Some Properties of Prime Submodules on Dedekind Module $\mathbb{Z}[\sqrt{-1}]$ Over Itself

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Abstract: Wardhana et al. gave the characterization of prime submodule and almost prime submodule of finitely generated module over principal ideal domain in 2016. In this article, we gave some properties of prime submodule of Dedekind module $\mathbb{Z}[\sqrt{-1}]$ over itself.

1 INTRODUCTION

Prime number is the key of Cryptography. Daun gave the definition of prime submodules in 1978 (Dauns, 1994) and later Khashan gave generalization of prime submodules in 2012 that called almost prime submodules (Khashan, 2012). Wardhana et al. gave characterization of prime submodules and almost prime submodules in finitely generated module over principal ideal domain (Wardhana et al., 2018). In our article we give some properties of prime submodules in more general case, which is in finitely generated module over Dedekind domain. Specifically, we give some properties of prime submodules in module $\mathbb{Z}\left[\sqrt{-1}\right]$ over itself.

The definition of Dedekind domain is given below.

Definition 1 An integral domain D is called a Dedekind domain if D satisfying the following three conditions:

a) Integral domain D is a Noetherian ring;
b) Integral domain D is integrally closed;
c) Every nonzero prime ideal of D is maximal

An example of Dedekind domain is integral domain $\mathbb{Z}\left[\sqrt{-1}\right] = \left\{a + b\sqrt{-1} | a, b \in \mathbb{Z}\right\}$. But $\mathbb{Z}\left[\sqrt{5}\right]$ is not Dedekind domain since $\mathbb{Z}\left[\sqrt{5}\right]$ is not integrally closed.

Definition 2 A fraction submodule N of R-module M, is a set $\{r \in R | rM \subset N\}$ and denoted by $(N: M)$. A submodule N is a prime submodule of M if for every $r \in R$ and for all $m \in M$ such that $rm \in N$ implies $r \in (N: M)$ or $m \in N$.

For an example, in $\mathbb{Z}$-module $\mathbb{Z}_n$. Submodule $\langle \bar{p} \rangle$ is a prime submodule of $\mathbb{Z}_n$ if and only if p is prime. Submodule $\langle \bar{0} \rangle$ is prime submodule by the definition. There are lot of properties of Dedekind domain that is very important. Especially next Theorem.

Theorem 1 Let D be a Dedekind domain, if I is ideal of D then I must be generated by two elements.

We can see the proof of this Theorem in Dauns (1994). By Theorem 1, every principal ideal domain (PID) is Dedekind domain since every ideal of PID is generated by one element, thus any principal ideal domain must be generated by two elements by choose 0 as the second generator. In this article, we used Dedekind domain $\mathbb{Z}\left[\sqrt{-1}\right] = \left\{a + \sqrt{-1}b | a, b \in \mathbb{Z}\right\}$. Let $R = \mathbb{Z}[-1]$ is an R-module over itself. It is easy to check that R is free module with $\{1\}$ as basis.

Corollary 1 A Principal Ideal Domain is a Dedekind domain.

If N is submodule of M, then N is not always a free submodule. Furthermore, N is not always generated by one element. For example, let $N = span\{2, 3\sqrt{-1}\}$. Generator of N is linearly dependent since $(3\sqrt{-1})2 + (-2)3\sqrt{-1} = 0$. But $\{2, 3\sqrt{-1}\}$ is minimal spanning set of N since $2 \notin span\{3\sqrt{-1}\}$ and $3\sqrt{-1} \notin span\{2\}$. This example shows us that even though the module M is free and generated by

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one element, we can find a submodule of M that is not free and generated by two elements.

Another important property of Dedekind domain is every nonzero ideal must be product of finitely prime ideal.

Theorem 2 If D is a Dedekind domain, then every nonzero ideal of D is product of finitely many prime ideals.

In this paper, R shall always denote the Dedekind Domain $\mathbb{Z}\left[\sqrt{-1}\right]$ and M is R-module $\mathbb{Z}\left[\sqrt{-1}\right]$.

2 RESULT

Given R-module M, we know that submodule N of M are not always generated by one element. But N can be generated by maximum two element.

Theorem 3 Every submodule N of an R-module M must be generated by at most two elements.

Proof Let N be generated by $\{x_1, x_2, \ldots, x_n\}$, $n > 2$. Let $x = a \pm \sqrt{-1}\, b \in N$. Then $a + \sqrt{-1}b = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n$. The equality of two complex number gives us a system linier equation (SLE) with two equation and $n > 2$ variable. According to basic linear algebra, our SLE always consistent with many solutions. Hence there is x_i, $i \in \{1,2,\ldots,n\}$, such that x_i linear combination of other $\{x_1, x_2, \ldots, x_n\} - \{x_i\}$. Therefore, we can eliminate x_i from $\{x_1, x_2, \ldots, x_n\}$, and $\{x_1, x_2, \ldots, x_n\} - \{x_i\}$ is still generated N. If $n - 1 > 2$ then we can use this method to eliminate other x_j until the generator of N is consist by two elements■

Now we will give some properties of submodule N that generated by one element. First, we need to know the fraction of submodule N.

Theorem 4 Let N be a submodule of R-module M. If submodule N is generated by $\{a + \sqrt{-1}b\}$, then $(N\colon M) = \langle a + \sqrt{-1}b \rangle$.

Proof Let $r \in \langle a + \sqrt{-1}b \rangle$ and $m \in M$. Suppose that $r = k(a + \sqrt{-1}b)$ for some k. Hence, $rm = k\left(a + \sqrt{-1}b\right)m \in N$. Since m is arbitrary, then we have $\langle a + \sqrt{-1}b \rangle \subseteq (N\colon M)$.

Conversely, let $r \in (N\colon M)$. Choose $1 \in M$, then $r1 \in N$. Hence $r = k(a + \sqrt{-1}b)$for some k. Therefore $r \in \langle a + \sqrt{-1}b \rangle$. So, we have $(N\colon M) \subseteq \langle a + \sqrt{-1}b \rangle$. ■

By Theorem 4, it is easy to find fraction submodule of any submodule that generated by one element. Theorem 2 also essential to recognize prime submodule in M.

Theorem 5 Let N be a submodule of R-module M that generated by $\{a + \sqrt{-1}b\}$. Submodule N is prime if and only if $a + \sqrt{-1}b$ is prime element in R.

Proof Let $p = a + \sqrt{-1}b$ is prime element of R. Let $r \in R$ and $m \in M$ such that $rm \in \langle p \rangle$. We have $rm = kp$, hence $p|rm$. Since p is prime, we have $p|r$ or $p|$. According to Theorem 3 we have $r \in (N\colon M)$ or $m \in N$. Hence N is prime submodule of M.

Conversely, let N be a prime submodule of M that generated by one element. Hence, we can write $N = \langle x \rangle$. Let $r \in R$ and $m \in M$ such that $rm \in N$. Then we have $rm = kx$ for some $k \in R$. Since N is prime then $r \in (N\colon M) = \langle x \rangle$ or $m \in N = \langle x \rangle$. Therefore $x|r$ or $x|m$, and hence x is prime element.■

In general, submodule N need not be generated by one element. In more general case we have the following properties of $(N\colon M)$.

Theorem 6 Let N be a submodule of R-module M. Then $(N\colon M) = N$.

Proof Let $r \in (N\colon M)$. Since $1 \in M$ then we have $r = r.1 \in N$. Therefore $(N\colon M) \subset N$. Conversely, let $x \in N$. Then for any $m \in M$ we have $xm \in N$, since N is also ideal of R. Hence $N = (N\colon M)$.■

Theorem 6 is more general properties of Theorem 4. Hence if N is generated by two elements, say $N = Ra + Rb$, then we have $(N\colon M) = Ra + Rb$.

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