Variant of Two Real Parameters Chun-Kim's Method Free Second Derivative with Fourth-order Convergence

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Abstract: Newton's method is one of the iterative methods that used to solve a nonlinear equation. In this paper, a new iterative method with two parameters was developed with variant modification of Newton's method using curvature and second-order Taylor series expansion, then its second derivative was approximated using equality of Newton-Steffensen's and Halley's Methods. The result of this study shows that this new iterative method has fourth-order convergence and involves three evaluation of functions with the efficiency index about 1.5874. In numerical simulation, we use several functions to test the performance of this new iterative method and the others compared iterative methods, such as: Newton's Method (MN), Newton-Steffensen's Method (MNS), Chun-Kim's Method (MCK) and Omran's Method (MO). The result of numerical simulation shows that the performance of this method is better than the others.

1 INTRODUCTION

Nonlinear equation is a mathematical representation that arises in the engineering and scientific problems. The number of assumptions and parameters used to construct equations will affect the complexity of nonlinear equations (Chapra, 1998). Therefore, mathematicians often difficult to determine the settlement of nonlinear equations. Generally, the problem arises when a complicated and complex nonlinear equation cannot be solved using analytical method. We can use repetitive computing techniques to find an alternative solution called as iterative method.

Classical iterative method that widely used by the researcher to solve nonlinear equations is Newton's method with general form,

$$x_{n+1} = x_n - \frac{f(x_n)}{f^{'}(x_n)}, \quad f^{'}(x_n) \neq 0. \tag{1}$$

Newton's method derived from cutting the first order Taylor's series with quadratic order convergence and the efficiency index about $2^{1/2} \approx 1.4142$ (Traub, 1964).

Lately, the researcher trying to develop iterative methods with cubic convergence order using several approaches, such as: adding new steps (Weerakoon and Fernando, 2000) and (Omran, 2013), second order Taylor series cutting (Traub, 1964), quadratic function (Amat et al., 2003); (Amat et al, 2008); (Melman, 1997); (Sharma, 2005); (Sharma, 2007), curvature curve (Chun and Kim, 2010), and the interpretation of two-point geometry (Ardelean, 2013).

Chun-Kim iterative method (Chun and Kim, 2010) was constructed by using curvature, this method express is,

$$x_{n+1} = x_n - \frac{f(x_n)^2 f^{"}(x_n) + 2f(x_n)(1+f^{'}(x_n)^2)}{(x^*_{n+1} - x_n) f^{"}(x_n + 2f^{'}(x_n)(1+f^{'}(x_n)^2)} \tag{2}$$

with x^*_{n+1} defined in equation (1). Equation (2) is an iterative method with a cubic convergence order with three evaluation functions, and the efficiency index is about $3^{1/3} \approx 1.4422$.

In this paper, a new method with two real parameters is generated from the development of the Chun-Kim Method (Chun and Kim, 2010) given in (2) using a second order Taylor sequence expansion. The new generated iterative method involves two real parameters θ and λ, this condition allows us to generate several other new iterative methods of either two, three or four by replacing the values of the real parameters.

Since the new iterative method that we generated still involves second derivative of its function, the use of the second derivative $f"(x_n)$ in the new iterative

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method can be avoided by reducing that second derivative using the similarity of two iterative methods as performed by (Kansal et al, 2016) and (Wartono et al, 2016). Besides, in some cases, reducing the second derivative by using the approach undertaken by (Kansal et al, 2016) and (Wartono et al, 2016) can increase the order of convergence of iterative methods so that the order of convergence of iterative methods becomes optimal (Kung and Traub, 1974).

At the end of this paper, numerical simulations are provided to test the performance of new iterative methods which include the efficiency index, the number of iterations, accuracy level and the accuracy that measured using absolute values of function and relative error. In numerical simulations, the performance measures of new iteration methods (M4) are compared with several other iterative methods, such as: Newton Method (Traub, 1964), Chun-Kim Method (Chun and Kim, 2010), Newton- Steffensen (Sharma, 2005), and the Omran Method (Omran, 2013).

2 METHODS

In this section, we use several theorems and definitions to construct the iterative method, to determine the convergence order, the number of iterations and computationally we computed convergence order. The theorems and definitions that we used are:

Theorem 2.1. (Conte, 1980) Given a function f which can be derived up to $n+1$ derivative for each x on the open interval D containing a, the Taylor expansion of $f(x)$ around a is written ,

$$f(x) = f(a) + f^{'}(a)(x-a) + \frac{f^{''}(a)}{2!}(x-a)^2 + \frac{f^{'''}(a)}{3!}(x-a)^3 + ... + \frac{f^{(n)}(a)}{n!}(x-n)^n + R_n(x) \quad (3)$$

with

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{(n+1)}, x < \xi < a \quad (4)$$

Convergence order is an indicator to measure the convergence velocity of an iterative method on approaching the roots of nonlinear equations. To calculated convergence order we use the Taylor's series expansion approach or by computation calculations. The following is given the convergence order definition.

Definition 2.2. Order of Convergence (Conte, 1980) and (Mathews, 1992). Let $f(x)$ be a function with the root of the equation α and $\{x_n\}_{n\in N}$ is a sequence of real numbers for $n \geq 0$ converging to α for p and there is a constant c, it is written,

$$\lim_{n\to\infty} \frac{|x_{n+1} - \alpha|}{|x_n - \alpha|^p} = c, \qquad p = 1, 2, 3, \ldots \quad (5)$$

Suppose $e_n = x_n - \alpha$ and $e_{n+1} = x_{n+1} - \alpha$ are the errors in the iterative n and $n+1$ respectively on an iterative method that produces the sequence $\{x_n\}$, then the error equation in the iteration $(n+1)$ provided by

$$e_{n+1} = ce_n^p + O(e_{n+1}^p), \quad (6)$$

where c is the asymptotic coefficient of the convergence order.

Definition 2.3. Computational Order of Convergence (Weerakoon and Fernando, 2000). Let α be the root of a nonlinear equation $f(x_n)$,with x_{n+1}, x_n and x_{n-1} are three successive iterations close enough to α , then the Computational Order of Convergence (COC) can be calculated using the formula

$$\rho \approx \frac{\ln|(x_{n+1} - \alpha)/(x_n - \alpha)|}{\ln|(x_n - \alpha)/(x_{n-1} - \alpha)|}. \quad (7)$$

Definition 2.4. The Efficiency Index (Traub, 1964). Let p be the order of convergence of an iteration method, then the efficiency index (I) is given by

$$I = p^{1/d} \quad (8)$$

with d is the number of function evaluations f (including derivatives) for each iteration.

3 RESULT AND DISCUSSION

3.1 New Iterative Method with Two Parameters

To describe the modification of the Newton method variant, the curve curvature equation in $(x_n, f(x))$ through the X axis at $(x_{n+1}, 0)$ we reconsidered in the form

$$(x_{n+1}-x_n)^2+2\frac{f'(x_n)(1+f'(x_n)^2)}{f''(x_n)}(x_{n+1}-x_n)+f(x_n)^2$$
$$+2f(x_n)\frac{1+f'(x_n)^2}{f''(x_n)}=0. \tag{9}$$

Equation (9) can be changed in the form

$$x_{n+1}=x_n-\frac{f(x_n)\left(f''(x_n)f(x_n)+2(1+f'(x_n)^2\right)}{((x_{n+1}^*-x_n)f''(x_n)+2f'(x_n)(1+f'(x_n)^2)}. \tag{10}$$

The right side of x_{n+1}^* is substituted by one-step iteration method. Accordingly, we reconsider again the Newton classical method (1) with a single parameter θ is written as follows,

$$x_{n+1}=x_n-\theta\frac{f(x_n)}{f'(x_n)} \tag{11}$$

then substituting (11) to (10), we have

$$x_{n+1}=x_n-\frac{f(x_n)f'(x_n)\left(f''(x_n)f(x_n)+2(1+f'(x_n)^2\right)}{2f'(x_n)^2(1+f'(x_n)^2)-\theta f(x_n)f''(x_n)}\ldots(12)$$

Equation (12) is a cubic convergence equation with the parameter θ and involves three functions evaluations with efficiency index is about $3^{1/3}\approx 1.4422$.

To increase the convergence order, the iterative method (12) we substituted into as Taylor series expansion as did by (Behl and Kanwar, 2013).

Next, we reconsider the Taylor series expansion of two-order $f(x_{n+1})$ around x_n written in the form,

$$f(x_{n+1})=f(x_n)+(x_{n+1}-x_n)f'(x_n)+\frac{(x_{n+1}-x_n)^2}{2!}f''(x_n) \tag{13}$$

If x_{n+1} close to α, then $f(x_{n+1})\approx 0$. By simplifying (13) we have

$$x_{n+1}=x_n-\frac{2f(x_n)}{f''(x_n)(x_{n+1}-x_n)+2f'(x_n)}, \tag{14}$$

furthermore, by replacing x_{n+1} on the right side (14) using (12) is obtained

$$x_{n+1}=x_n-\frac{2f(x_n)\left(\theta f(x_n)f''(x_n)-2f'(x_n)^2(1+f'(x_n)^2)\right)}{f'(x_n)\left(f(x_n)^2f''(x_n)^2+2\left(\theta+1+f'(x_n)^2\right)f(x_n)f''(x_n)-4f'(x_n)^2(1+f'(x_n)^2)\right)} \tag{15}$$

Equation (14) still contains second derivative $f''(x_n)$ in some cases it is becomes an issue in the computational process. Therefore, the second derivative $f''(x_n)$ from (15) is reduced using Newton-Steffensen Method (Sharma, 2005) with the parameters λ and Halley Method (Amat, et al., 2003), (Melman, 1997) written as follows, consecutively

$$x_{n+1}=x_n-\frac{f(x_n)^2}{f'(x_n)\left(f(x_n)-\lambda f(y_n)\right)}, \tag{16}$$

and

$$x_{n+1}=x_n-\frac{2f(x_n)f'(x_n)}{2f'(x_n)^2-f(x_n)f''(x_n)}. \tag{17}$$

Further, the second derivative $f''(x_n)$ can be determined by using similarities (16) and (17), then we get

$$f''(x_n)=2\lambda\frac{f'(x_n)^2f(y_n)}{f(x_n)^2}. \tag{18}$$

Equation (18) is a second derivative which derived from (16) and (17) containing one parameter λ, and by substituting (18) to (15), a new iterative method without second derivative is obtained

$$x_{n+1}=x_n-\frac{f(x_n)^2\left(\theta\lambda B-Af(x_n)\right)}{f'(x_n)\left((Bf'(x_n))^2+(A+\theta)f(x_n)(B-Af(x_n))\right)}, \tag{19}$$

with

$$A=1+f'(x_n)^2,\ B=\lambda f(y_n)$$

Equation (19) is the result of modification of the curvature curve using a second-order Taylor series expansion involving three evaluations of functions $f(x_n)$, $f(y_n)$, and $f'(x_n)$.

3.2 Convergence Order Analysis

In this section, we will determine the convergence order of the iterative method (19) as given in the following theorem.

Theorem 3.1. Let $\alpha\in D$ be the simplest root of the differentiable function $f:D\subset R\to R$ in an open interval D. If the initial value x_0 close to α, the iterative method (19) has a four-convergence order for $\theta=1$ and $\lambda=1$ which satisfies the error equation

$$e_{n+1}=\frac{1}{1+f'(\alpha)^2}\left(27f'(\alpha)^2c_2^3+(47f'(\alpha)^2+6)c_2c_3\right.$$
$$\left.+11f'(\alpha)^2c_4\right)e_n^4. \tag{20}$$

Proof. Let α be the root of $f(x)$, then $f(a)=0$.

Assume that $e_n=x_n-\alpha$ and $c_j=\frac{1}{j!}\frac{f^{(j)}(\alpha)}{f'(\alpha)}$.

Next, to expand function $f(x_n)$ around α using Taylor series expansion, we have

$$f(x_n) = f'(\alpha)\left(e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + O(e_n^5)\right) \quad (21)$$

and $f'(x_n) = f'(\alpha)\left(1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_3^3 + O(e_n^4)\right)$ (22)

from (21) and (22) we have

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2 e_n^2 + (2c_3 - 2c_2^2) e_n^3 + O(e_n^4). \quad (23)$$

and

$$\begin{aligned} A &:= f'(x_n)^2 + 1 \\ &= 1 + f'(\alpha)^2 (1 + 4c_2 e_n + (6c_3 + 4c_2^2) e_n^2 \\ &\quad + (12 c_2 c_3 + 8c_4) e_n^3 + + (16 c_2 c_4 + 9c_3^2 + 10 c_5) e_n^4 \\ &\quad + O(e_n{}^5)). \end{aligned} \quad (24)$$

Next, using (23) and $x_n = \alpha + e_n$ to determined y_n,

$$y_n = \alpha + c_2 e_n^2 - (2c_3 - 2c_2^2) e_n^3 + O(e_n^4),$$

(25)

The Taylor series expansion to $f(y_n)$ around α, produce

$$\begin{aligned} f(y_n) &= f'(\alpha)(c_2 e_n^2 + (2c_3 - 2c_2^2) e_n^3 + \\ &\quad (5c_2^3 - 7 c_2 c_3 + 3c_4) e_n^4 + O(e_n^5)), \end{aligned} \quad (26)$$

Based on (21), (22), (24), and (26) produce

$$\begin{aligned} & f'(x_n)^2 (\theta\lambda f(y_n) - A f(x_n)) \\ &= f'(\alpha)^3 (-(1 + f'(\alpha)^2) e_n^3 + \\ &\quad (\lambda\theta - 3 - 7 f'(\alpha)^2) e_n^4 + O(e_n^5)) \end{aligned} \quad (27)$$

and

$$\begin{aligned} & f'(x_n)(f'(x_n)^2 \lambda^2 f(y_n)^2 \\ & + (A + \theta) \lambda f(x_n) f(y_n) - A f(x_n)^2) \\ & = f'(\alpha)^3 \left(-(1 + f'(\alpha)^2\right) + e_n^2 \\ & \quad \left((\lambda - 8) f'(\alpha)^2 + \lambda - 4\right) c_2 e_n^3 \\ & + \left(\left(\lambda^2 + 5\lambda - 25\right) f'(\alpha)^2 + (\lambda(\theta + 1) - 5)) c_2^2\right) \\ & + ((2\lambda - 11) f'(\alpha)^2 + (2\lambda(\theta + 1) - 5)) c_3 \big) e_n^4 \\ & + O(e_n^5)\Big) \end{aligned} \quad (28)$$

Furthermore, using (27), $x_{n+1} = e_{n+1}$ and $x_n = e_n + \alpha$, we have

$$\begin{aligned} e_{n+1} &= (1 - 2\lambda) c_2 e_n^2 + \left(2(1 - \lambda) c_3 + \left(\lambda^2 \left(1 + \frac{\theta + f'(\alpha)^2}{1 + f'(\alpha)^2} \right) \right.\right. \\ &\quad \left.\left. - 4\lambda + 2 \right) c_2^2 \right) e_n^3 + \ldots + O(e_n^5)\Big). \end{aligned} \quad (29)$$

Equation (29) give us information that convergence order of the iterative method (19) increases for $\lambda = 1$ and $\theta = 1$. Therefore, by resubstituting $\lambda = 1$ and $\theta = 1$ to (29), the convergence order of (19) be

$$\begin{aligned} e_{n+1} &= \frac{1}{1 + f'(\alpha)^2} \Big(27 f'(\alpha)^2 c_2^3 + (47 f'(\alpha)^2 + 6) c_2 c_3 \\ &\quad + 11 f'(\alpha)^2 c_4 \Big) e_n^4 + O(e_n^5). \end{aligned} \quad (30)$$

Equation (30) shows that the iterative method (19) has a four-order convergence for $\lambda = 1$, $\theta = 1$ and involves three functions evaluation at each iteration, with the result of efficiency index $4^{1/3} \approx 1.5874$. Convergence order of the iterative method with the efficiency index of $4^{1/3}$ is said to be optimal (Kung and Traub, 1974).

The new iterative method with two real parameters θ and λ given in Equation (19) raises several other iterative methods by substituting the values of the real parameter.

If $\theta = 1$ and $\lambda = 0$, then we obtained the Newton Method (Traub, 1964)

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad (31)$$

If $\theta = 1$ and $\lambda = 1$, we obtained four order convergence iterative method,

$$x_{n+1} = x_n - \frac{f(x_n)^2 \left(f(x_n)(1 + f(x_n)^2) - f(y_n) \right)}{f'(x_n) \left((1 + f'(x_n)^2) f(x_n)^2 - (f'(x_n)^2 + 2) f(y_n) - f'(x_n)^2 f(y_n)^2 \right)}. \quad (32)$$

If $\theta = 0$ and $\lambda = 1$, we obtained a three-order convergence iterative method,

$$x_{n+1} = x_n - \frac{f(x_n)^3 (1 + f'(x_n)^2)}{f'(x_n) \left((1 + f'(x_n)^2)(f(x_n)^2 + f(x_n) f(y_n)) - f'(x_n)^2 f(y_n) \right)}. \quad (33)$$

3.3 Numerical Simulation

In this section, numerical simulations are performed to test the performance of new methods (19) for $\lambda = 1$ and $\theta = 1$ (M-4) and we compare them with some other iterative methods, such as: Newton Method (Traub, 1964), Method Chun-Kim (Chun and Kim, 2010), Newton-Steffensen Method (Sharma, 2005), and Omran Method (Omran, 2013) denoted by MN, MCK, MNS, and MO respectively.

The performance of an iterative method was measured from several indicators, they are the number of iterations that we used, the convergence order is either computed using the Taylor's series expansion or computational order of convergence (COC), and the accuracy of iteration. One of the most important indicators of an iterative method is the convergence order. In the process of determining the approximation roots, the convergence order of the iterative method gives effect to the number of iterations used and the accuracy of the iterative method. The higher the convergence order of an iterative method, the number of iterations used is less and the accuracy is better.

Therefore, the comparable indicators of the iterative methods are the convergence order, the number of iterations (IT), the absolute value of the function at the n^{th} iteration ($|f(x_n)|$), and the relative error $|x_n - x_{n-1}|$.

Numerical simulations are performed by applying compared iterative methods to some functions using software Maple 13 and 850 decimal digits (digits floating point arithmetic). Since the computation computed convergence order (COC) in each iteration method is calculated using at least three approximation roots value, i.e. at the $n-1$, n, and $n+1$ iterations, the accuracy is considered sufficient and can satisfy the condition is 10^{-95}.

Next, let x_0 be the initial guess value taken as close as possible to α (displayed until 20 decimal digits), then the iteration process can be performed. The iteration computing process is stopped if it meets the criteria,

$$|x_n - x_{n-1}| < \varepsilon, \qquad (34)$$

with $\varepsilon = 10^{-95}$.

The functions used in this numerical simulation are:

$f_1(x) = xe^{-x} - 0.1,\ \alpha \approx 0.11183255915896296483,$

$f_2(x) = e^x - 4x^2, \alpha \approx 4.30658472822069929833,$

$f_3(x) = \cos(x) - x, \alpha \approx 0.73908513321516064165,$

$f_4(x) = e^{-x^2+x+2} - \cos(x+1) + x^3 + 1,$
$\quad with \ \alpha \approx -1.00000000000000000000,$

$f_5(x) = \sin^2 x - x^2 + 1,\ \alpha \approx 1.40449164821534122603,$

$f_6(x) = \sqrt{x} - x,\ \alpha \approx 1.00000000000000000000.$

To determine the performance of an iterative method, we can see the number of iterations involved in a certain accuracy. In this simulation the number of iterations used (IT) and COC of the comparison methods is shown in Table 1.

Based on Table 1, we can conclude that the M-4 method is better than the other four methods, since it has fewer iterations. In addition, based on the value of COC can be concluded that M-4 method has four convergence order.

Next, we shown the value of $|f(x_n)|$ and $|x_n - x_{n-1}|$ from compared method to the 12 total of function evaluation (TNFE) which given in Table 2 and Table 3, consecutively.

Table 1: Number of iterations (IT) and COC.

$f(x)$	x_0	MN	MCK	MNS	MO	M-4
$f_1(x)$	-0.20	8 (1.9999)	5 (2.9999)	5 (2.9999)	5 (3.0000)	4 (3.9999)
	0.30	8 (1.9999)	5 (3.0000)	5 (3.0000)	5 (3.0000)	4 (3.9999)
$f_2(x)$	4.10	8 (1.9999)	5 (3.0000)	5 (3.0000)	5 (3.0000)	4 (3.9999)
	4.50	7 (1.9999)	5 (2.9999)	5 (2.9999)	5 (3.0000)	4 (3.9999)
$f_3(x)$	-0.10	8 (1.9999)	5 (3.0000)	5 (3.0000)	5 (3.0000)	4 (3.9999)
	1.50	7 (1.9999)	5 (2.9999)	5 (2.9999)	5 (3.0000)	4 (3.9999)
$f_4(x)$	-1.50	7 (2.0000)	5 (2.9999)	5 (3.0000)	5 (3.0000)	4 (3.9999)
	0.10	7 (2.0000)	7 (2.9999)	5 (2.9999)	5 (3.0000)	4 (3.9999)
$f_5(x)$	1.20	8 (1.9999)	5 (3.0000)	5 (3.0000)	5 (3.0000)	4 (3.9999)
	2.00	8 (1.9999)	5 (2.9999)	4 (2.9999)	5 (3.0000)	4 (3.9999)
$f_6(x)$	0.50	8 (1.9999)	6 (3.0000)	5 (3.0000)	5 (3.0000)	4 (3.9999)
	1.50	7 (1.9999)	5 (2.9999)	5 (2.9999)	5 (3.0000)	4 (3.9999)

Table 2: The function value $\left|f(x_n)\right|$ for TNFE = 12.

$f(x)$	x_0	MN	MCK	MNS	MO	M-4
$f_1(x)$	-0.20	3.0851 (e-36)	1.1432 (e-40)	9.0539 (e-54)	4.4068 (e-46)	1.4695 (e-142)
	0.30	1.0736 (e-42)	1.7153 (e-43)	1.2725 (e-45)	6.2110 (e-53)	9.3099 (e-176)
$f_2(x)$	4.10	2.6996 (e-45)	2.7159 (e-47)	1.4792 (e-57)	6.4848 (e-57)	6.0818 (e-159)
	4.50	3.1919 (e-52)	3.8771 (e-60)	2.4262 (e-66)	1.2177 (e-66)	7.2098 (e-200)
$f_3(x)$	-0.10	1.9402 (e-37)	1.4134 (e-17)	1.4119 (e-45)	3.1919 (e-47)	4.4549 (e-105)
	1.50	3.7607 (e-64)	9.4355 (e-50)	3.5077 (e-80)	3.9419 (e-77)	2.3513 (e-198)
$f_4(x)$	-1.50	5.7389 (e-66)	7.4069 (e-51)	5.1899 (e-92)	1.7321 (e-95)	6.3165 (e-173)
	0.10	3.0579 (e-64)	7.0708 (e-10)	2.9369 (e-69)	3.7202 (e-60)	2.4217 (e-137)
$f_5(x)$	1.20	2.0864 (e-47)	2.1333 (e-46)	7.4954 (e-60)	1.4432 (e-59)	9.6912 (e-165)
	2.00	2.2623 (e-32)	3.9133 (e-32)	8.2994 (e-41)	1.3569 (e-40)	4.9506 (e-112)
$f_6(x)$	0.50	1.5492 (e-43)	1.2886 (e-12)	2.2097 (e-55)	3.0629 (e-57)	4.1246 (e-128)
	1.50	1.0649 (e-66)	5.2351 (e-63)	2.4094 (e-84)	4.7864 (e-83)	4.6824 (e-225)

Table 3: Relative error $\left|x_n - x_{n-1}\right|$ for TNFE = 12.

$f(x)$	x_0	MN	MCK	MNS	MO	M-4
$f_1(x)$	-0.20	1.9116 (e-18)	4.3760 (e-14)	2.1608 (e-18)	7.8896 (e-16)	3.6099 (e-36)
	0.30	1.1277 (e-21)	5.0098 (e-15)	1.1235 (e-15)	4.1058 (e-18)	1.8111 (e-44)
$f_2(x)$	4.10	9.0319 (e-24)	8.5959 (e-17)	3.7720 (e-20)	6.1734 (e-20)	1.1433 (e-40)
	4.50	3.1057 (e-27)	4.4925 (e-21)	4.4484 (e-23)	3.5451 (e-23)	6.7084 (e-51)
$f_3(x)$	-0.10	7.2458 (e-19)	3.7159 (e-06)	2.5865 (e-15)	7.1335 (e-16)	1.6788 (e-26)
	1.50	3.1901 (e-32)	6.9968 (e-17)	7.5472 (e-27)	7.8465 (e-26)	8.0467 (e-50)
$f_4(x)$	-1.50	2.3956 (e-33)	1.5064 (e-17)	6.7780 (e-31)	4.7015 (e-32)	1.1494 (e-43)
	0.10	1.7487 (e-32)	6.8799 (e-4)	2.6022 (e-23)	2.8156 (e-20)	9.0447 (e-35)
$f_5(x)$	1.20	3.2751 (e-24)	4.2239 (e-16)	1.7005 (e-20)	2.1155 (e-20)	8.2906 (e-42)
	2.00	1.0784 (e-16)	2.4000 (e-11)	3.7903 (e-14)	4.4651 (e-14)	1.2464 (e-28)
$f_6(x)$	0.50	1.1133 (e-21)	2.1759 (e-4)	1.9194 (e-18)	4.6106 (e-19)	3.5839 (e-32)
	1.50	2.9189 (e-33)	3.4727 (e-21)	4.2562 (e-28)	1.1527 (e-27)	2.803 (e-56)

4 CONCLUSIONS

In this conclusion a new iteration method is given by the four-order convergence for $\lambda = 1$, $\theta = 1$ and the index efficiency $4^{1/3} \approx 1.5874$. The numerical simulations also provide information that the COC of the new iteration method is four given by Table 1. In addition, Tables 2 and 3 show the accuracy and the precision of the new method better than the Newton Method (Traub, 1964), Chun-Kim Method Chun and Kim, 2010), Newton-Steffensen Method (Sharma, 2005), and Omran Method (Omran, 2013).

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