}_F*)$. In line 27, the community labels $lb$ and $lb+1$ are inserted to the list $\mathcal{L}$ if both of them are not empty, i.e., contain at least one vertex. In line 27, the label of the next community to be created, $lb$, is updated and in line 28 the number of iterations, $it$, is incremented.

After defining a vertex partition $\mathcal{P}_F*$ in $F$, *SpecDiv* employs a local search strategy to improve it, until there is no move that improves the modularity value in line 31. Different from the local search performed in line 23, the vertices can be moved to any community. Finally, Algorithm 1 returns $\mathcal{P}_F*$ as output.

The next sections explain the proposed strategies to find disjoint and overlapping communities.

## 5.2 Finding Disjoint Vertex Partitions from Link Partitions

Algorithm 2 presents the strategy to find disjoint vertex communities having as input a link partition. It takes as input a connected graph $G$ and a link partition $\mathcal{P}'$ obtained by some vertex community detection algorithm applied to the line graph of $G$, here called $G'$.

In line 1, the algorithm assigns to $CV_i$ the community labels of the edges incident to vertex $i \in V$ in $G$, considering the link partition $\mathcal{P}'$. In line 2, to define a vertex partition, for each vertex $i$, $mode(CV_i)$ returns to $c_i$ the most frequent label in $CV_i$. The community labels identified in line 2 define a vertex partition $\mathcal{P}$ for $G$ in line 3. In line 4, a local search procedure that moves each vertex $i \in V$ from community labeled $c_i$ to that from $CV_i$ that results in the largest modularity gain is applied to $\mathcal{P}$. Again, we select the vertices to be moved in a random order. Finally, the algorithm returns the refined vertex partition $\mathcal{P}*$ in line 5.

Algorithm 2: Disjoint community detection.

**Input :** Connected graph $G = (V,E)$, link partition $\mathcal{P}'$
**Output:** $\mathcal{P}*$
1 $CV_i := \{c'_{(i,j)} \in \mathcal{P}'$ such that $(i,j) \in E\}$, for $i \in V$
2 $c_i = mode(CV_i)$, for $i \in V$
3 $\mathcal{P} := \{c_1, c_2, \ldots, c_{|V|}\}$
4 $\mathcal{P}* :=$ move each $i \in V$ from community labeled $c_i$ to that from $CV_i$ that results the largest modularity gain
5 **return** $\mathcal{P}*$

Algorithm 3: Overlapping community detection.

**Input :** Connected graph $G = (V,E)$, link partition $\mathcal{P}'$
**Output:** $\mathcal{PV}$
1 $CV_i := \{c'_{(i,j)} \in \mathcal{P}'$ such that $(i,j) \in E\}$, for $i \in V$
2 $\mathcal{PV} := \{CV_1, CV_2, \ldots, CV_{|V|}\}$
3 $\mathcal{X} :=$ set of community labels of vertices from $V$ that does not present overlapping in $\mathcal{P}'$
4 **forall** $i \in V$ *such that* $|CV_i| > 1$ **do**
5 **if** $|\{ci \in CV_i \cap \mathcal{X}\}| > 0$ **then** $\mathcal{X}i := \{ci \in CV_i \cap \mathcal{X}\}$;
6 **else** $\mathcal{X}i := CV_i$;
7 $CV_i :=$ $\{ld \in \mathcal{X}i | (\sum_{j \in V \text{ such that } c_j = ld} b_{ij}) > 0\}$
8 $\mathcal{PV} := \{CV_1, CV_2, \ldots, CV_{|V|}\}$
9 **return** $\mathcal{PV}$

## 5.3 Finding Overlapping Vertex Partitions from Link Partitions

In Algorithm 3, we describe the strategy to identify overlapping vertex communities from a link partition given as input, along with a connected graph $G$.

In line 1, $|V|$ sets of community labels according to the link partition given as input are created. In line 2, a list of overlapping communities with the vertex community labels assigned to $CV_i$ is defined.

[1] We required that $|Z|$ must be higher than $min\_size = 8$ due to conditions imposed by the number of iterations selected for the *implicitly restarted Arnoldi method* to converge. The local search procedures are capable of improving small communities with respect to the modularity value.

In line 3, set $\mathcal{X}$ with the community labels of the vertices of $G$ which are in only one community is constructed, i.e., vertices $i \in V$ such that $|CV_i| = 1$. We shall refer to vertices that belong to more than one community as overlapping vertices.

From lines 5 to 7, a post-processing strategy to define the community labels of each overlapping vertex $i, CV_i > 0$, is applied to $\mathcal{PV}$. In line 5, elements of $\mathcal{X}i$ are defined as the overlapping communities of vertex $i$, $CV_i$, in $\mathcal{X}$. If $CV_i \cap \mathcal{X}$ is empty, then $\mathcal{X}i := CV_i$ in line 6.

In line 7, the community labels of each overlapping vertex $i \in V$ are selected as those whose sum of the modularity contributions between $i$ and the vertices of that community is higher than 0. We require that a vertex is only moved to a community that exists in $\mathcal{X}$ to avoid the creation of communities composed only by overlapping vertices.

Finally, in line 10, the overlapping communities $\mathcal{PV}$ with the community labels of each vertex defined according to line 7 are returned.

## 6 CASE STUDY IN A TWITTER NETWORK

Before presenting the computational experiments, in this section, we study a practical application of overlapping community detection in a Twitter retweet network constructed by Ribeiro et al. (2018). The authors of this network also classified users that used hateful speech in Twitter.

The network presented here is the main connected component of a small subgraph induced by 5000 vertices and 10 hateful users selected randomly from the network. The order of the selected subgraph was due to computational performance restrictions of the proposed spectral divisive heuristic. The resulting numbers of vertices and edges after extracting the main connected component of this subgraph are 2885 and 4874, respectively.

Figure 1[2] illustrates the overlapping vertex communities $\mathcal{PV}$ found using Algorithm 3 having as input a link partition, obtained by *SpecDiv* applied to the line graph. In this figure, the colors of the vertices and edges identify the vertex and link communities. The overlapping vertices are multi-colored according to their community labels. The large-sized vertices defined by the '*' mark are the hateful users identified by Ribeiro et al. (2018). The large-sized vertices without marks corresponds to the top 1% vertices with the highest degrees.

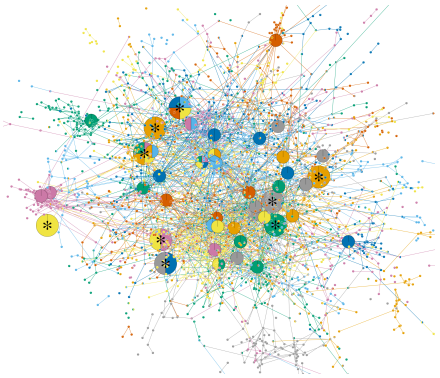


Figure 1: Overlapping communities found in a sample of a Twitter network.

The non-backtracking modularity of $\mathcal{PV}$, i.e., the modularity of the line graph, is 0.8384. The modularity of the partition obtained from the link partition by Algorithm 2 is 0.6371.

Approximately 36.05% of the vertices belong to overlapping communities. Among them, 62.31% have 2-overlapping community labels and only 5.1% have more than 4 communities. The average number of overlapping community labels is 2.59, with a standard deviation from the average of 0.91.

According to Figure 1, most of the top 1% highest degree nodes do not belong to overlapping communities and many of them belong to the largest communities. On the other hand, 4 of the 9 hateful users belong to overlapping communities. In particular, 2 of them belong to 4-overlapping communities. At least one hateful user in each community also belongs to another community that includes another hateful user. Overall, 44.44% of the hateful users belong to overlapping communities.

To ensure the consistency of the results, we replicated the aforementioned experiment on 5 different random subgraph sampling drawn from the original network. The average number of vertices and edges of the networks are 2868.8 and 4953.2, respectively. The average modularity values for the line graph and for the partition obtained from the link partition are respectively 0.6193 and 0.8359. The average number of overlapping communities is 2.6. On average, 37.99% of the vertices in the networks belong to overlapping communities. Among them, 62.45% are in 2-overlapping communities and only 5.23% belong to more than 4-overlapping communities, on average. These results are consistent with those presented in Figure 1.

In only one subgraph sampling the hateful users do not belong to any overlapping communities. The

[2]Some of the vertices and edges of the graph were suppressed to improve its readability.

average percentage of hateful users that belong to overlapping communities is 40.89% and presents the large standard deviation value of 23.54%. In fact, this experiment suggests that the subgraph sampling only affects the distribution of hateful users in the communities.

An interesting application from these empirical observations based on overlapping communities would be to study the influence of these overlapping hateful users on other users of the network.

# 7 COMPUTATIONAL EXPERIMENTS

This section presents the computational experiments with sparse LFR networks with disjoint and overlapping communities, generated using the software of Lancichinetti et al. (2008).

All the LFR networks were generated having as parameters: number of vertices equal to 1000, average and maximum degrees given by 3 and 10, respectively and exponential distribution equal to 2. Table 1 presents the remaining characteristics that distinguish the networks into sets: minimum and maximum number of vertices in the communities, *minc* and *maxc*, respectively; number of overlapping vertices, $OVn$; number of communities for the overlapping vertices, $OVm$; and degree of mixture of the communities, $\mu$.

Table 1: Characteristics of the generated LFR networks.

| Set | *minc* | *maxc* | $OVn$ | $OVm$ | $\mu$ |
|---|---|---|---|---|---|
| SD | 10 | 50 | 0 | - | $0.1, 0.2, \ldots, 0.8$ |
| LD | 20 | 100 | 0 | - | $0.1, 0.2, \ldots, 0.8$ |
| SOv | 10 | 50 | 100 | 2 | $0.1, 0.2, \ldots, 0.8$ |
| LOv | 20 | 100 | 100 | 2 | $0.1, 0.2, \ldots, 0.8$ |

The disjoint networks are referred to as in sets *SD* and *LD*, whereas the overlapping ones are said to be in *SOv* and *LOv* sets. Networks from *SD* and *SOv* are composed by small communities, whereas those in *LD* and *LOv* are composed by large communities.

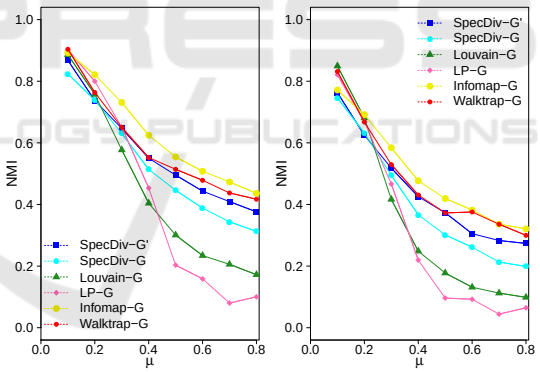
We generated 5 different networks for each set and value of $\mu$ described in Table 1 and reported the average results achieved over 5 independent executions. All the computational experiments were performed on a computer with a processor Intel Xeon E5-1620 with 3.70Gz and main memory of 32 GB. The disjoint and overlapping strategies were implemented in R with igraph library (Csardi and Nepusz, 2006). *SpecDiv* was implemented in C++ with ARPACK++ library. The parameter *max_label* of *SpecDiv* was set to be 10% of the number of vertices in the graph and the parameter *min_size* was set to 8.

## 7.1 Disjoint Communities

In this Section, we contrast the partitions found by vertex community detection algorithms with those obtained by the disjoint community detection strategy that takes as input a link partition from the line graph $G'$. To differentiate the strategies using line graphs from those using the original graphs as inputs, we add the suffixes "$-G'$" and "$-G$" to the names of the algorithms.

To assess a disjoint partition, we employed the Normalized Mutual Information (NMI) measure (Danon et al., 2005), which indicates the correlation between the partitions in a scale from 0 to 1, where 1 indicates a perfect correlation.

First, Figure 2 present the average NMI values achieved by *SpecDiv-G*, *SpecDiv-G'* and by the reference disjoint community detection algorithms Louvain method, LP, Infomap and Walktrap. According to these figures, *SpecDiv-G'* outperformed *SpecDiv-G* and presented better results than the algorithms with the exception of Infomap and Walktrap. Despite being worse than *SpecDiv-G'*, *SpecDiv-G* still obtained better results than Louvain and LP for networks with $\mu \geq 0.3$.



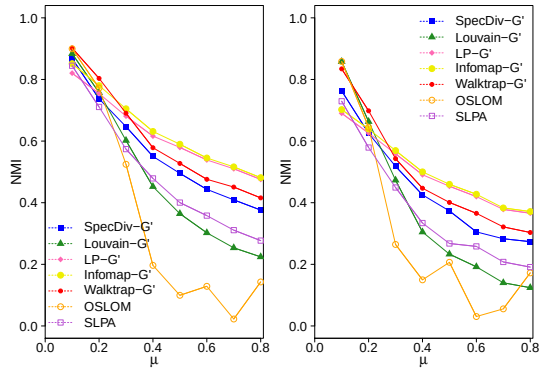

(a) Networks from *SD*. (b) Networks from *LD*.

Figure 2: Average NMI values over 5 executions of the methods for detecting disjoint communities.

Figure 3 displays the average NMI values achieved by *SpecDiv-G'* and by the disjoint community detection strategy taking as input the link partition obtained by the reference algorithms applied to the line graphs. It also presents the results of OSLOM[3] and of the disjoint version of SLPA.

The results of *SpecDiv*, Louvain, LP, Infomap and Walktrap improved on average 9.93%, 14.95%,

[3]OSLOM identified overlapping in some of the disjoint networks. We chose one community label from each overlapping vertex to present its results.




(a) Networks from *SD*. (b) Networks from *LD*.

Figure 3: Average NMI values over 5 executions of the methods for detecting disjoint communities using line graphs.

170.25%, 2.64% and 2.6%, respectively, when compared to those for networks from *SD* presented in Figure 2(a) and 16.62%, 20.32%, 258.57%, 4.64% and 1.63%, respectively, when compared to those for networks from *LD* presented in 2(b). LP, Infomap and Walktrap were able to find better results than *SpecDiv-G'* by using the introduced link strategy. Moreover, the disjoint communities obtained from the link partitions, with the exception of Louvain, were better than those found by OSLOM and SLPA for $\mu \geq 0.3$.

The maximum standard deviation values from the presented averages results of LP and OSLOM were 0.19 and 0.27, respectively. The remaining algorithms found results consistently closer to the average, reporting standard deviations values lower than or equal to 0.05.

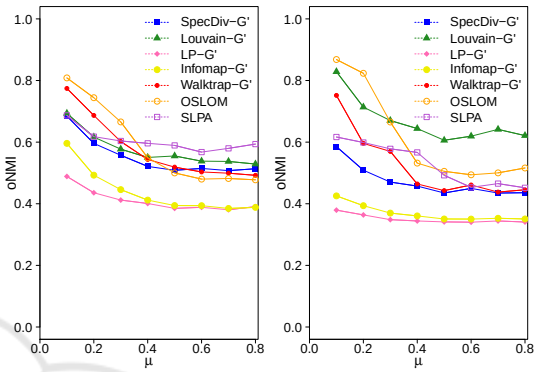
These results allow us to conclude that even though the introduced link strategy is mainly designed to detect overlapping communities, it can improve the detection of disjoint communities in networks.

## 7.2 Overlapping Communities

The traditional NMI employed in the previous section is not capable of comparing overlapping communities. In this sense, we employed the extension of the NMI introduced by Lancichinetti et al. (2009), called oNMI, to assess the communities with respect to the ground truth provided by the LFR software. The implementation is due to McDaid et al. (2011).

In this section, we contrast the oNMI values achieved by the introduced overlapping strategy having as input link partitions obtained by *SpecDiv*, Louvain method, Infomap, LP and Walktrap with those achieved by the state-of-the-art overlapping methods OSLOM and SLPA[4].

Figure 4 reports the oNMI values obtained by the algorithms. According to Figure 4(a), for $\mu \leq 0.3$, OSLOM obtained the communities with the highest oNMI values. For $\mu \geq 0.4$, SLPA found the communities with the highest oNMI values. *SpecDiv-G'*, Louvain-$G'$ and Walktrap-$G'$ achieved better results than OSLOM for networks with $\mu \geq 0.5$ and *SpecDiv-G'* was better than Walktrap-$G'$ in networks with $\mu \geq 0.6$.




(a) Networks from *SOv*. (b) Networks from *LOv*.

Figure 4: Average oNMI values over 5 executions of the methods for detecting overlapping communities.

On the other hand, Figure 4(b) shows that in networks with large overlapping communities, OSLOM found the communities with the highest oNMI values for $\mu \leq 0.2$ and Louvain-$G'$ method found the communities with the best oNMI values for $\mu \geq 0.3$.

The standard deviations of OSLOM and SLPA from the average values in the large community networks, however, were in the intervals $[0.02, 0.06]$ and $[0.02, 0.23]$, respectively. Thereby, SLPA might not find consistently good results in all its executions. On the other hand, the standard deviations of the remaining algorithms were lower than or equal to 0.05. These values suggest the robustness of the algorithms based on link partitions.

We, nonetheless, highlight that these networks are very sparse and thus are different from the traditional denser LFR networks considered in most of the experiments performed in the literature.

## 7.3 Running Times

Table 2 summarizes the average running times of the algorithms to detect disjoint and overlapping communities. The numbers presented between parentheses are the standard deviation values from the averages.

[4]The threshold parameter of SLPA was set to 0.1.

Table 2: Average running times of the algorithms in seconds.

| Set | *SpecDiv-G'* | Louvain-$G'$ | LP-$G'$ | Infomap-$G'$ | Walktrap-$G'$ | OSLOM | SLPA |
|---|---|---|---|---|---|---|---|
| *SD* | 370.88 (51.17) | 32.05 (12.25) | 45.92 (9.74) | 45.17 (12.58) | 38.49 (15.4) | 37.86 (9.14) | 1.74 (0.12) |
| *LD* | 346.55 (38.36) | 38.31 (10.53) | 57.57 (4.89) | 56.61 (5.6) | 45.5 (13.2) | 41.02 (4.34) | 1.78 (0.16) |
| *SOv* | 451.1 (70.05) | 15.89 (3.59) | 21.99 (2.03) | 22.63 (3.31) | 18.94 (4.64) | 36.56 (8.24) | 2.8 (0.22) |
| *LOv* | 439.99 (30.43) | 18.88 (4.06) | 26 (2.52) | 26.81 (2.57) | 22.23 (4.9) | 39.58 (3.79) | 2.8 (0.28) |

Despite the good results achieved by *SpecDiv-G'* in disjoint and overlapping communities, it is the most time-consuming algorithm. Even though the fastest algorithm to find overlapping communities is SLPA, it obtained disjoint partitions worse than *SpecDiv-G'*.

We did not present the results of *SpecDiv-G*, Louvain-*G*, LP-*G*, Infomap-*G* and Walktrap-*G* because they presented worse results than *SpecDiv-G'*, Louvain-$G'$, LP-$G'$, Infomap-$G'$ and Walktrap-$G'$, respectively, and are not capable of finding overlapping communities. However, we must point out that, with the exception of *SpecDiv-G*, the remaining algorithms required less than 2$s$ to find disjoint communities. *SpecDiv-G* required on average 48.8$s$ and 47.55$s$ to find disjoint communities for *SD* and *LD*, respectively.

## 8 CONCLUSIONS

Overlapping community detection in networks is a challenging problem explored by a few studies in the literature. In particular, a research direction to approach it consists of applying community detection algorithms to the line graphs of the networks under investigation.

In this paper, we propose a strategy to detect overlapping communities in networks founded on detecting link partitions in line graphs and hence defining the overlapping communities by an algorithm whose input is link partitions. As part of the strategy, we suggest a spectral community detection algorithm named *SpecDiv*.

The results of the experiments performed using LFR networks showed that the overlapping communities found by the strategy based on line graphs are competitive with state-of-the-art overlapping methods.

Additional experiments were carried out using disjoint LFR networks. As a result, the proposed strategy obtained better vertex partitions from the link partitions than those obtained by *SpecDiv* and by the reference algorithms applied to the original graphs.

In a case study using a Twitter network, we found that most of the highest degree vertices do not belong to overlapping communities. On the other hand, almost half of the users that use hateful speech belong to overlapping communities. We have also observed that hateful users belong to overlapping communities which includes other hateful users. This observation suggests the study of the influence of these users on the rest of the network.

A major advantage of the introduced overlapping strategy is its capability of obtaining overlapping communities from any vertex community detection algorithm.

The presented strategy is intended to be employed in sparse networks when the number of vertices is approximately the number of edges. A lot of large-scale networks have that characteristic. The high cost of the methods based on line graphs makes them unsuitable for large dense networks. To better approach this type of networks, the strategy must be parallelized to be more efficient in multi-core computers.

We suggest as a future work to reduce substantially the number of edges of line graphs by considering edge contraction strategies in the original or the line graph itself. Another research direction is the extension of the presented methods to support directed graphs.

## ACKNOWLEDGEMENTS

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