Numerical Solution of Sasando String Motion Model

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Keywords: Numerical solution, CTCS, String Motion, Sasando

Abstract: This study describes the problem of string motion on the Sasando musical instrument. This research focuses on the numerical solution of Sasando string motion model which is a partial differential equation. The method used to find out the numeric solution is CTCS (Central Time Central Space) method. The explicit formula of discretization and Von Neuman stability analysis are considered here. The results show the solution of $\boldsymbol{u}\ (\boldsymbol{x}, \boldsymbol{t})$ is stable, which means the movement of the string toward the equilibrium value $(\mathbf{0}, \mathbf{0})$.

1 INTRODUCTION

Applied mathematics is needed in every part of life especially to help solve problems related to mathematical models. Mathematical modeling is a field of mathematics that seeks to represent and explain real problems into mathematical equations. For example, the vibration phenomenon that occurs along the strings of Sasando can be analyzed by using mathematical model. The mathematical model for this problem has been done by Kusumastuti, et al(Kusumastuti & Brylliant, 2017).

Sasando is one of the traditional musical instruments originated from Rote Island of Ndao Regency of East Nusa Tenggara. The distinctive and beautiful sound makes many people, not only local people but also the foreign community interested to examine how this instrument is able to create beautiful sounds. The problem of string motion on the Sasando musical instrument is a difficult problem to analyze directly. Therefore, an effort is needed to understand the problem, one of which is to establish a proper measure that can represent the problem in its real state. In mathematics, this is known as the mathematical model. The mathematical model for motion problems in the Sasando musical instrument is seen as an abstraction of the strings problem on the complex Sasando musical instrument presented in the form of a mathematical language. By modeling the strings on the Sasando musical instrument, it can be seen the vibration pattern of the string on the Sasando instrument in the mathematical equation. Thus, the understanding of the strings on the Sasando musical instrument becomes more systematic and easier to analyze further.

Purwanto, have done research on the analysis and synthesis of sound signals generated from the instrument of semi-acoustic guitar. In the study the strings of the guitar were picked and then the sound produced was recorded using the SOUND FORGE program. The recording sound data is then analyzed by FFT to obtain the sound signal spectrum, and the components of the composer of the sound signal, such as fundamental frequency, harmonic frequency, amplitude, and amplitude ratio. Then the sound signal construction based on the components that have been obtained. And the results show that by adding the damping factor to the sound signal model make a similar sound with the original sound from guitar. (Purwanto, et al., 2006)

This study is a follow-up study of previous research by Kusumastuti, et al who has analyzed the construction of a strap motion model on a sasando instrument. In this work, numerical analysis is done using finite difference method, i.e. Central Time Central Space (CTCS). Triatmodjo states that finite difference method is usually used to find the numerical solution of partial differential equations. Finite difference schemes approximate the solution by discretization the partial differential equation.

Kusumastuti, A., Khudzaifah, M., Widayani, H. and Zuhriah, A.
Numerical Solution of Sasando String Motion Model.
DOI: 10.5220/0008525005370541
In *Proceedings of the International Conference on Mathematics and Islam (ICMIs 2018)*, pages 537-541
ISBN: 978-989-758-407-7

CTCS scheme is a numerical approach using a central difference to the time and center difference to space. (Triatmodjo, 2002)

From the above background explanation, the authors have an idea to study the numerical solution of Sasando string motion model using CTCS Method.

2 SASANDO STRING MODEL

In general, the string position of Sasando musical instrument as shown in Figure 1:

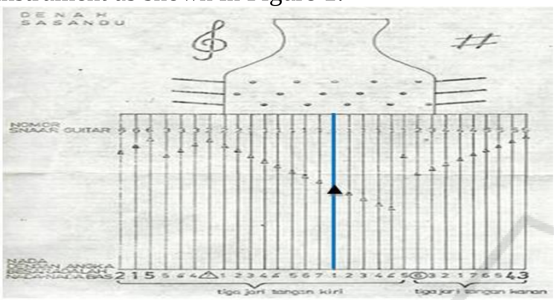


Figure1:Sasando String Position(Kusumastuti & Brylliant, 2017)

Based onFigure1, string that make base tone *do is in the middle, i.e.* string 16th(marked in blue).The distance between these two hooks is 35 cm. The location of the buffer holder is in the middle, so the lengths on both sides of the string are the same (l). The type of wire material used is nylon which has a modulus of elasticity(E)as$5\times10^9 N/m^2$. The modulus of elasticity constant of the Sasando strings shows the degree of flexibility of the strings.

Kusumastuti, et al generated a string mathematical model on the musical instrument Sasando which classified as hyperbolic PDE given by:

$$\frac{\partial^2 u}{\partial t^2} - \left(\frac{1}{2}c^2 + 2\frac{c}{l}\right)\frac{\partial^2 u}{\partial x^2} + k_d\frac{\partial u}{\partial t} = 0 \qquad (1)$$

The construction of a string mathematical model on the Sasando musical instrument is formed from the use of the laws of physics. Fingerpicking given to the Sasando strings generate potential energy (Ep) and kinetic energy (Ek) along the string.

The potential energy (Ep) of the Sasando strand represents the total of each potential energy occurring on the Sasando strings. In the case of the Sasando strings being picked, there are several potential energies that occur, among which are :

1. Spring Potential Energy(Ep_{spring}) as:

$$Ep_{spring} = \frac{1}{2}k_p l^2\left(\frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 + 1\right) \qquad (2)$$

2. Stress potential energy(Ep_{stress}) as:

$$Ep_{stress} = 2E\left(\frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 + 1\right) \qquad (3)$$

3. Friction potential energy$(Ep_{friction})$ as:

$$Ep_{friction} = -k_b\eta\left(\frac{\partial u}{\partial t}\right)u \qquad (4)$$

From equation (2), (3), and (4) can be generated potential energy for Model (Ep_M)which is the overall potential energy, so can be written as

$$Ep_M = \left(\frac{1}{2}k_p l^2 + 2E\right)\left(\frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 + 1\right) + \left(-k_b\eta\left(\frac{\partial u}{\partial t}\right)u\right) \qquad (5)$$

When the strings of Sasando are struck an oscillatory motion occurs on the strings, so there is kinetic energy on the Sasando strings. Kinetic energy on the Sasando strings is defined as follows:

$$Ek_{Model} = \frac{1}{2}\rho l\left(\frac{\partial u}{\partial t}\right)^2 \qquad (6)$$

The next step is to determine the Lagrange equation. The Lagrange equation is defined as the difference between the kinetic energy model (Ep_{Model}) and the model potential energy(Ek_{Model}), so that the Lagrange equation is obtained as follows:

$$L = \frac{1}{2}\rho l\left(\frac{\partial u}{\partial t}\right)^2 - \left(\frac{1}{2}k_p l^2 + 2E\right)\left(\frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 + 1\right) + \left(k_b\eta\left(\frac{\partial u}{\partial t}\right)u\right) \qquad (7)$$

By deriving the Lagrange equation in equation (7), we find the equation (1).

Based on equation (1) there are several parameters, namely:

1. The length of the strings of Sasando (l), is the distance Between the two ends of the Sasando string is 16 cm.
2. The damper constant (k_d) that affects the wavelength during the insulated string is 1.5.
3. The speed of elasticity (c) is $1\ m/s^2$ and the function $f(x)$ is expressed as follows:

$$f(x) = \begin{cases} \dfrac{2hx}{l}, & 0 \le x \le \dfrac{1}{2}l \\ 2h - \dfrac{2hx}{l}, & \dfrac{1}{2}l \le x \le l \end{cases}$$

with initial condition as

$$u(x,0) = f(x) \text{ for } 0 < x < l \text{ and}$$

$$\frac{\partial}{\partial t}u(x=0,t) = \frac{\partial}{\partial t}u(x=l,t) = 0$$

and boundary condition as

$$u(0,t) = 0 \text{ and } u(l,t) = 0 \text{ for } t > 0$$

(Kusumastuti & Brylliant, 2017)

3 MAIN RESULTS

3.1 Discretization

Strauss in (Strauss, 1983)states that the finite difference method is a common method to solve differential equation, ordinary or partial differential equations. This method based on the Taylor series expansion. For partial differential equation with two independent variables, i.e. space (x) and time (t) the stability and convergence depend on the used scheme. For hyperbolic PDE with proper initial and boundary condition, central scheme for space is commonly used to approximate the second order derivative of space. Then, forward, backward, or central scheme could be used for time variable. Implementation of forward scheme of time in simulation more simple than backward scheme which used inverse matrix calculation.

Let $x = j.\Delta x$ with $j = 0,1,2,3 \ldots \frac{l}{\Delta x}$ and $t = n.\Delta t$ with $n = 0,1,2,3 \ldots$ As stated in(Burden & Faires, 2011), the Euler explicit scheme for $\frac{\partial u}{\partial t}$ at $\left(x = x_j, t = t_n\right)$ can be written as

$$\left.\frac{\partial u}{\partial t}\right|_{t=n.\Delta t}^{n=j.\Delta x} \approx \frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} \tag{8}$$

We central scheme for second order derivative of time. Central Time Central Space (CTCS) scheme to approximate second order derivative of time and space can be written as:

$$\frac{\partial^2 u}{\partial t^2} = \frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{\Delta t^2} \tag{9}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} \tag{10}$$

Substitution of (8), (9), and (10)approximation to hyperbolic PDE (1) and do some tedious calculation give the discretization scheme as

$$u_j^{n+1} = A_1 u_j^n - A_2 u_j^{n-1} + A_3 u_{j+1}^n + A_4 u_{j-1}^n \tag{11}$$

where

$$A_1 = \frac{4l\Delta x^2 - 2c^2 l\Delta t^2 - 8c^2\Delta t^2}{2l\Delta x^2 + lk_d\Delta x^2\Delta t}$$

$$A_2 = \frac{2 - k_d\Delta t}{2 + k_d\Delta t}$$

$$A_3 = \frac{c^2 l\Delta t^2 + 4c^2\Delta t^2}{2l\Delta x^2 + lk_d\Delta x^2\Delta t}$$

$$A_4 = \left(\frac{c^2 l\Delta t^2 + 4c^2\Delta t^2}{2l\Delta x^2 + lk_d\Delta x^2\Delta t}\right)$$

3.2 Stability Analysis

The scheme stability was done using Von Neumann stability test. It said stable if$\rho \le 1$, Substitution of$u_j^n = \rho^n e^{i\alpha j}$, $\forall\, i = \sqrt{-1}$to equation (3.5) give us

$$\rho^{n+1}e^{i\alpha j} = A_1\rho^n e^{i\alpha j} - A_2\rho^{n-1}e^{i\alpha j} + A_3\rho^n e^{i\alpha(j+1)} + A_4\rho^n e^{i\alpha(j-1)}$$

or equivalent with equation below

$$\rho = A_1 - A_2\rho^{-1} + A_3 e^{i\alpha} + A_4 e^{-i\alpha}$$

Since$e^{\pm i\alpha} = cos\ \alpha \pm i\ \sin\alpha$, then

$$\rho = A_1 - A_2\rho^{-1} + 2A_3\cos\alpha$$

multiplied by ρ, then we get

$$\rho^2 = A_1\rho - A_2 + 2A_3\cos\alpha\ \rho$$

$$\rho^2 = (A_1 + 2A_3\cos\alpha)\rho - A_2$$

$$\rho^2 = \left(A_1 + \left(\frac{2c^2 l\Delta t^2 cos\alpha + 8c^2\Delta t^2\cos\alpha}{2l\Delta x^2 + lk_d\Delta x^2\Delta t}\right)\right)\rho - A_2$$

$$\rho^2 = \left(\frac{4l\Delta x^2 - 2c^2 l\Delta t^2 - 8c^2\Delta t^2 + 2c^2 l\Delta t^2 cos\alpha + 8c^2\Delta t^2\cos\alpha}{2l\Delta x^2 + lk_d\Delta x^2\Delta t}\right)\rho - A_2$$

or

$$\rho^2 - \left(\frac{4l\Delta x^2 - 2c^2 l\Delta t^2 - 8c^2\Delta t^2 + 2c^2 l\Delta t^2 cos\alpha + 8c^2\Delta t^2\cos\alpha}{2l\Delta x^2 + lk_d\Delta x^2\Delta t}\right)\rho + A_2 = 0 \tag{3.6}$$

So, the roots of (3.6) are

$$\rho_{1,2} = \frac{\left(\frac{4l\Delta x^2 - 2c^2 l\Delta t^2 - 8c^2\Delta t^2 + 2c^2 l\Delta t^2 cos\alpha + 8c^2\Delta t^2\cos\alpha}{2l\Delta x^2 + lk_d\Delta x^2\Delta t}\right)}{2} \pm \frac{\sqrt{\left(\frac{4l\Delta x^2 - 2c^2 l\Delta t^2 - 8c^2\Delta t^2 + 2c^2 l\Delta t^2 cos\alpha + 8c^2\Delta t^2 cos\alpha}{2l\Delta x^2 + lk_d\Delta x^2\Delta t}\right)^2 - 4A_2}}{2} \tag{12}$$

Since equation (12) contains cosα, in this case we choose the discrete point which is $\cos\alpha = -1, \cos\alpha = 0$,and$\cos\alpha = 1$. From tedious calculation obtained sufficient condition for$|\rho_1| \le 1$is

$$\Delta t \le \frac{1}{k_d}$$

While sufficient condition for$|\rho_2| \le 1$is$\Delta t \le 0$.If $k_d = 1.5$, the$\Delta t \le 0.67$ guarantee that discretization scheme will be stable.

3.3 Numerical Simulation

With the value of l = 16 cm, k_d= 1.5, c = 1 m/s^2, substituted to equation (3.5), then the numerical scheme is

$$u_j^{n+1} = \left(\frac{8\Delta x^2 - 4\Delta t^2 - 1\Delta t^2}{2\Delta x^2 + 3\Delta x^2\Delta t}\right)u_j^n - \left(\frac{2 - 1.5\Delta t}{2 + 1.5\Delta t}\right)u_j^{n-1} + \left(\frac{4\Delta t^2 + 1\Delta t^2}{8\Delta x^2 + 6\Delta x^2\Delta t}\right)u_{j+1}^n + \left(\frac{4\Delta t^2 + 1\Delta t^2}{8\Delta x^2 + 6\Delta x^2\Delta t}\right)u_{j-1}^n \tag{13}$$

The graph simulations were performed using MATLAB R2017b, with a numerical solution for equation (13). Taken $\Delta t = 0.1$, resulting in graph output as follows:

Table 1. Numerical Solutions of equation (1) with $\Delta t = 0.1$

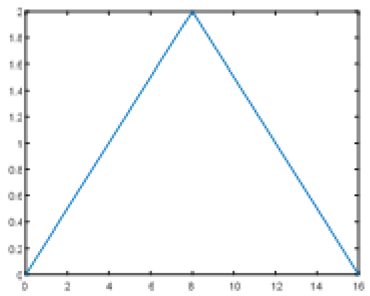
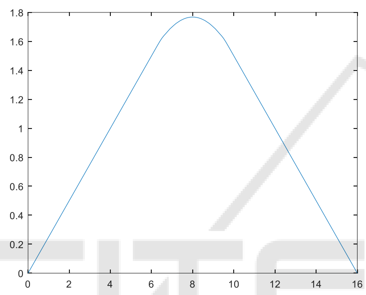
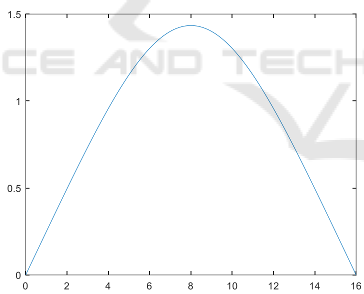
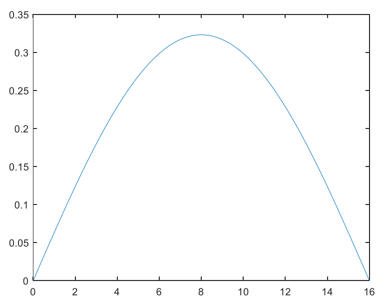
$t = n\Delta t$	Output
$t = 1$	
$t = 21$	
$t = 51$	
$t = 101$	

Table 1 shows the simulation results of the graph for equation (13) with $\Delta t = 0.1$, describing the movement of the Sasando stretch deviation which changes to the value of t where $t = 1,2,3,\ldots,n$.

From Table 1 above shows that the higher the t value, the Sasando strings toward the equilibrium point (0,0).

The graphical output when $\Delta t = 0.1$ for equation (13) using Matlab is as follows:

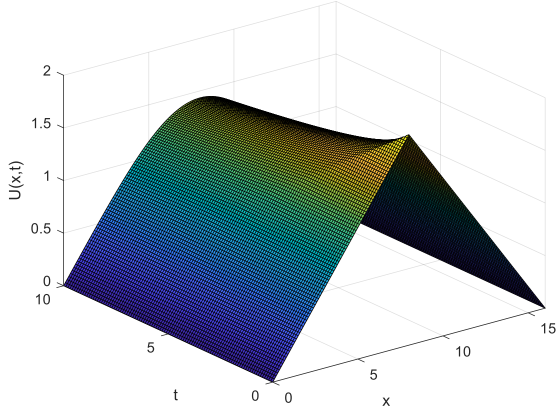


Figure2:Three-dimension solution of Sasando string motion with $\Delta t = 0.1$

Figure 2 shows the simulation results of the numerical solution of the wavelength model on the Sasando musical instrument when $\Delta t = 0.1$ illustrates the movement of the deviation which changes to the values of x and t, and from the graph shows that the result of the graph is stable where the movement of the string goes to the equilibrium point (0.0).

While the graph output when $\Delta t = 0.2$ for equation (13) using Matlab is as follows:

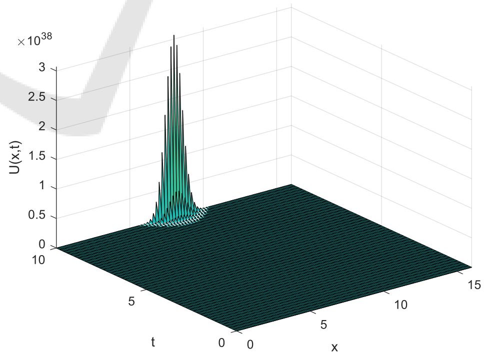


Figure 3: Three-dimension solution of Sasando string motion with $\Delta t = 0.2$

Figure 3 shows the simulation results of the numerical solution of the wavelength model on the Sasando musical instrument when $\Delta t = 0.2$ describes the deviation movements that are changing against the values of x and t, and from the graph shows that the results are unstable.

From the results of Table 1, Figure 2 and Figure 3, it can be concluded that the movement of strings on the Sasando musical instrument is stable with the condition $\Delta t \leq 0.1$ which means t higher, Sasando's strings move closer to the equilibrium point (0.0).

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