The Partition Dimension of Bridge Graphs from Homogeneous Caterpillars and Cycle

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Keywords: Partition Dimension, Resolving Partition, Bridge Graph, Caterpillar.

Abstract: The partition dimension of graph is one of open problem in the graph theory. Investigation of the problem can be solved by operations of graph. Several operations that are known is the partition dimension of corona product, cartesian product, subdivision operation. In this paper, the partition dimension is investigated by a bridge operation. Let G_1, G_2 be two connected graphs and let $u \in V(G_1)$ and $u \in V(G_2)$. The Bridge graph, $B(G_1, G_2, uv)$ is a graph which obtained from two graphs G_1, G_2 with a linking u to v. This paper is devoted to find the partition dimension of the bridge graph from homogeneous caterpillars and the cycle graphs.

1 INTRODUCTION

On the transport network, there is always one path as the main road. On the main road, there will be several paths as branch roads. The design of the main path with a number of intersections originating from the main road in graph theory is known as caterpillar graph. In the caterpillar graph, the main path is known as the backbone edge and the paths of an intersection are called the leaves edges. In addition to the transportation path, the graph can be used to design a robot navigation network (B. Shanmukha et al. 2002, S. Khuller et al., 1996).

The interesting problem in graph theory which is an open problem until now is a partition dimension of the graph. The problem of partition dimension is the problem to determine the classes of vertices such that each vertex is distinguished from each other. The researchers in dimensional partitions used in various methods namely research in the certain classes of the graph, or the operations of graphs. Some operations which were used are corona (Yero et al, 2011), subdivision (Amrullah et al., 2013, Amrullah et al., 2015) and cartesian operations (Yero et al., 2010). One of the operations that have not yet appeared in partition dimension research is a bridge operation.

2 BASIC CONCEPTS

First, we introduce several notations and basic concepts to investigate the partition dimension. Let $G = (V, E)$ be a connected graph, $u, v \in V(G)$. The distance $d(u, v)$ from vertex u to vertex v is the length of a shortest path between u and v. Let $L = \{v_1, v_2, \cdots, v_k\}$ be a subset of $V(G)$, Then the distance $d(u, L)$ from a vertex v to L is $\min\{d(v, v_i) | v_i \in L\}$. Let $\Pi = \{L_1, L_2, L_3, \dots L_k\}$ be a k-partition of $V(G)$. The representation $r(v|\Pi)$ of vertex v with respect to Π is the vector $(d(v, L_1), d(v, L_2), \dots, d(v, L_k))$. The partition Π is called a resolving partition of G if $r(w|\Pi) \neq r(v|\Pi)$ for all distinct $w, v \in V(G)$. The partition dimension of G, denoted by $pd(G)$, is the cardinality of a minimum resolving partition of G. If two vertices u and v are in the same partition class under Π, then we write $u \sim_\pi v$, otherwise $u \nsim_\pi v$. If $d(v, L_j) \neq d(u, L_j)$ for some $j \in [1, k]$, then we shall say that u and v are distinguished by L_i or u and v are distinguishable. Let $v \in L_i$, if $d(v, L_i) = 1$ for any $L_j \neq L_i$ then v is called a dominant vertex under Π. Let L_t be a partition class distinguishing two vertices u, v where $t \in [1, p]$. Vertices x and y in L_t are called the distance defining vertices of u and v in L_t if $d(u, L_t) = d(u, x)$, $d(v, L_t) = d(v, y)$ and $d(u, x) \neq d(v, y)$.

Amrullah, ., A., S., S., H., Turmuzi, M. and Y. S., A.
The Partition Dimension of Bridge Graphs from Homogeneous Caterpillars and Cycle.
DOI: 10.5220/0008519602170221
In *Proceedings of the International Conference on Mathematics and Islam (ICMIs 2018)*, pages 217-221
ISBN: 978-989-758-407-7

Let G_1, G_2 be two connected graphs, $u \in V(G_1)$ and $v \in V(G_2)$. The bridged graph of G_1 and G_2 by $e = uv$, $B(G_1, G_2, uv)$, is a graph obtained from graph G_1 and G_2 which linking the vertex u in $V(G_1)$ to the vertex v in G_2 . In this paper, we examine the bridge graphs $B(G_1, G_2, uv)$ where G_1 is the homogeneous caterpillar dan G_2 is a cycle graph. The Homogeneous caterpillar, $C(m,n)$, is a graph obtained by attaching $w_{i,1}, w_{i,2}, \ldots w_{i,n}$ leaves to each vertex v_i of the path P_m , for $i \in [1,m]$. The cycle graph, $C_{n,}$ is a connected graph which each vertex has one degree with $V(C_n) = \{c_1, c_2, \ldots, c_n\}$. This paper is devoted to find the partition dimension of $B(G_1, G_2, uv)$ where G_2 is a cycle graph.

In the following lemmas, we introduce several properties which are useful in this research. Lemma 2.1 and Corollary 2.2 are given by G. Chartrand et al., (2000).

Lemma 2.1. (G. Chartrand et al., 2000) Let G be a connected with a resolving partition Π. If $d(u,w) = d(v,w)$ for all $w \in V(G) - \{u,v\}$, then vertices u, v must be in distinct partition classes of Π.

A lower bound of partition dimension of graph given by a direct consequence of Lemma 2.1

Corollary 2.2. (G. Chartrand et al., 2000) Let G be a connected graph, if G has a vertex having k leaves then $pd(G) \geq k$.

The next Lemma 2.2 gives the partition dimension of path P_n (G. Chartrand et al., 1998)

Lemma 2.2. (G. Chartrand, et al. 1998) Let G be a connected graph of order $n \geq 2$. Then $pd(G) = 2$ if and only if $G = P_n$.

The Lemma 2.2 shows that the other graphs have the partition dimension at least three.

A homogeneous caterpillar $C(m,n)$ is a graph obtained by attaching m vertices to each vertex v_i of the path P_n, for $i \in [1,n]$. All vertices of degree one are called leaves. All leaves attached to v_i are labelled by $w_{i,1}, w_{i,1}, \ldots, w_{i,m}$ Darmaji et al. (2009) gave the partition dimension of a homogeneous caterpillar in the following theorem.

Theorem 2.1 Let $G = C(m,n)$ be a homogeneous caterpillar with $m \geq 3, n \geq 2$. Then,

$$pd(G) = \begin{cases} m & \text{if } m \leq n \\ m+1 & \text{Otherwise} \end{cases}.$$

3 MAIN RESULTS

In first result, we give partition dimension of bridge graph obtaining from a homogeneous caterpillar $C(m,n)$ for $m \in \{1,2\}$ and cycle C_3.

Lemma 3.1. If $G_1 = C(m,n)$ and $G_2 = C_k$, $m \in \{1,2\}$, $n \geq 2, k \geq 3$, and $u \in V(G1)$, $v \in V(G2)$, then $pd(B(G_1, C_3, uv)) = 3$.

Proof
Since $G_2 = C_k$, we obtain that $B(G_1, G_2, uv)$ is not a path. So, we have $pd(B(G_1, G_2, uv)) \geq 3$
Next, let $\Pi = \{L_1, L_2, L_3\}$ be a partition of $V(B(G_1, G_2, uv))$, look at Figure 1.

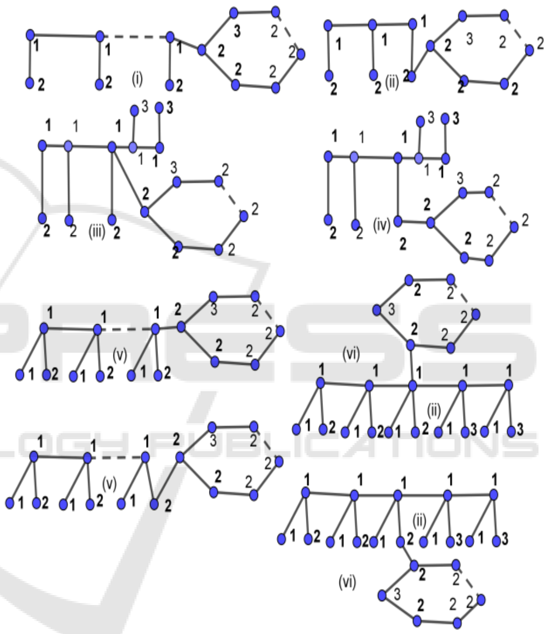

Figure 1: The resolving partition of $B(G_1, C_3, uv)$ where (i,ii,v,vi) $G_1 = C(1,n)$ and (iii,iv,vii) $G_1 = C(2,n)$ with $n \geq 1$.

By definition of Π at Figure 1, it is easy to say that Π is a resolving partition of $B(G_1, C_3, uv)$. So, we have $pd(B(G_1, C_3, uv)) = 3$. □

Lemma 3.2. If $G_1 = C(3,n)$ and $G_2 = C_k$, $n \geq 2$, $k \geq 3$ and $u \in V(G_1), v \in V(G_2)$, then

$$pd\big(B(G_1, G_2, uv)\big) = \begin{cases} 3 & \text{if } n \in [1.2] \text{ or } (n = 3 \text{ and } u \text{ is a leaf of } G_1), \\ 4 & \text{otherwise} \end{cases}$$

Proof.
To easy our notation, let $H = B(G_1, G_2, uv)$. Without loss of generality, let $= c_1$. This proof considers three cases.

Case 1. For $m \in \{1,2\}$, Since H is not a path, then $pd(H) \geq 3$. Let $\Pi = \{L1, L2, L3\}$ be a partition of $V(H)$, look at Figure 2.

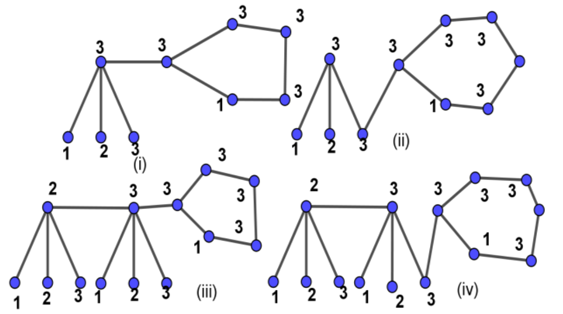

Figure 2: The resolving partition of $B(G_1, C_3, uv)$ where (i,ii) $G_1 = C(3,1)$ and (iii,iv) $G_1 = C(3,2)$.

By definition of Π at Figure 2, it is clear that Π is a resolving partition of $B(G_1, C_3, uv)$. Let x, y in $B(G_1, C_3, uv)$. If $x, y \in L_3 \subset V(C_k)$ then they are distinguished by c_2 or L_2. If $x, y \in V(C(m,n))$ then they are distinguished by some vertex which adjacent to x or y, c_2 or L_2. If $x \in V(C(m,n))$ and $y \in V(C_k)$ then they are distinguishe L_2. So, we have $pd(B(G_1, C_3, uv)) = 3$.

Case 2. For $m = 3$ and u is a leaf of G_1, since H is not a path, then $pd(H) \geq 3$. Let $\Pi = \{L_1, L_2, L_3\}$ be a partition of $V(H)$, look at Figure 3.

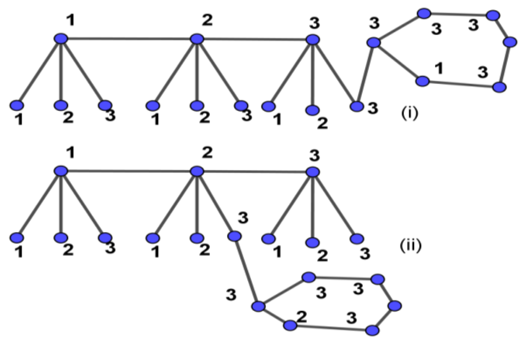

Figure 3: The resolving partition of $B(C(3,3), C_k, uv)$ where u is a leaf of $C(3,3)$.

Base on the definition of Π at Figure 2, it is clear to say that Π is a resolving partition of $B(C(3,3), C_k, uv)$. So, we have $d(B(C(3,3), C_3\ uv)) = 3$, where u is aleaf of G_1.

Case 3. We will show that $pd(H) \geq 4$. For a contradiction that $pd(H) < 4$, let $\Pi = \{L_1, L_2, L_3\}$ be a resolving partition of H. Since each vertex v_i is adjacent to three leaves $w_{i,j}$, without loss of generality v_i, $w_{i,j}$ in a partition class L_i for $i, j \in [1,3]$. Now we consider the vertices c_i for $i \in [1,3]$. Since each v_i is a dominant vertex then the vertices c_1, c_2, c_3 contain at most in two partition classes.

If $c_1 \in L_3$, then $r(c_1 | \Pi) \in \{(2,1,0), (2,2,0), (1,2,0), (1,1,0)\}$. Since $r(v_3 | \Pi) = (1,1,0)$, $r(w_{3,3} | \Pi) = (2,2,0)$, $r(w_{2,3} | \Pi) = (2,1,0)$, and $r(w_{1,3} | \Pi) = (1,2,0)$, then $r(c_1 | \Pi)$ will same to one of representation of $v_3, w_{1,3}, w_{2,3}$ and $w_{3,3}$. This implies that $c_1 \notin L_3$. If $c_1 \in L_2$, then $r(c_1 | \Pi) \in \{(2,0,1), (1,0,1)\}$. Since $r(v_2 | \Pi) = (1,0,1)$, $r(w_{3,2} | \Pi) = (2,0,1)$, then $r(c_1 | \Pi)$ will same to one of representation of v_2, and $w_{3,2}$. This implies that $c_1 \notin L_2$. If $c_1 \in L_1$, then $r(c_1 | \Pi) \in \{(0,1,1), (0,2,1)\}$. Since $r(v_1 | \Pi) = (0,1,1)$, $r(w_{3,1} | \Pi) = (0,2,1)$, then $r(c_1 | \Pi)$ will same to one of representation of v_1, and $w_{3,1}$. This implies that $c_1 \notin L_1$.

These implies $c_1 \notin L_1$ or $c_1 \notin L_2$ or $c_1 \notin L_3$, contradiction. As the consequences $pd(H) \geq 4$.

To show the upper bound of $pd(H)$, we define a new partition $\Pi = \{L_1, L_2, L_3, L_4\}$ of $V(H)$ where $L_1 = \{v_1, v_2, \ldots, v_m\} \cup \{w_{i,1} | 1 \leq i \leq m\}$, $L_2 = \{w_{i,2} | 1 \leq i \leq m\} \cup \{c_2\}$, $L_3 = \{w_{i,3} | 1 \leq i \leq m\} \cup \{c_1\}$ and $L_4 = \{c_3\}$, look at Figure 4.

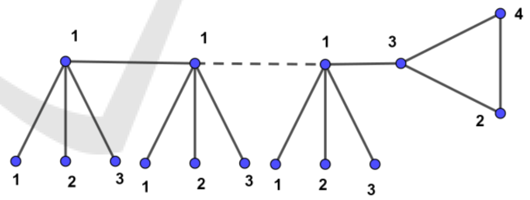

Figure 4: The resolving partition of $B(C(3,n), C_3, uv)$ with $n \geq 3$.

By definition of Π at Figure 3, it is easy to say that Π is a resolving partition of $B(C(3,n), C_3, uv)$ with $n \geq 3$. So, we have $pd(B(C(3,n), C_3, uv)) = 4$. □

Theorem 3.1. If C_m, C_n are two cycles for $m, n \geq 3$, then $pd(B(C_m, C_n, uv)) = 3$ for $u \in V(C_m)$, $v \in V(C_n)$.

Proof.
Let $V(C_m) = \{u_1, u_2, \cdots, u_m\}$ and $V(C_n) = \{v_1, v_2, \ldots, v_n\}$. Let u_1v_1 be a bridge of $B(C_m, C_n, u_1$

v_1). Define $L = \{L_1, L_2, L_3\}$ is a partition of $V(B(P_1, P_2, u_1v_1))$ where $L_1 = \{u_1, u_2, \dots, u_{m-1}\} \cup \{v_n\}$, $L_2 = \{v_{n-1}\}$ and $L_3 = \{v_1, v_2, \dots, v_{n-2}\} \cup \{u_m\}$. For $x, y \in L_1$, if $d(x, L_3) = d(y, L_3)$ then x, y are distinguished by v_{n-1}. The otherwise, they are distinguished by L_3. For $x, y \in L_3$, if $d(x, L_1) = d(y, L_1)$ then x, y are distinguished by v_{n-1}. The otherwise, they are distinguished by L_1. The partition class L_2 is a singleton. These implies that L is a resolving partition of $B(C_m, C_n, uv)$. Thus, we obtain $pd(B(C_m, C_n, uv)) = 3$.

Lemma 3.3 Let $G_1 = C(m,n)$ be a homogeneous caterpillar with $m \geq 4$, $m \leq n$, and $G_2 = C_k$ be a cycle with orde $k \geq 3$. If $u \in V(G_1)$ is not a leaf and $v \in V(G_2)$, then

$$pd(B(G_1, \mathrm{G}_2, uv)) = \begin{cases} m & if\ m \leq n \\ m+1 & otherwises \end{cases}$$

Proof:
Consider this proof in two cases.
Case 1. For $m \leq n$, since there is a vertex v_i which is adjacent to three leaves $w_{i,i}$ then we obtain $pd(B(G_1, G_2, uv)) \geq m$. Let $\Pi = \{L_1, L_2, \dots, L_m\}$ of a recolving partition of G_1 where $L_i = \{v_i, w_{j,i} | 1 \leq j \leq m\}$ for $i \in [1, n]$. Let $v = v_t$ for some $t \in [2, n]$. Define a new partition $\Pi' = \{L'_1, L'_2, \dots, L'_m\}$ of $V(B(G_1, G_2, uv))$ where $L'_i = L_i$ for $i \notin \{t-1, t\}$, $L'_{t-1} = L_{t-1} \cup \{c_n\}$ and $L'_t = L_t \cup \{c_i | 1 \leq i \leq n-1\}$. We will show that $\Pi' = \{L'_1, L'_2, \dots, L'_m\}$ is a resolving partitito of $B(G_1, G_2, uv)$.

Let x, y in $V(B(G_1, G_2, uv))$. If x, y are the leaves, let $x = w_{i,t}$ dan $y = w_{i,t}$ then they are distinguished by $v_i \in L_i$. If ($x = c_i\ or\ x = w_{i,t}$ for some i, j) and $y = v_{i,}$, then $y = v_j$ is a dominant vertex but x is not a dominant vertex, so they are distinguished. If $x = w_{i,t}$ and $y = c_j$, then they are distinguished by L_1 because $d(x, L_1) < d(y, L_1)$.
If $x = c_i$ and $y = c_k$ for $k \neq j$ then they are distinguished by L_1 or L_n.
Case 2. For $m > n$, since there are at least $m+1$ verteces v_i which is adjacent to m leaves $w_{i,j}$, then we obtain $pd(B(G_1, G_2, uv)) \geq m+1$.
Let $u = v_r$ for some $r \in [1, n]$.
Define a partition $\Pi_1 = \{L_1, L_1, \dots, L_{m+1}\}$ of $V(B(G_1, G_2, uv))$ where $L_1 = \{v_t | 1 \leq t \leq r\} \cup \{w_{t,1} | 1 \leq t \leq n\} \cup \{v_t | 1 \leq t \leq n-1\}$, $L_2 = \{v_t | r+1 \leq t \leq n\} \cup \{w_{t,2} | 1 \leq t \leq n\}$, $L_{m+1} = \{c_n\}$ and, $L_i = \{w_{t,i} | 1 \leq t \leq m\}$ for $i \in [3, m]$. We will show that Π_1 is a resolving partitito of $B(G_1, G_2, uv)$.

Let x, y in $V(B(G_1, G_2, uv))$. If $d(x, c_n) \neq d(y, c_n)$ then they are distinguished by $c_n \in L_{m+1}$. If $d(x, c_n) = d(y, c_n)$, then they are distinguished by L_3. □

Lemma 3.4 Let $G_1 = C(m,n)$ be a homogeneous caterpillar with $m \geq 4$, and $G_2 = C_k$ be a cycle with orde $k \geq 3$. If $u \in V(G_1)$ is a leaf and $v \in V(G_2)$, then

$$pd(B(G_1, \mathrm{G}_2, uv)) = \begin{cases} m & if\ m \leq n+1 \\ m+1 & others \end{cases}$$

Proof
We consider this proof in two cases.
Case 1. For $m \leq n+1$, since there is a vertex v_i which is adjacent to m leaves $w_{i,j}$, then we obtain $pd(B(G_1, \mathrm{G}_2, uv)) \geq m$. Let $\Pi = \{L_1, L_2, \dots, L_m\}$ of a resolving partition of G_1 where $Li = \{v_i, w_{j,i} | 1 \leq j \leq m\}$ for $i \in [1, n]$. Let $v = v_t$ for some $t \in [2, n]$. Define a new partition $\Pi' = \{L'_1, L'_2, \dots, L'_m\}$ of $V(B(G_1, G_2, uv))$ where $L'_1 = L_1$ for $i \notin \{t-1, t\}$, $L'_{t-1} = L_{t-1} \cup \{c_n\}$ and $L'_t = L_t \cup \{c_i | 1 \leq i \leq n-1\}$. We will show that $\Pi' = \{L'_1, L'_2, \dots, L'_m\}$ is a resolving partition of $B(G_1, G_2, uv)$.

Let x,y in $V(B(G_1, G_2, uv))$. If x, y are the leaves, let $x = w_{i,t}$ dan $y = w_{j,t}$ then they are distinguished by $v_i \in L_i$.
If ($x = c_i$ or $x = w_{i,t}$ for some i, j) and $y = v_{i,}$ then $y = v_j$ is a dominant vertex but x is not a dominant vertex. If $x = w_{i,t}$ and $y = c_j$ then they are distinguished by L_1 because $d(x, L_1) < d(y, L_1)$. If $x = c_j$ and $y = c_k$ for $k \neq j$ then they are distinguished by L_1 or L_n.
These imply that the vertices x, y are distinguished.
Case 2. For $m > n+1$, since there are at least $m+1$ vertices v_i which is adjacent to m leaves $w_{i,j}$, then we obtain $pd(B(G_1, G_2, uv)) \geq m+1$.

Let $u = w_{r,1}$ for some $r \in [1, n]$.
Define a partition $\Pi_1 = \{L_1, L_2, \dots, L_{m+1}\}$ of $V(B(G_1, G_2, uv))$ where $L_1 = \{v_t | 1 \leq t \leq r\} \cup \{w_{t,1} | 1 \leq t \leq n\} \cup \{c_t | 1 \leq t \leq n-1\}$, $L_2 = \{v_t | r+1 \leq t \leq n\} \cup \{w_{t,2} | 1 \leq t \leq n\}$, $L_{m+1} = \{c_n\}$ and, $L_i = \{w_{t,i} | 1 \leq t \leq m\}$ for $i \in [3, m]$. Let x, y in $V(B(G_1, G_2, uv))$. If $d(x, c_n) \neq d(y, c_n)$, then they are distinguished by $c_n \in L_{m+1}$. If $d(x, c_n) = d(y, c_n)$, then they are distinguished by L_3. □

The following theorem gives the upper bound of partition dimension of the bridge graph from any connected graph and a cycle C_n.

Theorem 3.2. Let G_1 be a connected graph and $G_2 = C_n$ be a cycle with orde $n \geq 3$. If $u \in V(G_1)$ and $v \in V(G_2)$, then $pd(B(G_1, G_2, uv)) \leq pd(G) + 1$.

Proof.
Let $V(C_n) = \{v_1, v_2, \dots, v_n\}$ and uv be a bgidge of $B(G_1, G_2, uv)$ with $v = v_1$. Suppose $L = \{L_1, L_2, \dots, L_n\}$ is a resolving partition of G and $u \in L_n$ and $\Pi = \{L'_1, L'_2, \cdots, L'_n, L'_{n+1}\}$ be a partition of $B(G_1, G_2, uv)$ where $L'_i = L_i$ for $i \in \{1,2.., n-1\}$, $L'_n = L_n \cup \{v_1, v_2, \dots, v_{n-1}\}$ and $L'_{n+1} = \{v_n\}$.

Let x, y be two distinct vertices of $B(G_1, G_2, uv)$. We consider x, y in three cases.

Case 1. the vertices x, y in $V(B(G_1, G_2, uv)) \backslash V(Cn)$. If $d(x, v_1) \neq d(y, v_1)$, then they are distinguished by L'_{n+1}. If $d(x, v_1) = d(y, v_1)$, then consider a partition class L_t in G_1 which is distinguishing x, y. Since $L'_t = L_t$ and the vertices x, y in $V\big(B(G_1, G_2, uv)\big) \backslash V(Cn)$, then the vertices x, y are distinguished by L'_t.

Case 2. the vertices x, y in $V(B(G_1, G_2, uv)) \backslash V(G1)$.If $d(x, u) \neq d(y, u)$, then they are distinguished by L'_1. $If\ d(x, u) = d(y, u)$, then they are distinguished by L'_{n+1}.

Case 3. the vertex x in $V(B(G_1, G_2, uv)) \backslash V(Cn)$ and y in $V(B(G_1, G_2, uv)) \backslash V(G1)$. By definition a partition Π, we only have x, y in L_n. If $d(x, v_n) \neq d(y, v_n)$, then they are distinguished by L'_{n+1}. If $d(x, v_n) = d(y, v_n)$, then we consider $d(x, L'_1) = p$. Since $L'_1 \subset V(B(G_1, G_2, uv_1)) \backslash V(Cn)$, we have $d(y, L'_1) > p$. This implies that the vertices x, y are distinguished by L'_1.

As the consequences that $\Pi = \{L'_1, L'_2, \cdots, L'_n, L'_{n+1}\}$ is a resolving partition of $B(G_1, G_2, uv)$. So, we have $pd(B(G_1, G_2, uv)) \leq pd(G)$+1. □

4 CONCLUSIONS

In this paper, we obtained the partition dimension of the bridge graphs, $pd(B(G_1, G_2, uv))$ from two special graph namely the homogeneous caterpillar as G_1 and a cyclic graph as G_2. The results show that the partition dimension $m - 1 \leq pd(B(G_1, G_2, uv)) \leq m + 1$ where partition dimension of the homogeneous caterpillar is m.

ACKNOWLEDGEMENTS

This research was supported by Research Grand ”Program Hibah Riset Dasar Ristek DIKTI” 2018, Ministry of Research, Technology and Higher Education.

REFERENCES

Amrullah, Assiyatun, H., Baskoro, E. T., Uttunggadewa, S., and Simanjuntak, R., 2013. The Partition Dimension for a Subdivision of Homogeneous Caterpillars. *AKCE International Journal of Graphs and Combinatorics* 10.3, 317-325.

Amrullah, Darmaji and Baskoro, E. T., 2015. The partition dimension for homogeneous firecrackers. *Far East Journal of Applied Mathematics*, 90, 77 - 98.

Chartrand, G., Salehi, E., and Zhang, P., 2000. The partition dimension of graph. *Aequationes Mathematicae*, 59, 45-54.

Chartrand, G., Salehi, E., and Zhang, P., 1998. On the partition dimension of graph. *Congressus Numerantium*, 130, 157-168.

Darmaji, Uttunggadewa, S., Simanjuntak R., and Baskoro, E. T., 2009. The partition dimension of a complete multipartite graph, a special caterpillar and a windmill. *J. Combin. Math. Combin. Comput*. 71, 209-215.

Khuller, S., Raghavachari, B., and Rosenfeld, A., 1996. Landmarks in graphs. *Disc. Appl. Math*. 70, 217–229.

Shanmukha, B., Sooryanarayana B. and Harinath, K. S., 2002. Metric dimension of wheels. *Far East J. Appl. Math*. 8 (3), 217–229.

Yero, Ismael G., and Juan A. Rodrguez-Velzquez, 2010. A note on the partition dimension of Cartesian product graphs. *Applied Mathematics and Computation* 217.7, 3571-3574.

Yero, Ismael G., Dorota K., and Juan A. Rodrguez-Velzquez, 2011. On the metric dimension of corona product graphs. *Computers & Mathematics with Applications*, 61.9, 2793-2798.