Modeling the Esscher Premium Principle for a System of Elliptically Distributed Risks

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Abstract: The Esscher premium principle provides an important framework for allocating a certain loaded premium for some claim (risk) in order to manage the risks of insurance companies. In this paper, we show how to model the celebrated Esscher premium principle for a system of elliptically distributed dependent risks, where each risk is greater or equal than its value-at-risk. Furthermore, we present calculations of the proposed multivariate risk measure, investigate its properties and formulas, and show how special elliptical models can be implemented in the theory.

1 INTRODUCTION

Recently, there is a growing interest in multivariate risk measures. The motivation behind considering a multivariate risk measure is that it provides more accurate measurements of risks that are mutually dependent on each other. There are several attempts to obtain such multivariate measures (Jouini et al., 2004; Molchanov and Cascos, 2016; Cousin and Di Bernardino, 2014; Feinstein and Rudloff, 2017; Landsman et al., 2016; Shushi, 2018). For instance, Landsman et al. (2016) introduced the multivariate tail conditional expectation (MTCE) with the following form

$$MTCE_q(\mathbf{X}) = E\left(\mathbf{X}|\mathbf{X} > VaR_q(\mathbf{X})\right).$$

Here $\mathbf{X} = (X_1, X_2, ..., X_n)^T$ is $n \times 1$ vector of risks that are mutually depending on each other, and $VaR_q(\mathbf{X}) = \left(VaR_{q_1}(X_1), VaR_{q_2}(X_2), ..., VaR_{q_n}(X_n)\right)^T$ is $n \times 1$ vector, where $VaR_{q_i}(X_i)$ is the value at risk of X_i under the $q_i - th$ quantile, $q_i \in (0,1)$. In this notation, for two n-variate random vectors $\mathbf{X}$ and $\mathbf{Y}$, $\mathbf{X} \geq \mathbf{Y}$ means that $\{X_i \geq_{a.s.} Y_i, i = 1, ..., n\}$. The multivariate tail covariance measure was also introduced in the literature by (Landsman et al., 2018), and obtained for the class of elliptical distributions. The Esscher premium principle is a widely used measure in risk measurement and portfolio theory, which allows to quantify insurance premiums (Kamps, 1998; Van Heerwaarden et al., 1989; Landsman, 2004; Shushi, 2017; Chi et al., 2017). In the theory of risks there exist vast number of different models to calculate insurance premiums (Goovaerts et al., 1984; Wang and Dhaene, 1998; Déniz et al., 2000) . The Esscher premium was first introduced in the seminal paper of Buhlmann (Bühlmann, 1980). In his paper, Buhlmann claimed that actuaries think about premiums as a measure of risks, which are considered random. Unlike actuarial premiums, economical premiums depend also on market conditions which can be characterized by another random risk. In this paper we focus on actuarial premiums, and thus we are not taking into account any market conditions.

Let X be a random risk. Then, the Esscher premium of X takes the following form

$$\pi_\lambda(X) = \frac{E\left(Xe^{\lambda X}\right)}{M_X(\lambda)}, \tag{1}$$

where $\lambda > 0$, $M_X(\lambda) = E\left(e^{\lambda X}\right)$ is the moment generating function (MGF) of X, and $E\left(Xe^{\lambda X}\right) < \infty$.

The Wang's premium (Wang et al., 2002) introduced as an exponential tilting of some risk, X, induced by another risk, Y,

$$\pi_\lambda(X,Y) = \frac{E\left(Xe^{\lambda Y}\right)}{M_Y(\lambda)}, \tag{2}$$

and the Esscher premium is the special case of $\pi_\lambda(X,X) = \pi_\lambda(X)$. Furthermore, (2) has actuarial

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sense behind the quantitative measure. For a portfolio consisting $n \times 1$ risks, $\mathbf{X}$, the measure $\pi_\lambda(X_i, S)$ quantifies the amount of risk of X_i to the aggregate risks, which is the sum of the risks, i.e., $S = X_1 + X_2 + ... + X_n$.

In Shushi (2018), a multivariate conditional version of the Esscher premium has been introduced which takes into account only the tail of the multivariate distribution of the vector of risks. In this paper, we generalize the Esscher premium into a conditional framework such that each risk X_i is greater than its value-at-risk measure and consider the most general case which is a portfolio that consists of $n-$variate dependent risks.

The multivariate conditional Esscher premium (MCEP) takes the following form

$$\pi_{\alpha,\lambda}(\mathbf{X}) = \frac{E\left(\mathbf{X}e^{\lambda^T\mathbf{X}}|\mathbf{X} > VaR_q(\mathbf{X})\right)}{E\left(e^{\lambda^T\mathbf{X}}|\mathbf{X} > VaR_q(\mathbf{X})\right)},\ \lambda_i > 0, \quad (3)$$

$$\lambda = (\lambda_1, ..., \lambda_n)^T, i = 1, 2, ..., n.$$

The motivation behind this multivariate risk measure is that it provides a conservative premium principle, in the sense that it quantifies the premium under the assumption that the $i-$th loss is greater than its value-at-risk, $X_i \geq VaR_q(X_i), i = 1, 2, ...$. and therefore the MCEP measure is greater than or equal to the Esscher premium:

$$\pi_\lambda(\mathbf{X}) \leq \pi_{\alpha,\lambda}(\mathbf{X}),$$

where $\pi_\lambda(\mathbf{X}) = (\pi_\lambda(X_1), \pi_\lambda(X_2), ..., \pi_\lambda(X_n))^T$.

We define the conditional analog to the Wang's premium, as follows:

$$\pi_{\alpha,\lambda}(X_i, S) = \frac{E\left(X_i e^{\lambda S}|S > \sum_{i=1}^n VaR_q(X_i)\right)}{E\left(e^{\lambda S}|S > \sum_{i=1}^n VaR_q(X_i)\right)}.$$

In the next Section, we give a concise definition of the family of elliptical distributions, and in Section 3 we analyze the proposed MCEP measure by providing its main properties and their implications. In Section 4 we compute the MCEP for a system of mutually dependent elliptically distributed risks, and in Section 5 we give examples. Section 6 offers a discussion to the paper.

2 THE CLASS OF ELLIPTICAL DISTRIBUTIONS

The class of elliptical distributions consists many important distributions such as the normal, Student-t, logistic, and Laplace distributions. In fact, it is a natural generalization of the normal distribution (Cambanis et al., 1981). This class has attempting properties which will be shown in the sequel.

Let $\mathbf{X}$ be $n \times 1$ random vector following elliptical distribution, $\mathbf{X} \backsim E_n(\mu, \Sigma, g_n)$. Then, the pdf of $\mathbf{X}$ is

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{c_n}{\sqrt{|\Sigma|}} g_n\left(\frac{1}{2}(\mathbf{x}-\mu)^T\Sigma^{-1}(\mathbf{x}-\mu)\right), \mathbf{x} \in R, \quad (4)$$

where c_n is the normalizing constant, $g(u)$, $u \geq 0$, is called the density generator of $\mathbf{X}$, μ is an $n \times 1$ location vector, and Σ is an $n \times n$ scale matrix.

The characteristic function of $\mathbf{X}$ takes the following form

$$\varphi_{\mathbf{X}}(\mathbf{t}) = \exp(i\mathbf{t}^T\mu)\psi_{g_n}(\frac{1}{2}\mathbf{t}^T\Sigma\mathbf{t}), \quad (5)$$

some function $\psi_{g_n}(u) : [0, \infty) \to R$, called the characteristic generator.

The marginal distributions of the elliptical distribution are also elliptical with the same characteristic generator. For a random vector $\mathbf{X}$ such that

$$\mathbf{X} = (\mathbf{X}_1, \mathbf{X}_2)^T \backsim E_n\left(\begin{pmatrix}\mu_1 \\ \mu_2\end{pmatrix}, \begin{pmatrix}\Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22}\end{pmatrix}, g_n\right),$$

where $\mathbf{X}_1$ and $\mathbf{X}_2$ are m ($m < n$) and $n-m$ random vectors, the characteristic function of $\mathbf{X}$, $\varphi_{\mathbf{X}}(\mathbf{t})$, takes the form

$$\varphi_{\mathbf{X}}(\mathbf{t}) = \exp(i(\mathbf{t}_1^T\mu_1 + \mathbf{t}_2^T\mu_2)) \cdot \psi\left(\frac{1}{2}\begin{pmatrix}\mathbf{t}_1 \\ \mathbf{t}_2\end{pmatrix}^T \Sigma \begin{pmatrix}\mathbf{t}_1 \\ \mathbf{t}_2\end{pmatrix}\right),$$

$$\mathbf{t}_1 \in R^m, \mathbf{t}_2 \in R^{n-m}.$$

Then, for the marginal $\mathbf{X}_1$ of $\mathbf{X}$ we take $\mathbf{t}_2 = \mathbf{0}$ where $\mathbf{0}$ is vector of $n-m$ zeros,

$$\varphi_{\mathbf{X}}((\mathbf{t}_1, \mathbf{0}_2)^T) = \exp(i\mathbf{t}_1^T\mu_1) \cdot \psi\left(\frac{1}{2}\mathbf{t}_1^T\Sigma_{11}\mathbf{t}_1\right) \quad (6)$$

$$= \varphi_{\mathbf{X}_1}(\mathbf{t}_1).$$

As can be clearly seen, the above equation takes the same form as (5) with vector of locations μ_1, scale matrix Σ_{11}, and characteristic generator $\psi(u)$. Therefore, $\mathbf{X}_1 \backsim E_m(\mu_1, \Sigma_{11}, g_m)$.

For $m \times n$ matrix B with rank $m \leq n$ and $m \times 1$ vector $\mathbf{c}$, the transformation $B\mathbf{X} + \mathbf{c}$ is $m-$variate elliptical random vector, i.e., $B\mathbf{X} + \mathbf{c}$ is distributed $E_m(\mu^* = B\mu + \mathbf{c}, \Sigma^* = B\Sigma B^T, g_m)$. This can be shown by the form of elliptical characteristic function. From (5) it follows that

$$\begin{aligned}\varphi_{B\mathbf{X}+\mathbf{c}}(\mathbf{t}) &= e^{i\mathbf{t}^T\mathbf{c}}\varphi_{\mathbf{X}}(B\mathbf{t}) \\ &= \exp(i\mathbf{t}^T(B\mu + \mathbf{c})) \cdot \psi(\frac{1}{2}(B\mathbf{t})^T\Sigma(B\mathbf{t})) \\ &= \exp(i\mathbf{t}^T\mu^*) \cdot \psi(\frac{1}{2}\mathbf{t}\Sigma^*\mathbf{t}).\end{aligned}$$

From this property, we immediately establish that the marginal distribution is also elliptical as has been shown previously.

Let matrix B be

$$B = \begin{pmatrix} I_{m\times m} & 0_{(n-m)\times m} \\ 0_{m\times(n-m)} & 0_{(n-m)\times(n-m)} \end{pmatrix},$$

where $I_{m\times m}$ is $m\times m$ identity matrix, $0_{(n-m)\times m}$ is $(n-m)\times m$ matrix with zero components, $0_{(n-m)\times(n-m)}$ is $(n-m)\times(n-m)$ matrix with zero components, and $0_{(n-m)\times m} = 0^T_{m\times(n-m)}$. Then, random vector $B\mathbf{X}$ is the marginal random vector $\mathbf{X}_1$. Furthermore, in the case that $B = \mathbf{b}$ is a $n\times 1$ vector, then $\mathbf{b}^T\mathbf{X}$, representing weighted-sum, is distributed $E_1(\mathbf{b}^T\mu, \mathbf{b}^T\Sigma\mathbf{b}, g_1)$.

3 THE PROPERTIES OF THE MCEP MEASURE

Let us now show some important and desirable properties of the MCEP measure for the elliptical model.

Proposition 1. *Proposition 1. Let $\mathbf{X} \backsim E_n(\mu,\Sigma,g_n)$ be a system of n elliptically distributed risks. Then, the MCEP follows the properties:*

1. *Translation Invariance: For any random vector of risks $\mathbf{X}$ and any vector of constants $\alpha \in R^n$*
$$\pi_{\alpha,\lambda}(\mathbf{X}+\alpha) = \pi_{\alpha,\lambda}(\mathbf{X}) + \alpha. \tag{7}$$
2. *Independence of risks:* If the vector of risks $\mathbf{X}$ has independent components. Then
$$\pi_{\alpha,\lambda}(\mathbf{X}) = \begin{pmatrix} \pi_{q,\lambda_i}(X_1) \\ \pi_{q,\lambda_2}(X_2) \\ \dots \\ \pi_{q,\lambda_n}(X_n) \end{pmatrix}. \tag{8}$$
3. *Monotonicity:* Suppose $\mathbf{Y},\mathbf{X}$, are $n\times 1$ random vectors of risks and $\mathbf{Y} \overset{a.s}{\geq} \mathbf{X}$. Then
$$\pi_{\alpha,\lambda}(\mathbf{Y}-\mathbf{X}) \geq \mathbf{0}, \tag{9}$$
where $\mathbf{0}$ is vector of n zeros.
4. *Semi-Positive Homogeneity: For some positive constant $a>0$*, The MCEP follows the following equality
$$\pi_{\alpha,\lambda}(a\mathbf{X}) = a\pi_{\alpha,\lambda}(\mathbf{X}). \tag{10}$$
5. *Semi-subadditivity of $\pi_{\alpha,\lambda}(\mathbf{X})$ for elliptical distributions:* Consider an $(2n)\times 1$ elliptical random vector $\mathbf{X}$ with the partition $\mathbf{X} = \left(\mathbf{X}_1^T, \mathbf{X}_2^T\right)^T$, $\mathbf{X}_1 = (X_1,\dots,X_n)^T, \mathbf{X}_2 = (X_{n+1},\dots,X_{2n})^T$. Then, the following inequality hold
$$\pi_{\alpha,\lambda}(\mathbf{X}_1+\mathbf{X}_2) \leq \pi_{\alpha,\lambda}(\mathbf{X})_1 + \pi_{\alpha,\lambda}(\mathbf{X})_2, \tag{11}$$
where $\pi_{\alpha,\lambda}(\mathbf{X}) = \left(\pi_{\alpha,\lambda}(\mathbf{X})_1^T, \pi_{\alpha,\lambda}(\mathbf{X})_2^T\right)^T$.

The motivation behind the semi-subadditivity property can be found in Landsman et al. (Landsman et al., 2016) . In our case (11) means that combining risks provides less premium than separating them.

Proof.

1. The translation invariance property can be proved after some algebraic calculations. We notice that $VaR_q(\mathbf{X}+\alpha) = \alpha + VaR_q(\mathbf{X})$, so
$$\begin{aligned} &\pi_{\alpha,\lambda}(\mathbf{X}+\alpha) \\ &= \frac{E\left((\mathbf{X}+\alpha)e^{\lambda^T(\mathbf{X}+\alpha)} | \mathbf{X}+\alpha > VaR_q(\mathbf{X}+\alpha)\right)}{E\left(e^{\lambda^T(\mathbf{X}+\alpha)} | \mathbf{X}+\alpha > VaR_q(\mathbf{X}+\alpha)\right)} \\ &= \frac{E\left((\mathbf{X}+\alpha)e^{\lambda^T\mathbf{X}} | \mathbf{X} > VaR_q(\mathbf{X})\right)}{E\left(e^{\lambda^T\mathbf{X}} | \mathbf{X} > VaR_q(\mathbf{X})\right)} \\ &= \alpha + \pi_{\alpha,\lambda}(\mathbf{X}). \end{aligned}$$
2. Since we assumed that $(X_1, X_2, \dots, X_n)$ are mutually independent random risks the probability density function (pdf) of $\mathbf{X}$ is the multiplication for the pdf's of the i-th component of $\mathbf{X}$, so
$$\begin{aligned} \pi_{\alpha,\lambda}(\mathbf{X}) &= \frac{E\left(\mathbf{X}e^{\lambda^T\mathbf{X}} | \mathbf{X} > VaR_q(\mathbf{X})\right)}{E\left(e^{\lambda^T\mathbf{X}} | \mathbf{X} > VaR_q(\mathbf{X})\right)} \\ &= \frac{1}{\prod_{i=1}^{n} E\left(X_i e^{\lambda_i X_i} | X_i > VaR_q(X_i)\right)} \\ &\cdot \begin{pmatrix} E\left(X_1 e^{\lambda^T\mathbf{X}} | X_1 > VaR_q(X_1)\right) \\ E\left(X_2 e^{\lambda^T\mathbf{X}} | X_2 > VaR_q(X_2)\right) \\ \dots \\ E\left(X_n e^{\lambda^T\mathbf{X}} | X_n > VaR_q(X_n)\right) \end{pmatrix} \\ &= \begin{pmatrix} \pi_{q,\lambda_i}(X_1) \\ \pi_{q,\lambda_2}(X_2) \\ \dots \\ \pi_{q,\lambda_n}(X_n) \end{pmatrix}. \end{aligned}$$
3. Notice that as $\mathbf{Y}$ is greater than (a.s.) $\mathbf{X}$, we can define a random vector in which its components get only non-negative values, $\mathbf{V} = \mathbf{Y}-\mathbf{X} \geq \mathbf{0}$, where $\mathbf{0}$ is $n\times 1$ vector of zeros. Then, as $\mathbf{V}$ is non-negative random vector $VaR_q(\mathbf{U}) \geq \mathbf{0}$, and thus $\pi_{q,\lambda}(\mathbf{V}) \geq \mathbf{0}$,
$$\pi_{\alpha,\lambda}(\mathbf{Y}-\mathbf{X}) = E\left(\mathbf{V} | \mathbf{V} > VaR_q(\mathbf{V})\right) \geq \mathbf{0}.$$
4. Similar to the Esscher premium, the MCEP is not positive homogenous, but, the semi-positive ho-

mogeneity property holds, as follows:

$$\begin{aligned}\pi_{\alpha,\lambda}(a\mathbf{X}) &= \frac{E\left(a\mathbf{X}e^{a\lambda^T\mathbf{X}}|a\mathbf{X} > aVaR_q(\mathbf{X})\right)}{E\left(e^{a\lambda^T\mathbf{X}}|a\mathbf{X} > aVaR_q(\mathbf{X})\right)} \\ &= a\frac{E\left(\mathbf{X}e^{a\lambda^T\mathbf{X}}|\mathbf{X} > VaR_q(\mathbf{X})\right)}{E\left(e^{a\lambda^T\mathbf{X}}|\mathbf{X} > VaR_q(\mathbf{X})\right)} \\ &= a\pi_{\alpha,a\lambda}(\mathbf{X}).\end{aligned}$$

5. The proof is similar to the proof of the semi-subadditivity of the MTCE for elliptical distributions shown in Landsman et al. (2016). From (McNeil et al., 2005), Theorem 6.8, we notice that for any matrix B with $n\times(2n)$ dimensions, in the case of elliptical random vector $\mathbf{X}$,

$$VaR_q(B\mathbf{X}) \le BVaR_q(\mathbf{X}). \tag{12}$$

In our case $B = \begin{pmatrix} I_{n\times n} & I_{n\times n}\end{pmatrix}$.Then, since $\left\{B\mathbf{X} > BVaR_q(\mathbf{X})\right\} = \left\{\mathbf{X} > VaR_q(\mathbf{X})\right\}$ we have

$$\begin{aligned}\pi_{\alpha,\lambda}(\mathbf{X}_1+\mathbf{X}_2) &= E\left(B\mathbf{X}e^{B\lambda^T\mathbf{X}}|B\mathbf{X} > VaR_q(B\mathbf{X})\right) \\ &\le E\left(B\mathbf{X}e^{B\lambda^T\mathbf{X}}|B\mathbf{X} > BVaR_q(\mathbf{X})\right) \\ &= BE\left(\mathbf{X}e^{\lambda^T\mathbf{X}}|\mathbf{X} > VaR_q(\mathbf{X})\right)\end{aligned}$$

so

$$\pi_{\alpha,\lambda}(\mathbf{X}_1+\mathbf{X}_2) \le \pi_{\alpha,\lambda}(\mathbf{X})_1 + \pi_{\alpha,\lambda}(\mathbf{X})_2.$$

■

4 DERIVATION OF MCEP FOR ELLIPTICAL MODELS

In risk measurement, the family of elliptical distributions is important since this family has desirable properties which were shown in the previous Section. This class is used to model loss distributions of some random risks associated with this family (Landsman, 2004; Valdez and Chernih, 2003; Xiao and Valdez, 2015). Therefore, it is natural to derive the conditional Esscher premium for the family of elliptical distributions.

Before we derive the MCEP measure for elliptical models, we define a cumulative generator $\overline{G}_n(u)$, (Landsman and Valdez, 2003), which takes the following form

$$\overline{G}_n(u) = \int_u^\infty g_n(q)dq. \tag{13}$$

Furthermore, let us define a shifted cumulative generator $\overline{G}^*_{n-1}(u)$ (Landsman et al., 2016)

$$\overline{G}^*_{n-1}(u) = \int_u^\infty g_n(q+a)dq, a \ge 0,\ n > 1,$$

under the condition that $\overline{G}^*_{n-1}(0) < \infty$. For the sequel, let us define the random vector of risks $\mathbf{X} \backsim E_n(\mu,\Sigma,g_n)$, and a standard random vector $\mathbf{Z} = \Sigma^{-1/2}(\mathbf{X}-\mu) \backsim E_n(\mathbf{0},I,g_n)$. Furthermore, define $\zeta_q = \Sigma^{-1/2}(VaR_q(\mathbf{X})-\mu)$, $\mathbf{x}_q = VaR_q(\mathbf{X})$, and $\zeta_{q,-i} = \left(\zeta_{q_1,1},...,\zeta_{q_{i-1},i-1},\zeta_{q_{i+1},i+1},\zeta_{q_n,n}\right)^T$, and we introduce the tail function of $(n-1)-$variate random vector $\mathbf{Y}_i$, $\overline{F}_{\mathbf{Y}_i}(\mathbf{y})$,

$$\begin{aligned}&\overline{F}_{\mathbf{Y}_i}(\mathbf{y}) \\ &= \int_{\mathbf{y}}^\infty f_{\mathbf{Y}_i}(\mathbf{u})d\mathbf{u},\ \mathbf{u},\mathbf{y} \in R^{n-1},\ d\mathbf{u} = du_1du_2...du_n,\end{aligned}$$

where $f_{\mathbf{Y}_i}(\mathbf{u})$ is the elliptical pdf

$$\begin{aligned}&f_{\mathbf{Y}_i}(\mathbf{y}) \\ &= c^*_{n-1,i}\overline{G}^*_{n-1,i}\left(\frac{1}{2}\mathbf{y}^T\mathbf{y}\right) = c^*_{n-1,i}\overline{G}_n\left(\frac{1}{2}\mathbf{y}^T\mathbf{y}+\frac{1}{2}\zeta^2_{q,i}\right),\end{aligned}$$

$i = 1,2,...,n$.

Lemma 1. *Lemma 1. If $M_{\mathbf{X},q}(\lambda) < \infty$, the conditional moment generating function of $\mathbf{X}$ is given by*

$$M_{\mathbf{X},q}(\lambda) = e^{\lambda^T\mu}\psi_{g_n}\left(\frac{1}{2}\lambda^T\Sigma\lambda\right)\frac{\overline{F}_\theta(\zeta_q)}{\overline{F}_{\mathbf{X}}(\mathbf{x}_q)}, \tag{14}$$

where $\overline{F}_\theta$ is the tail function of a random vector θ with the pdf

$$f_\theta(\mathbf{t}) = \psi_{g_n}\left(\frac{1}{2}\mathbf{t}^T\Sigma\mathbf{t}\right)^{-1}e^{\lambda^T\Sigma^{1/2}\mathbf{t}}\cdot c_ng_n\left(\frac{1}{2}\mathbf{t}^T\mathbf{t}\right). \tag{15}$$

Proof. From the definition of $M_{\mathbf{X},q}(\lambda)$, we have

$$\begin{aligned}&M_{\mathbf{X},q}(\lambda) \\ &= \frac{c_n\int\limits_{VaR_q(\mathbf{X})}^\infty e^{\lambda^T\mathbf{x}}\cdot|\Sigma|^{-1/2}g_n\left(\frac{1}{2}(\mathbf{x}-\mu)^T\Sigma^{-1}(\mathbf{x}-\mu)\right)d\mathbf{x}}{\overline{F}_{\mathbf{X}}(\mathbf{x}_q)}\end{aligned}$$

after the transformation $\mathbf{z} = \Sigma^{-1/2}(\mathbf{x}-\mu)$, we have

$$\begin{aligned}M_{\mathbf{X},q}(\lambda) &= \frac{c_n\int\limits_{\zeta_q}^\infty e^{\lambda^T(\mu+\Sigma^{1/2}\mathbf{z})}\cdot|\Sigma|^{-1/2}g_n\left(\frac{1}{2}\mathbf{z}^T\mathbf{z}\right)|\Sigma|^{1/2}d\mathbf{z}}{\overline{F}_{\mathbf{X}}(\mathbf{x}_q)} \\ &= \frac{c_ne^{\lambda^T\mu}\int\limits_{\zeta_q}^\infty e^{\lambda^T\Sigma^{1/2}\mathbf{z}}\cdot g_n\left(\frac{1}{2}\mathbf{z}^T\mathbf{z}\right)d\mathbf{z}}{\overline{F}_{\mathbf{X}}(\mathbf{x}_q)}.\end{aligned}$$

Taking into account (15), we conclude that

$$c_n\int_{\zeta_q}^\infty e^{\lambda^T\Sigma^{1/2}\mathbf{z}}\cdot g_n\left(\frac{1}{2}\mathbf{z}^T\mathbf{z}\right)d\mathbf{z} = \psi_{g_n}\left(\frac{1}{2}\lambda^T\Sigma\lambda\right)\overline{F}_\theta(\zeta_q),$$

and finally,

$$M_{\mathbf{X},q}(\lambda) = c_n e^{\lambda^T \mu} \psi_{g_n}\left(\frac{1}{2}\lambda^T \Sigma \lambda\right) \frac{\overline{F}_\theta(\zeta_q)}{\overline{F}_{\mathbf{X}}(\mathbf{x}_q)}.$$

□

We note that the proof of Lemma 1 is based on the same method introduced in (Landsman et al., 2013) which derived the TCE and TV of the elliptical distributions, respectively.

Theorem 1. *Theorem 1. Suppose that the conditional moment generating function of* $\mathbf{X}$, $M_{\mathbf{X},q}(\lambda)$, *exist, and that* $E\left(X_i e^{\lambda^T \mathbf{X}} | \mathbf{X} > VaR_q(\mathbf{X})\right) < \infty \ \forall i = 1,2...$ *. Then, the MCEP for the multivariate elliptical distribution,* $n > 1$, *takes the form*

$$\pi_{\alpha,\lambda}(\mathbf{X}) = \mu + \Sigma^{1/2} \chi_{q,\lambda}. \tag{16}$$

Here $\chi_{q,\lambda}$ *is* $n \times 1$ *vector of that depends on the* $q-th$ *percentile*

$$\chi_{q,\lambda} = \begin{pmatrix} \chi_{1,q} & \chi_{2,q} & \dots & \chi_{n,q} \end{pmatrix}^T, \tag{17}$$

where each component of (17) is

$$\chi_{i,q} = \frac{\psi_{\overline{G}^*_{n-1,i}}\left(\frac{1}{2}\left(\lambda^T\Sigma^{1/2}\right)^T_{-i}\left(\lambda^T\Sigma^{1/2}\right)_{-i}\right)}{\psi_{g_n}\left(\frac{1}{2}\lambda^T\Sigma\lambda\right)} \frac{\overline{F}_{\theta_i^*}(\zeta_{q,-i})\, c_n}{\overline{F}_\theta(\zeta_q)\, c^*_{n-1,i}} + \left(\lambda^T\Sigma^{1/2}\right)_i \frac{\psi_{\overline{G}_n}\left(\frac{1}{2}\lambda^T\Sigma\lambda\right)}{\psi_{g_n}\left(\frac{1}{2}\lambda^T\Sigma\lambda\right)} \frac{\overline{F}_{\theta^{**}}(\zeta_q)\, c_n}{\overline{F}_\theta(\zeta_q)\, c^*_n},$$

with the pdf's of $\theta^{**} \in R^n$ *and* $\theta_i^* \in R^{n-1}$, $i = 1,2,...,n$,

$$f_{\theta^{**}}(\mathbf{t}) = \psi_{\overline{G}_n}\left(\frac{1}{2}\mathbf{t}^T\Sigma\mathbf{t}\right)^{-1} e^{\lambda^T\Sigma^{1/2}\mathbf{t}} \cdot c^*_n \overline{G}_n\left(\frac{1}{2}\mathbf{t}^T\mathbf{t}\right), \mathbf{t} \in R^n, \tag{18}$$

and

$$f_{\theta_i^*}(\mathbf{u}) = \psi_{\overline{G}^*_{n-1,i}}\left(\frac{1}{2}\mathbf{u}^T\mathbf{u}\right)^{-1} e^{\left(\lambda^T\Sigma^{1/2}\right)_{-i}\mathbf{u}} c^*_{n-1,i}\overline{G}^*_{n-1,i}\left(\frac{1}{2}\mathbf{u}^T\mathbf{u}\right), \mathbf{u} \in R^{n-1}, \tag{19}$$

with the cumulative generator

$$\overline{G}^*_{n-1,i}(u) = \int_u^\infty g_n\left(q + \frac{1}{2}z^2_{q,i}\right) dq.$$

where c^*_n *and* $c^*_{n-1,i}$ *are the normalizing constants of (18) and (19), respectively.*

Proof. From the definition of $\pi_{\alpha,\lambda}(\mathbf{X})$, we have

$$\pi_{\alpha,\lambda}(\mathbf{X}) = \frac{1}{M_{\mathbf{X},q}(\lambda)} \cdot \frac{c_n \int_{VaR_q(\mathbf{X})}^{\infty} \mathbf{x} e^{\lambda^T\mathbf{x}} \cdot |\Sigma|^{-1/2} g_n\left(\frac{1}{2}(\mathbf{x}-\mu)^T\Sigma^{-1}(\mathbf{x}-\mu)\right) d\mathbf{x}}{\overline{F}_{\mathbf{X}}(\mathbf{x}_q)}.$$

Now, substituting $\mathbf{z} = \Sigma^{-1/2}(\mathbf{x}-\mu)$, we obtain

$$\begin{aligned}\pi_{q,\lambda}(\mathbf{X}) &= \frac{c_n \int_{\zeta_q}^{\infty} (\mu + \Sigma^{1/2}\mathbf{z}) e^{\lambda^T(\mu+\Sigma^{1/2}\mathbf{z})} \cdot |\Sigma|^{-1/2} g_n\left(\frac{1}{2}\mathbf{z}^T\mathbf{z}\right) |\Sigma|^{1/2} d\mathbf{z}}{e^{\lambda^T\mu}\psi_{g_n}\left(\frac{1}{2}\mathbf{t}^T\Sigma\mathbf{t}\right)\overline{F}_\theta(\zeta_q)} \\ &= \mu + \frac{c_n\Sigma^{1/2}\int_{\zeta_q}^{\infty} \mathbf{z} e^{\lambda^T\Sigma^{1/2}\mathbf{z}} g_n\left(\frac{1}{2}\mathbf{z}^T\mathbf{z}\right) d\mathbf{z}}{\psi_{g_n}\left(\frac{1}{2}\mathbf{t}^T\Sigma\mathbf{t}\right)\overline{F}_\theta(\zeta_q)} \\ &= \mu + \Sigma^{1/2}\chi_{q,\lambda},\end{aligned}$$

with $\chi_{q,\lambda}$ an $n \times 1$ vector of the form

$$\chi_{q,\lambda} = \frac{c_n}{\psi_{g_n}\left(\frac{1}{2}\mathbf{t}^T\Sigma\mathbf{t}\right)\overline{F}_\theta(\zeta_q)} \begin{pmatrix} \alpha_{1,q} & \alpha_{2,q} & \dots & \alpha_{n,q} \end{pmatrix}^T, \tag{20}$$

where

$$\alpha_{i,q} = \int_{\zeta_q}^{\infty} z_i e^{\lambda^T\Sigma^{1/2}\mathbf{z}} g_n\left(\frac{1}{2}\mathbf{z}^T\mathbf{z}\right) d\mathbf{z}.$$

From (13), and after some algebraic calculations, we have

$$\alpha_{i,q} = -\int_{\zeta_{q,-i}}^{\infty} d\mathbf{z}_{n-1,-i} e^{\left(\lambda^T\Sigma^{1/2}\right)_{-i}\mathbf{z}_{n-1,i}} \int_{\zeta_{q,i}}^{\infty} e^{\left(\lambda^T\Sigma^{1/2}\right)_i z_i} \cdot d_i\overline{G}_n\left(\frac{1}{2}\mathbf{z}^T_{n-1,-i}\mathbf{z}_{n-1,-i} + \frac{1}{2}z_i^2\right).$$

where $\mathbf{z}_n = (z_1,...,z_n)^T$, $\mathbf{z}_{n-1,-i} = (z_1,z_2,...,z_{i-1},z_{i+1},...,z_n)^T$, and $\zeta_{q,i}$ is the $i-th$

element of vector ζ_q.

$$\alpha_{i,q} = \int_{\zeta_{q,-i}}^{\infty} d\mathbf{z}_{n-1,-i} e^{\left(\lambda^T\Sigma^{1/2}\right)_{-i}\mathbf{z}_{n-1,i}} [e^{\left(\lambda^T\Sigma^{1/2}\right)_i z_{i,q}}$$

$$\overline{G}_n\left(\frac{1}{2}\mathbf{z}_{n-1,-i}^T\mathbf{z}_{n-1,-i} + \frac{1}{2}z_{i,q}^2\right)$$

$$+\left(\lambda^T\Sigma^{1/2}\right)_i \int_{\zeta_{q,i}}^{\infty} e^{\left(\lambda^T\Sigma^{1/2}\right)_i z_i}$$

$$\cdot \overline{G}_n\left(\frac{1}{2}\mathbf{z}_{n-1,-i}^T\mathbf{z}_{n-1,-i} + \frac{1}{2}z_i^2\right) dz_i]$$

$$= e^{\left(\lambda^T\Sigma^{1/2}\right)_i z_{i,q}} \int_{\zeta_{q,-i}}^{\infty} e^{\left(\lambda^T\Sigma^{1/2}\right)_{-i}\mathbf{z}_{n-1,i}}$$

$$\overline{G}_n\left(\frac{1}{2}\mathbf{z}_{n-1,-i}^T\mathbf{z}_{n-1,-i} + \frac{1}{2}z_{i,q}^2\right) d\mathbf{z}_{n-1,-i}$$

$$+\left(\lambda^T\Sigma^{1/2}\right)_i \int_{\zeta_q}^{\infty} e^{\lambda^T\Sigma^{1/2}\mathbf{z}_n}\overline{G}_n\left(\frac{1}{2}\mathbf{z}_n^T\mathbf{z}_n\right) d\mathbf{z}_n.$$

Finally, from the random vectors θ_i^* and θ^{**}, we obtain

$$\alpha_{i,q} = \frac{1}{c_{n-1,i}^*}\psi_{\overline{G}_{n-1,i}^*}\left(\frac{1}{2}\left(\lambda^T\Sigma^{1/2}\right)_{-i}^T\left(\lambda^T\Sigma^{1/2}\right)_{-i}\right)$$
$$\cdot \overline{F}_{\theta_i^*}\left(\zeta_{q,-i}\right)$$
$$+\frac{\left(\lambda^T\Sigma^{1/2}\right)_i}{c_n^*}\psi_{\overline{G}_n}\left(\frac{1}{2}\lambda^T\Sigma\lambda\right)\overline{F}_{\theta^{**}}\left(\zeta_q\right).$$

□

Remark 1. *Remark. The calculation of the components $\chi_{i,q}$ can be computed explicitly for special members of the elliptical distributions (e.g., the normal, logistic and Laplace distributions), in the same way, that was obtained in (Dhaene et al., 2008).*

Corollary 1. *Corollary 1. Suppose that $\mathbf{X} = \left(\mathbf{X}_1^T, \mathbf{X}_2^T\right)^T \backsim E_{2n}(\mu = \begin{pmatrix} \mu_{\mathbf{X}_1} \\ \mu_{\mathbf{X}_2} \end{pmatrix}, \Sigma, g_{2n})$ where $\left(\mathbf{X}_1^T, \mathbf{X}_2^T\right)^T$, $\mathbf{X}_1, \mathbf{X}_2 \in R^n$, has uncorrelated components (i.e. Σ is a diagonal matrix). Then, the MCEP takes the form*

$$\pi_{\alpha,\lambda}(\mathbf{X}) = \mu + \sigma\chi_{q,\lambda}. \tag{21}$$

Here $\sigma = diag\left(\sqrt{\sigma_{11}}, ..., \sqrt{\sigma_{nn}}\right)^T$ where σ_{ii} is the variance of the i-th random variable of $\mathbf{X}$, and $\chi_{i,q}$ is expressed as follows:

$$\chi_{i,q} = \frac{\psi_{\overline{G}_{n-1,i}^*}\left(\frac{1}{2}(\lambda\sigma)_{-i}^T(\sigma\lambda)_{-i}\right)}{\psi_{g_n}\left(\frac{1}{2}\lambda^T\Sigma\lambda\right)}\frac{\overline{F}_{\theta_1^*}\left(z_q\mathbf{1}_{2n-1}\right)c_n}{\overline{F}_{\theta}\left(z_q\mathbf{1}_{2n}\right)c_{n-1,i}^*}$$
$$+\lambda_i\sqrt{\sigma_{ii}}\frac{\psi_{\overline{G}_n}\left(\frac{1}{2}\lambda^T\Sigma\lambda\right)}{\psi_{g_n}\left(\frac{1}{2}\lambda^T\Sigma\lambda\right)}\frac{\overline{F}_{\theta^{**}}\left(z_q\mathbf{1}_{2n}\right)c_n}{\overline{F}_{\theta}\left(z_q\mathbf{1}_{2n}\right)c_n^*},$$

where $\mathbf{1}_k = (1, 1, ..., 1)$ is vector of k ones, and $z_q = VaR_q(Z)$, $Z \backsim E_1(0, 1, g_1)$.

Proof. As $\mathbf{X}_1$ and $\mathbf{X}_2$ are uncorrelated random vectors, Σ is a diagonal matrix, so

$$\zeta_q = \Sigma^{-1/2}\left(VaR_q(\mathbf{X}) - \mu\right) = z_q\mathbf{1}_{2n}.$$

This gives us the following expression of $\pi_{q,\lambda}(\mathbf{X})$

$$\pi_{q,\lambda}(\mathbf{X}) = \mu + \chi_{q,\lambda}\sigma.$$

□

Lemma 2. *Lemma 2. For the random vector $(X_1, X_2)^T$*

$$(X_1, X_2)^T \backsim E_2\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}, g_2\right).$$

Then, the conditional Wang Esscher premium $\pi_{q,\lambda}(X_1, X_2)$ is

$$\pi_{q,\lambda}(X_1, X_2) = \mu_1 + \frac{\sigma_1 c_2}{c_1^*}\frac{\psi_{\overline{G}_1}\left(\frac{1}{2}\lambda_2^2\sigma_{22}\right)\overline{F}_{\theta^{**}}\left(\zeta_{2,q}\right)}{\psi_{g_1}\left(\frac{1}{2}\lambda^2\sigma_{22}\right)\overline{F}_{\theta}\left(\zeta_{2,q}\right)} \tag{22}$$

where $\sigma_i = \sqrt{\sigma_{ii}}$.

Proof. From the definition of $\pi_{\alpha,\lambda}(X_1, X_2)$, we have

$$\pi_{\alpha,\lambda}(X_1, X_2)$$
$$= E\left(X_1 e^{\lambda X_2} | X_2 > VaR_q(X_2)\right)$$
$$= \frac{1}{M_{X_2,q}(\lambda)}\cdot$$
$$\frac{c_2\int_{-\infty}^{\infty}\int_{VaR_q(X_2)}^{\infty} x_1 e^{\lambda x_2}\cdot|\Sigma|^{-1/2} g_2\left(\frac{1}{2}(\mathbf{x}-\mu)^T\Sigma^{-1}(\mathbf{x}-\mu)\right)d\mathbf{x}}{\overline{F}_{X_2}(x_{2,q})}$$
$$= \frac{c_2\int_{-\infty}^{\infty}\int_{\zeta_{2,q}}^{\infty}\left(\mu_1 + z_1\sigma_1\right)e^{\lambda(\mu_2 + z_2\sigma_2)}\cdot g_2\left(\frac{1}{2}\mathbf{z}^T\mathbf{z}\right)d\mathbf{z}}{e^{\lambda\mu_2}\psi_{g_1}\left(\frac{1}{2}\lambda^2\sigma_{22}\right)\overline{F}_{\theta}\left(\zeta_{2,q}\right)}$$
$$= \mu_1 + \sigma_1 c_2\frac{\int_{-\infty}^{\infty}\int_{\zeta_{2,q}}^{\infty} e^{\lambda\sigma_2 z_2}\cdot\overline{G}_2\left(\frac{1}{2}\mathbf{z}^T\mathbf{z}\right)d\mathbf{z}}{\psi_{g_1}\left(\frac{1}{2}\lambda^2\sigma_{22}\right)\overline{F}_{\theta}\left(\zeta_{2,q}\right)}.$$

Then, after some calculations, and by using the marginality property of the elliptical distributions,

$$\pi_{\alpha,\lambda}(X_1,X_2)=\mu_1+\frac{\sigma_1 c_2}{c_1^*}\frac{c_2^*\int\limits_{-\infty}^{\infty}\int\limits_{\zeta_{2,q}}^{\infty}e^{\lambda\sigma_2 z_2}\cdot\overline{G}_2\left(\frac{1}{2}\mathbf{z}^T\mathbf{z}\right)d\mathbf{z}}{\psi_{g_1}\left(\frac{1}{2}\lambda^2\sigma_{22}\right)\overline{F}_\theta\left(\zeta_{2,q}\right)}$$
$$=\mu_1+\frac{\sigma_1 c_2}{c_1^*}\frac{c_1^*\int\limits_{\zeta_{2,q}}^{\infty}e^{\lambda\sigma_2 z_2}\cdot\overline{G}_1\left(\frac{1}{2}z_2^2\right)dz_2}{\psi_{g_1}\left(\frac{1}{2}\lambda^2\sigma_{22}\right)\overline{F}_\theta\left(\zeta_{2,q}\right)},$$

and from the characteristic function of the elliptical distributions

$$\pi_{\alpha,\lambda}(X_1,X_2)=\mu_1+\frac{\sigma_1 c_2}{c_1^*}\frac{\psi_{\overline{G}_1}\left(\frac{1}{2}\lambda_2^2\sigma_{22}\right)\overline{F}_{\theta^{**}}\left(\zeta_{2,q}\right)}{\psi_{g_1}\left(\frac{1}{2}\lambda^2\sigma_{22}\right)\overline{F}_\theta\left(\zeta_{2,q}\right)}$$

□

Theorem 2. *Theorem 2. Let $\mathbf{X}\backsim E_n(\mu,\Sigma,g_n)$ and let $S=X_1+X_2+\ldots+X_n$, so*

$$(X_i,S)^T \backsim E_2\left(\left(\begin{array}{c}\mu_i\\ \sum\limits_{j=1}^{n}\mu_j\end{array}\right),\left(\begin{array}{cc}\sigma_{ii} & \sum\limits_{j=1}^{n}\sigma_{ij}\\ \sum\limits_{j=1}^{n}\sigma_{ij} & \sigma_{SS}\end{array}\right),g_2\right), \tag{23}$$

where $\sigma_{SS}=\sum\limits_{i,j=1}^{n}\sigma_{ij}$. Then

$$\pi_{q,\lambda}(X_i,S)=\mu_i+\frac{\sigma_i c_2}{c_1^*}\frac{\psi_{\overline{G}_1}\left(\frac{1}{2}\lambda^2\sigma_{SS}\right)\overline{F}_{\theta^{**}}\left(\zeta_{S,q}\right)}{\psi_{g_1}\left(\frac{1}{2}\lambda^2\sigma_{SS}\right)\overline{F}_\theta\left(\zeta_{S,q}\right)},$$

where $\zeta_{S,q}=VaR_q\left(S\right)$.

Proof. From the marginal properties of the elliptical distributions, we know that the distribution of $(X_i,S)^T$ is (23). Then, from Lemma 2, we immediately have

$$\pi_{\alpha,\lambda}(X_i,S)=\mu_i+\frac{\sigma_i c_2}{c_1^*}\frac{\psi_{\overline{G}_1}\left(\frac{1}{2}\lambda^2\sigma_{SS}\right)\overline{F}_{\theta^{**}}\left(\zeta_{S,q}\right)}{\psi_{g_1}\left(\frac{1}{2}\lambda^2\sigma_{SS}\right)\overline{F}_\theta\left(\zeta_{S,q}\right)}.$$

□

5 EXAMPLES

In this Section, we show several special members of the elliptical family where the MCEP can be computed. For computing the MCEP we need to compute $\chi_{q,\lambda}$, (17).

5.1 Normal Distribution

Suppose that $\mathbf{X}\backsim N_n(\mu,\Sigma)$. Then $g_n(u)=e^{-u}$, so $c_n g_n(\frac{1}{2}\mathbf{x}^T\mathbf{x})=\phi_n(\mathbf{x})=(2\pi)^{-n/2}\exp(-\frac{1}{2}\mathbf{x}^T\mathbf{x})$ and $\Phi_n(\mathbf{x})$ is the $n-th$ multivariate standard normal pdf and cdf, respectively. In this case $c_n=(2\pi)^{-n/2}$, $\overline{G}_n(u)=e^{-u}=g_n(u)$. Thus $\overline{G}_{n-1,i}^*\left(\frac{1}{2}\mathbf{y}^T\mathbf{y}\right)=\exp(-\frac{1}{2}\left(\mathbf{y}^T\mathbf{y}+\zeta_{q,i}^2\right))$ so

$$f_{\theta^{**}}(\mathbf{t})=f_\theta(\mathbf{t})\propto\exp\left(\frac{1}{2}\mathbf{t}^T\Sigma\mathbf{t}+\lambda^T\Sigma^{1/2}\mathbf{t}-\frac{1}{2}\mathbf{t}^T\mathbf{t}\right)$$
$$,\mathbf{t}\in R^n,$$

and

$$f_{\theta_i^*}(\mathbf{u})\propto\exp\left(\frac{1}{2}\mathbf{u}^T\mathbf{u}+\left(\lambda^T\Sigma^{1/2}\right)_{-i}\mathbf{u}-\frac{1}{2}\mathbf{u}^T\mathbf{u}\right)$$
$$,\mathbf{u}\in R^{n-1}.$$

5.2 Logistic Distribution

Suppose that $\mathbf{X}$ has a logistic distribution. Then its pdf is

$$f_{\mathbf{X}}(\mathbf{x})=\frac{c_n}{\sqrt{|\Sigma|}}\frac{\exp\left(-\frac{1}{2}(\mathbf{x}-\mu)^T\Sigma^{-1}(\mathbf{x}-\mu)\right)}{\left[1+\exp\left(-\frac{1}{2}(\mathbf{x}-\mu)^T\Sigma^{-1}(\mathbf{x}-\mu)\right)\right]^2},$$

and we write $\mathbf{X}\backsim Lo_n(\mu,\Sigma)$ (Gupta et al., 2013). In this case the density generator is

$$g_n(u)=\frac{\exp(-u)}{[1+\exp(-u)]^2},$$

and c_n is

$$c_n=(2\pi)^{-n/2}\left[\sum_{j=0}^{\infty}(-1)^{j-1}j^{1-n/2}\right]^{-1},$$

see (Landsman and Valdez, 2003), and the cumulative generator $\overline{G}_n(u)$ is

$$\overline{G}_n(u)=\int\limits_u^\infty\frac{e^{-x}}{[1+e^{-x}]^2}dx=\frac{e^{-u}}{1+e^{-u}}.$$

Then, $f_\theta(\mathbf{t}), f_{\theta^{**}}(\mathbf{t})$, and $f_{\theta_i^*}(\mathbf{u})$ are, respectively,

$$f_\theta(\mathbf{t})\propto\psi_{g_n}\left(\frac{1}{2}\mathbf{t}^T\Sigma\mathbf{t}\right)^{-1}e^{\lambda^T\Sigma^{1/2}\mathbf{t}}\frac{\exp\left(-\frac{1}{2}\mathbf{t}^T\mathbf{t}\right)}{\left[1+\exp\left(-\frac{1}{2}\mathbf{t}^T\mathbf{t}\right)\right]^2}.$$

$$f_{\theta^{**}}(\mathbf{t})\propto\psi_{\overline{G}_n}\left(\frac{1}{2}\mathbf{t}^T\Sigma\mathbf{t}\right)^{-1}e^{\lambda^T\Sigma^{1/2}\mathbf{t}}\frac{\exp\left(-\frac{1}{2}\mathbf{t}^T\mathbf{t}\right)}{1+\exp\left(-\frac{1}{2}\mathbf{t}^T\mathbf{t}\right)}$$

and

$$f_{\theta_i^*}(\mathbf{u}) \propto \psi_{\overline{G}^*_{n-1,i}}\left(\frac{1}{2}\mathbf{u}^T\mathbf{u}\right)^{-1} e^{\left(\lambda^T\Sigma^{1/2}\right)_{-i}\mathbf{u}} \cdot \frac{\exp\left(-\frac{1}{2}\mathbf{u}^T\mathbf{u}+\frac{1}{2}z_{q,i}^2\right)}{1+\exp\left(-\frac{1}{2}\mathbf{u}^T\mathbf{u}+\frac{1}{2}z_{q,i}^2\right)}, \mathbf{u}\in R^{n-1}.$$

We note that while $\psi_{\overline{G}_n}$ and $\psi_{\overline{G}^*_{n-1,i}}$ can be difficult to calculate, these characteristic functions are reduced when applying them in the MCEP $\pi_{q,\lambda}(\mathbf{X})$. In fact, for the $i-th$ component of $\chi_{q,\lambda}$

$$\chi_{i,q} = \frac{c_n}{\overline{F}_\theta(\zeta_q)}[e^{\left(\lambda^T\Sigma^{1/2}\right)_i z_{i,q}+\frac{1}{2}z_{i,q}^2} \int_{\zeta_{q,-i}}^{\infty} \frac{\exp\left(-\frac{1}{2}\mathbf{u}^T\mathbf{u}+\left(\lambda^T\Sigma^{1/2}\right)_{-i}\mathbf{u}\right)}{1+\exp(-\frac{1}{2}\mathbf{u}^T\mathbf{u}+\frac{1}{2}z_{i,q}^2)}d\mathbf{u} + \int_{\zeta_q}^{\infty}\frac{\exp\left(-\frac{1}{2}\mathbf{z}^T\mathbf{z}+\lambda^T\Sigma^{1/2}\mathbf{z}\right)}{1+\exp\left(-\frac{1}{2}\mathbf{z}^T\mathbf{z}\right)}d\mathbf{z}].$$

5.3 Laplace Distribution

We say that $\mathbf{X}$ is multivariate Laplace random vector if its pdf has the form (Fang, 2017)

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{\Gamma(n/2)}{2\pi^{n/2}\Gamma(n)}\exp\left(-\left((\mathbf{x}-\mu)^T\Sigma^{-1}(\mathbf{x}-\mu)\right)^{1/2}\right)$$

and we write $\mathbf{X} \backsim La_n(\mu,\Sigma)$. Then, the density generator and the characteristic generator are, respectively, $g_n(u) = e^{-\sqrt{2u}}$ and

$$\psi_{g_n}(u) = \frac{1}{1+u}.$$

In this case $\overline{G}_n(u)$ is

$$\overline{G}_n(u) = \int_u^\infty e^{-\sqrt{2x}}dx = \left(1+\sqrt{2u}\right)e^{-\sqrt{2u}},$$

Then, $f_\theta(\mathbf{t}), f_{\theta^{**}}(\mathbf{t})$, and $f_{\theta_i^*}(\mathbf{u})$ are, respectively,

$$f_\theta(\mathbf{t}) \propto \left(1+\frac{1}{2}\mathbf{t}^T\Sigma\mathbf{t}\right)e^{\lambda^T\Sigma^{1/2}\mathbf{t}-\sqrt{\mathbf{t}^T\mathbf{t}}}.$$

$$f_{\theta^{**}}(\mathbf{t}) \propto \psi_{\overline{G}_n}\left(\frac{1}{2}\mathbf{t}^T\Sigma\mathbf{t}\right)^{-1}e^{\lambda^T\Sigma^{1/2}\mathbf{t}-\sqrt{\mathbf{t}^T\mathbf{t}}}\left(1+\sqrt{\mathbf{t}^T\mathbf{t}}\right), \mathbf{t}\in R^n,$$

and

$$f_{\theta_i^*}(\mathbf{u}) \propto \psi_{\overline{G}^*_{n-1,i}}\left(\frac{1}{2}\mathbf{u}^T\mathbf{u}\right)^{-1} e^{\left(\lambda^T\Sigma^{1/2}\right)_{-i}\mathbf{u}} \cdot \left(1+\sqrt{\frac{1}{2}\mathbf{u}^T\mathbf{u}+\frac{1}{2}z_{q,i}^2}\right) \exp\left(-\sqrt{\frac{1}{2}\mathbf{u}^T\mathbf{u}+\frac{1}{2}z_{q,i}^2}\right), \mathbf{u}\in R^{n-1},$$

Notice that although $\psi_{\overline{G}_n}$ and $\psi_{\overline{G}^*_{n-1,i}}$ can be difficult to calculate they can be reduced when applying them in the MCEP $\pi_{q,\lambda}(\mathbf{X})$. For the $i-th$ component of $\chi_{q,\lambda}$

$$\chi_{i,q} = \frac{c_n}{\overline{F}_\theta(\zeta_q)}[e^{\left(\lambda^T\Sigma^{1/2}\right)_i z_{i,q}} \cdot \int_{\zeta_{q,-i}}^{\infty}\exp\left(\left(\lambda^T\Sigma^{1/2}\right)_{-i}\mathbf{u}-\sqrt{\mathbf{u}^T\mathbf{u}+z_{i,q}^2}\right)d\mathbf{u} + \int_{\zeta_q}^{\infty}\exp\left(-\sqrt{\mathbf{z}^T\mathbf{z}}+\lambda^T\Sigma^{1/2}\mathbf{z}\right)d\mathbf{z}].$$

6 DISCUSSION

In this paper, we have shown how to model the Esscher premium principle for a system of mutually dependent risks with the underlying elliptical model, which is common in the world of risk measurement and actuarial science. Furthermore, we derived the conditional moment generating function for the family of multivariate elliptical distribution, in which the MTCE measure is a special case,

$$MTCE_q(\mathbf{X}) = \frac{\partial}{\partial\lambda}M_{\mathbf{X},q}(\lambda)|_{\lambda=\mathbf{0}}.$$

The MCEP measure quantifies the premium of a vector of dependent risks under the condition that an event outside a given probability level has occurred. We derived a general formula of the MCEP for the elliptical distributions

$$\pi_{\alpha,\lambda}(\mathbf{X}) = \mu+\Sigma^{1/2}\chi_{q,\lambda}.$$

We then derived the MCEP for aggregate risks, based on the Wang's premium with exponential tilting,

$$\pi_{\alpha,\lambda}(X_i,S) = \mu_i+\frac{\sigma_1}{c_1^*}\frac{\psi_{\overline{G}_1}\left(\frac{1}{2}\lambda^2\sigma_{SS}\right)\overline{F}_{\theta^*}(\zeta_{S,q})}{\psi_{g_1}\left(\frac{1}{2}\lambda^2\sigma_{SS}\right)\overline{F}_{\theta^{**}}(\zeta_{S,q})}.$$

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REFERENCES

Bühlmann, H. (1980). An economic premium principle. *ASTIN Bulletin: The Journal of the IAA*, 11(1):52–60.

Cambanis, S., Huang, S., and Simons, G. (1981). On the theory of elliptically contoured distributions. *Journal of Multivariate Analysis*, 11(3):368–385.

Chi, Y., Lin, X. S., and Tan, K. S. (2017). Optimal reinsurance under the risk-adjusted value of an insurer's liability and an economic reinsurance premium principle. *North American Actuarial Journal*, 21(3):417–432.

Cousin, A. and Di Bernardino, E. (2014). On multivariate extensions of conditional-tail-expectation. *Insurance: Mathematics and Economics*, 55:272–282.

Déniz, E. G., Vazquez Polo, F., and Bastida, A. H. (2000). Robust bayesian premium principles in actuarial science. *Journal of the Royal Statistical Society: Series D (The Statistician)*, 49(2):241–252.

Dhaene, J., Henrard, L., Landsman, Z., Vandendorpe, A., and Vanduffel, S. (2008). Some results on the cte-based capital allocation rule. *Insurance: Mathematics and Economics*, 42(2):855–863.

Fang, K. W. (2017). *Symmetric Multivariate and Related Distributions: 0*. Chapman and Hall/CRC.

Feinstein, Z. and Rudloff, B. (2017). A recursive algorithm for multivariate risk measures and a set-valued bellman's principle. *Journal of Global Optimization*, 68(1):47–69.

Goovaerts, M., De Vylder, F., and Haezendonck, J. (1984). Insurance premiums.

Gupta, A. K., Varga, T., and Bodnar, T. (2013). *Elliptically contoured models in statistics and portfolio theory*. Springer.

Jouini, E., Meddeb, M., and Touzi, N. (2004). Vector-valued coherent risk measures. *Finance and stochastics*, 8(4):531–552.

Kamps, U. (1998). On a class of premium principles including the esscher principle. *Scandinavian Actuarial Journal*, 1998(1):75–80.

Landsman, Z. (2004). On the generalization of esscher and variance premiums modified for the elliptical family of distributions. *Insurance: Mathematics and Economics*, 35(3):563–579.

Landsman, Z., Makov, U., and Shushi, T. (2016). Multivariate tail conditional expectation for elliptical distributions. *Insurance: Mathematics and Economics*, 70:216–223.

Landsman, Z., Makov, U., and Shushi, T. (2018). A multivariate tail covariance measure for elliptical distributions. *Insurance: Mathematics and Economics*, 81:27–35.

Landsman, Z., Pat, N., and Dhaene, J. (2013). Tail variance premiums for log-elliptical distributions. *Insurance: Mathematics and Economics*, 52(3):441–447.

Landsman, Z. M. and Valdez, E. A. (2003). Tail conditional expectations for elliptical distributions. *North American Actuarial Journal*, 7(4):55–71.

McNeil, A. J., Frey, R., Embrechts, P., et al. (2005). *Quantitative risk management: Concepts, techniques and tools*, volume 3. Princeton university press Princeton.

Molchanov, I. and Cascos, I. (2016). Multivariate risk measures: a constructive approach based on selections. *Mathematical Finance*, 26(4):867–900.

Shushi, T. (2017). Skew-elliptical distributions with applications in risk theory. *European Actuarial Journal*, 7(1):277–296.

Shushi, T. (2018). Stein's lemma for truncated elliptical random vectors. *Statistics & Probability Letters*, 137:297–303.

Valdez, E. A. and Chernih, A. (2003). Wang's capital allocation formula for elliptically contoured distributions. *Insurance: Mathematics and Economics*, 33(3):517–532.

Van Heerwaarden, A. E., Kaas, R., and Goovaerts, M. J. (1989). Properties of the esscher premium calculation principle. *Insurance: Mathematics and Economics*, 8(4):261–267.

Wang, S. and Dhaene, J. (1998). Comonotonicity, correlation order and premium principles. *Insurance: Mathematics and Economics*, 22(3):235–242.

Wang, S. et al. (2002). 43 a set of new methods and tools for enterprise risk capital management and portfolio optimization.

Xiao, Y. and Valdez, E. A. (2015). A black–litterman asset allocation model under elliptical distributions. *Quantitative Finance*, 15(3):509–519.