H_∞ Measurement-feedback Tracking with Preview

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Keywords: H_∞ Control, Discrete-time Systems, Tracking Preview.

Abstract: Finite-horizon H_∞ output-feedback tracking control for linear discrete time-varying systems is explored along with the stationary infinite-horizon case. We consider three tracking patterns depending on the nature of the reference signal i.e : whether it is perfectly known in advance, measured on-line or previewed in a fixed time-interval ahead. For each of the above three cases a solution is found where, given a specific reference signal, the controller plays against nature which chooses the initial condition and the energy-bounded disturbances. The problems are solved based on a specially devised bounded real lemma for systems with tracking signals. The finite-horizon case is extended to the stationary one where similar results are achieved.

1 INTRODUCTION

In the present paper we address the problem of H_∞ output-feedback tracking control with preview of discrete-time linear systems, in both the finite and the infinite time settings.

The stability analysis and control design for systems with stochastic uncertainties have received much attention in the past (see (Gershon and U. Shaked, 2005) and the references therein), where mainly continuous-time non retarded systems were considered. In the late 80's, a renewed interest in the control and estimation designs of these systems has been encountered and solutions to the stochastic control and filtering problems of both: continuous-time and discrete-time systems, have been derived that ensure a worst case performance bound in the H_∞ sense (see (Gershon and U. Shaked, 2005) for an extensive review). Systems whose parameter uncertainties are modeled as white noise processes in a linear setting have been treated in (Dragan and Morozan, 1997a) (Hinriechsen and Pritchard, 1998), for the continuous-time case and in (Dragan and Morozan, 1997b), (Bouhtouri et al., 1999) for the discrete-time case. Such models of uncertainties are encountered in many areas of applications (see (Gershon and U. Shaked, 2005) and the references therein) such as: nuclear fission and heat transfer, population models and immunology. In control theory such models are encountered in gain scheduling when the scheduling parameters are corrupted with measurement noise.

Tracking feedback-control has been a central problem in both the frequency domain and in the state-space domain over the last decades. Numerous variations of this problem, by large, have been published in both the continuous-time and in the discrete-time settings (Yang and Zhang, 2008), (Wencheng Luo and Ling, 2005),(Kim and Tao, 2002), (Wei-qian You, 2010) and (Gao and Chen, 2008),(Verriest and Florchinger, 1995). The problem of tracking control with preview has been solved in the deterministic case by (Shaked and deSouza, 1995). In the latter work a previewed tracking signal appears in the system dynamics allowing for three patterns of preview. The first case deals with the inclusion of the tracking signal where no preview is given [the simplest case] while in the second case the preview signal is known for a given fixed finite time interval in the future which is smaller the system dynamic horizon. The third pattern deals with the case where the previewed signal is known along the full finite-horizon of the system dynamics. The latter problems were solved in (Shaked and deSouza, 1995) by applying a game theoretical approach where a saddle point strategy is adopted.

The solution of the finite-horizon, stochastic counterpart, state-feedback tracking control with preview was obtained in (Gershon and U. Shaked, 2005),(Gershon and U. Shaked, 2010), where three preview patterns of the tracking signal were considered. These include the simple case of on-line measurement of the tracking signal and two patterns where this signal is either previewed with a fixed time interval ahead or perfectly known in advance along the system horizon (Gershon and U. Shaked, 2010) (see

Gershon, E.
H_∞ Measurement-feedback Tracking with Preview.
DOI: 10.5220/0006853903250331
In *Proceedings of the 15th International Conference on Informatics in Control, Automation and Robotics (ICINCO 2018) - Volume 1*, pages 325-331
ISBN: 978-989-758-321-6

also (Gershon and U. Shaked, 2005) for further details). For all these patterns the solutions were obtained using a game theory approach where the controller plays against nature which chooses the system energy-bounded disturbance and the initial condition.

In the present paper, we extend the work of (Gershon and U. Shaked, 2010) which deals with the stochastic state-feedback tracking control [zeroing, of course, the stochastic terms] to the case where there is no full access to the state-vector and a dynamic output-feedback strategy must be applied. Here, we treat the case where correlated parameter uncertainties appear in both the system dynamics and the measurement matrices. An optimal output-feedback tracking strategy is derived which minimizes the standard H_∞ performance index, for the three tracking patterns of the reference signal. In both, the finite-horizon horizon case and the stationary one, a min-max strategy is applied that yields an index of performance that is less than or equal to a certain cost. We address the problem via two approaches: In the finite-horizon case we apply the Difference LMI (DLMI) method (Gershon and U. Shaked, 2005) for the solution of the Riccati inequality obtained, and in the stationary case we apply a special Lyapunov function which leads to an LMI based tractable solution.

Notation: Throughout the paper the superscript 'T' stands for matrix transposition, $\mathcal{R}^n$ denotes the n dimensional Euclidean space, $\mathcal{R}^{n\times m}$ is the set of all $n\times m$ real matrices, $\mathcal{N}$ is the set of natural numbers and the notation $P>0$, (respectively, $P\geq 0$) for $P\in\mathcal{R}^{n\times n}$ means that P is symmetric and positive definite (respectively, semi-definite). We denote by $L^2(\Omega,\mathcal{R}^n)$ the space of square-integrable $\mathcal{R}^n-$ valued functions on the probability space $(\Omega,\mathcal{F},\mathcal{P})$, where Ω is the sample space, $\mathcal{F}$ is a σ algebra of a subset of Ω called events and $\mathcal{P}$ is the probability measure on $\mathcal{F}$. By $(\mathcal{F}_k)_{k\in\mathcal{N}}$ we denote an increasing family of σ-algebras $\mathcal{F}_k\subset\mathcal{F}$. We also denote by $\tilde{l}^2(\mathcal{N};\mathcal{R}^n)$ the n-dimensional space of nonanticipative stochastic processes $\{f_k\}_{k\in\mathcal{N}}$ with respect to $(\mathcal{F}_k)_{k\in\mathcal{N}}$ where $f_k\in L^2(\Omega,\mathcal{R}^n)$. On the latter space the following l^2-norm is defined:

$$||\{f_k\}||^2_{\tilde{l}_2}=E\{\Sigma_0^\infty ||f_k||^2\}=\Sigma_0^\infty E\{||f_k||^2\}<\infty,\quad \{f_k\}\in\tilde{l}_2(\mathcal{N};\mathcal{R}^n), \tag{1}$$

where $||\cdot||$ is the standard Euclidean norm. We denote by Tr$\{\cdot\}$ the trace of a matrix and by δ_{ij} the Kronecker delta function. Throughout the manuscript we refer to the notation of exponential l^2 stability, or internal stability, in the sense of (Bouhtouri et al., 1999) (see Definition 2.1, page 927, there). By $[Q_k]_+$, $[Q_k]_-$ we denote the causal and non causal parts respectively, of a sequence $\{Q_i,\ i=1,2,...,N\}$.

2 PROBLEM FORMULATION

Given the following linear discrete time-varying system:

$$\begin{aligned} x_{k+1} &= A_k x_k + B_{2,k}u_k + B_{1,k}w_k + B_{3,k}r_k \\ y_k &= C_{2,k}x_k + D_{21,k}n_k, \end{aligned} \tag{2a-b}$$

where $x_k\in R^n$ is the state vector, $y_k\in\mathcal{R}^z$ is the measurement vector, $w_k\in R^p$ is a deterministic exogenous disturbance, $r_k\in R^r$ is deterministic reference signal which can be measured on line or previewed, $u_k\in R^l$ is the control input signal and x_0 is an unknown initial state and where we denote

$$z_k = C_k x_k + D_{2,k}u_k + D_{3,k}r_k,\ z_k\in R^q,\ k\in[0,N] \tag{3}$$

and we assume, for simplicity that:

$$[C_k^T \quad D_{3,k}^T \quad D_{2,k}^T]D_{2,k} = [0\ 0\ \tilde{R}_k], \tilde{R}_k>0.$$

Our objective is to find a control law $\{u_k\}$ that minimizes the energy of $\{z_k\}$ by using the available knowledge on the reference signal, for the worst-case of the process disturbances $\{w_k\},\{n_k\}$ and the initial condition x_0. We, therefore, consider, for a given scalar $\gamma>0$, the following performance index:

$$\begin{aligned} \tilde{J}_E(r_k,u_k,w_k,n_k,x_0) &\stackrel{\Delta}{=} \{||C_N x_N + D_{3,N}r_N||^2\} \\ &+ \{||z_k||_2^2 - \gamma^2[||w_k||_2^2 + ||n_k||_2^2]\} - \gamma^2 x_0^T R^{-1}x_0, \\ R^{-1} &\geq 0. \end{aligned} \tag{4}$$

Similarly to (Gershon and U. Shaked, 2010) we consider three tracking problems differing on the information pattern over $\{r_k\}$:

1) H_∞-tracking with full preview of $\{r_k\}$: The tracking signal is perfectly known for the interval $k\in[0,N]$.

2) H_∞-tracking with no preview of $\{r_k\}$: The tracking signal measured at time k is known for $i\leq k$.

3) H_∞-tracking with fixed-finite preview of $\{r_k\}$: At time k, r_i is known for $i\leq min(N,k+h)$ where h is the preview length.

In all the above three cases we seek a control law $\{u_k\}$ of the form

$$u_k = H_y y_k + H_r r_k$$

where H_y is a causal operator and where the causality of H_r depends on the information pattern of the reference signal. The design objective is to minimize

$$\max\tilde{J}_E(r_k,u_k,w_k,n_k,x_0) \quad \forall\{w_k\},\{n_k\},\quad \{u_k\}\in l_2[0,N-1],\quad x_o\in R^n,$$

where for all of the three tracking problems we derive a controller $\{u_k\}$ which plays against it's adversaries $\{w_k\},\{n_k\}$ and x_0.

3 FINITE-HORIZON OUTPUT-FEEDBACK TRACKING

We bring first the deterministic counterpart of the state-feedback result of (Gershon and U. Shaked, 2010) which is based on a game theory approach and which constitutes the first step in the solution of the output-feedback control problem. We consider the system of (2a) and (3) and obtain the following result:

Lemma 1: *Consider the system of (2a), (3) and $\tilde{J}_E$ of (4) with the term of n_k excluded. Given $\gamma > 0$, the state-feedback tracking game possesses a saddle-point equilibrium solution iff there exists $Q_i > 0, \forall i \in [0,N]$ that solves the following Riccati-type equation*

$$\begin{array}{l} Q_k = A_k^T M_{k+1} A_k + C_k^T C_k - A_k^T M_{k+1} B_{2,k} \Phi_k^{-1} \\ B_{2,k}^T M_{k+1} A_k, \ Q(N) = C_N^T C_N, \\ M_{k+1} \stackrel{\Delta}{=} Q_{k+1}[I - \gamma^{-2} B_{1,k} B_{1,k}^T Q_{k+1}]^{-1}, \\ \quad \Phi_k = B_{2,k}^T M_{k+1} B_{2,k} + \tilde{R}_k. \end{array} \tag{5a-c}$$

and satisfies

$$\begin{array}{c} R_{k+1} > 0, \quad k \in [0 \quad N-1], \quad \gamma^2 R^{-1} - Q_0 > 0, \\ where \ R_{k+1} \stackrel{\Delta}{=} \gamma^2 I - B_{1,k}^T Q_{k+1} B_{1,k}. \end{array} \tag{6a-c}$$

When a solution exists, the saddle-point strategies are given by:

$$\begin{array}{l} x_0^* = (\gamma^2 R^{-1} - Q_0)^{-1} \theta_0, \\ w_k^* = R_{k+1}^{-1} B_{1,k}^T [\theta_{k+1} + Q_{k+1}(A_k x_k + B_{2,k} u_k + B_{3,k} r_k)] \\ u_k^* = -\Phi_k^{-1}\{B_{2,k}^T M_{k+1}[A_k x_k + B_{3,k} r_k + Q_{k+1}^{-1} \theta_{k+1}^c]\} \end{array} \tag{7a-c}$$

where the causal part of θ_{k+1} is

$$\theta_{k+1}^c = [\theta_{k+1}]_+ \tag{8}$$

and where θ_k satisfies

$$\begin{array}{l} \theta_k = \bar{A}_k^T \theta_{k+1} + \bar{B}_k r_k, \quad \theta_N = C_N^T D_{3,N} r_N, \\ \bar{A}_k = Q_{k+1}^{-1} S_{k+1}^{-1} A_k, \ S_{k+1} \stackrel{\Delta}{=} M_{k+1}^{-1} + B_{2,k} T_{k+1}^{-1} B_{2,k}^T, \\ \bar{B}_k = \bar{A}_k^T Q_{k+1} B_{3,k} + C_k^T D_{3,k}, \quad T_{k+1} \stackrel{\Delta}{=} \tilde{R}_k, \end{array} \tag{9}$$

The game value is then given by:

$$\begin{array}{r} \tilde{J}_E(r_k, u_k^*, w_k^*, x_0^*) = \|S_{k+1}^{-\frac{1}{2}}(Q_{k+1}^{-1}\theta_{k+1} + B_{3,k} r_k)\|_2^2 \\ -\|Q_{k+1}^{-\frac{1}{2}} \theta_{k+1}\|_2^2 + \|D_{3,k} r_k\|_2^2 \\ + \|D_{3,N} r_N\|^2 + \theta_0^T (\gamma^2 R^{-1} - Q_0)^{-1} \theta_0. \end{array} \tag{10}$$

Proof: (Gershon and U. Shaked, 2010) The proof is based on adapting the standard completing to squares arguments to the case. We bring in the Appendix the Sufficiency part of the proof, which is needed for the derivation of the BRL of the next section. We note also, at this point, that the solution of the output-feedback control problem is also based on these derivations.

Remark 1: It is important to note that the signal of θ_k in (9) is admitted in the above derivation because of the tracking signal which affects the dynamics of (2a) (Gershon and U. Shaked, 2010). This signal accounts for the nature of the tracking pattern, where it's causal part (i.e $[\theta_{k+1}]_+$) appears in the structure of the controller in accordance with the preview patterns.

Remark 2: Applying the result of Lemma 1 on the specific pattern of the reference signal it is shown in (Gershon and U. Shaked, 2010) that the saddle-point controller strategy depends on the causal part of θ_{k+1} (i.e $[\theta_{k+1}]_+$), where θ_{k+1} is given in (9). The latter dependency on θ_{k+1} appears also in the structure of the control signal in the output-feedback case.

The solution of the output-feedback control problem involves 2 steps where the latter one is a filtering problem of order n. A second Riccati equation is thus achieved by applying the BRL to the dynamic equation of the estimation error. The latter imposes augmentation of the system to 2n order. This augmented system contains also a tracking signal component and therefore one needs to apply a special BRL for systems with tracking signal. We thus bring first the following lemma:

3.1 BRL for Systems with Tracking Signal

We consider the following system:

$$\begin{array}{c} x_{k+1} = A_k x_k + B_{1,k} w_k + B_{3,k} r_k \\ z_k = C_k x_k + D_{3,k} r_k, \quad z_k \in R^q, \ k \in [0,N] \end{array} \tag{11a,b}$$

which is obtained from (2,a) and (3) by setting $B_{2,k} \equiv 0$ and $D_{2,k} \equiv 0$. We consider the following index of performance:

$$\begin{array}{l} J_B(r_k, w_k, x_0) \stackrel{\Delta}{=} \{\|C_N x_N + D_{3,N} r_N\|^2 + \|z_k\|_2^2\} - \\ \{\gamma^2[\|w_k\|_2^2]\} - \gamma^2 x_0^T R^{-1} x_0, \quad R^{-1} \geq 0. \end{array} \tag{12}$$

We arrive at the following theorem:

Theorem 1: *Consider the system of (11a,b) and J_B of (12). Given $\gamma > 0$, J_B of (12) satisfies $J_B \leq \tilde{J}(r,\varepsilon)$, $\forall \{w_k\} \in l_2[0,N-1]$, $x_o \in R^n$, where*

$$\tilde{J}(r,\varepsilon) = \sum_{k=0}^{N-1} \theta_{k+1}^T \{B_{1,k} R_{k+1}^{-1} B_{1,k}^T\} \theta_{k+1} + \sum_{k=0}^{N-1} r_k^T (D_{3,k}^T D_{3,k}) r_k$$

$$+ ||D_{3,N} r_N||^2 + 2 \sum_{k=0}^{N-1} \theta_{k+1}^T \tilde{Q}_{k+1}^{-1} (\bar{M}_{k+1}^{-1})^{-1} B_{3,k} r_k + \theta_0^T \varepsilon^{-1} \theta,$$

if there exists $\tilde{Q}_k$ that solves the following Riccati-type equation

$$\tilde{Q}_k = A_k^T \bar{M}_{k+1} A_k + C_k^T C_k,$$
$$\gamma^2 I - B_{1,k}^T \tilde{Q}_{k+1} B_{1,k} > 0, \ \tilde{Q}(0) = \gamma^2 R^{-1} - \varepsilon I, \qquad (13a,c)$$

for some $\varepsilon > 0$ where

$$\tilde{\theta}_k = \hat{A}_k^T \tilde{\theta}_{k+1} + \hat{B}_k r_k, \quad \tilde{\theta}_N = C_N^T D_{3,N} r_N,$$
$$\hat{A}_k = \tilde{Q}_{k+1}^{-1} \bar{M}_{k+1} A_k, \ \hat{B}_k = \hat{A}_k^T \tilde{Q}_{k+1} B_{3,k} + C_k^T D_{3,k},$$
$$\hat{B}_k = \hat{A}_k^T \tilde{Q}_{k+1} B_{3,k} + C_k^T D_{3,k},$$
$$\bar{M}_{k+1} \stackrel{\Delta}{=} \tilde{Q}_{k+1}[I - \gamma^{-2} B_{1,k} B_{1,k}^T \tilde{Q}_{k+1}]^{-1}. \qquad (14a\text{-}e)$$

Proof: Unlike the state-feedback tracking control in (Gershon and U. Shaked, 2010), the solution of the BRL does not acquire saddle-point strategies (Since the input signal u_k is no longer an adversary). It can, however, be readily derived based on the first part of the sufficiency of Lemma 1 by setting $B_{2,k} \equiv 0$ and $D_{2,k} \equiv 0$. In the Appendix we bring the proof of the BRL as a derivation of the proof of the state-feedback tracking control solution (which is not included in (Gershon and U. Shaked, 2010). The latter proof is also essential for the derivation of the output-feedback tracking control solution.

Remark 3: The choice of $\varepsilon > 0$ in $\tilde{Q}(0)$ of (13b) reflects on both, the cost value (i.e $\tilde{J}(r,\varepsilon)$) of (13b) and the minimum achievable γ. If one chooses $0 < \varepsilon << 1$ then, the cost of $\tilde{J}(r,\varepsilon)$ increases while the solution of (13a) is easier to achieve, which results in a smaller γ. The choice of large ε, on the other hand, causes the reverse effect, which leads to a larger γ.

3.2 The Output-feedback Tracking Control

We consider the system of (2a,b) and (3). Like in the state-feedback case (Gershon and U. Shaked, 2010) we seek a control law $\{u_k\}$, based on the information of the reference signal $\{r_k\}$ that minimizes the tracking error between the system output and the tracking trajectory, for the worst case of the initial condition x_0, the process disturbances $\{w_k\}$ and $\{n_k\}$. We, therefore, consider the performance index of (4) and we assume that (5a) has a solution $Q_{k+1} > 0$ over $[0,N]$ where (6a,b) are satisfied. The solution of the output-feedback problem is stated in the following theorem, for the a priori case, where u_k can use the information on $\{y_i, \ 0 \leq i < k\}$:

Theorem 2: *Consider the system of (2a,b), (3) and $\tilde{J}_E$ of (4). Given $\gamma > 0$, the output-feedback tracking control problem, where $\{r_k\}$ is known a priori for all $k \leq N$ (the full preview case) possesses a solution if there exists $\hat{P}_k \in \mathcal{R}^{2n \times 2n} > 0, K_{o,k} \in \mathcal{R}^{n \times z}, \forall i \in [0,N]$ that solves the following Difference LMI (DLMI):*

$$\begin{bmatrix} \hat{P}_k^{-1} & \tilde{A}_k^T & 0 & \tilde{C}_{1,k}^T \\ \tilde{A}_k & \hat{P}_{k+1} & \gamma^{-1}\tilde{B}_{1,k} & 0 \\ 0 & \gamma^{-1}\tilde{B}_{1,k}^T & I & 0 \\ \tilde{C}_{1,k} & 0 & 0 & I \end{bmatrix} \geq 0, \qquad (15a,b)$$

where $P_0 = \hat{Q}_0^{-1} \stackrel{\Delta}{=} \gamma^{-2} \begin{bmatrix} R & R \\ R & R + \varepsilon I_n \end{bmatrix}$ *with a forward iteration, starting from the above initial condition of P_0, where R is defined in (4), where $\tilde{A}_k$, $\tilde{B}_{1,k}$, $\tilde{C}_k$ are defined in (23) and where $\hat{Q}_k = \hat{P}_k^{-1}$ is given in (24).*

Proof: Using the expression that is achieved in the Appendix for $\tilde{J}_E(r_k,u_k,w_k,x_0)$ in the state-feedback case, the index of performance is now given by:

$$\tilde{J}_E(r_k,u_k,w_k,n_k,x_0) = \tilde{J}_E(r_k,u_k,w_k,x_0)$$
$$-\gamma^2 ||n_k||_2^2 = -\gamma^2 ||\bar{w}_k||_2^2 - \gamma^2 ||x_0 - x_0^*||^2_{R^{-1} - \gamma^{-1} Q_0}$$
$$+ \textstyle\sum_{k=0}^{N-1} [\{||\bar{u}_k + \hat{C}_{1,k} x_k||^2\}]_+ - \gamma^2 ||n_k||_2^2 + \bar{J}(r). \qquad (16)$$

We note that in the full preview case $[\theta_{k+1}]_+ = \theta_{k+1}$. Using the following definitions:

$$\bar{u}_k = \Phi_{k+1}^{1/2} u_k + \Phi_{k+1}^{-1/2} B_{2,k}^T M_{k+1}(B_{3,k} r_k + Q_{k+1}^{-1} \theta_{k+1})$$
$$\bar{w}_k = \gamma^{-1} R_{k+1}^{1/2} w_k - \gamma^{-1} R_{k+1}^{-1/2} B_{1,k}^T [Q_{k+1}(A_k x_k + B_{2,k} u_k$$
$$+ B_{3,k} r_k) + \theta_{k+1}], \qquad (17)$$

we note that

$$\bar{w}_k = \gamma^{-1} R_{k+1}^{1/2}(w_k - w_k^*), \quad \bar{u}_k = \Phi_{k+1}^{1/2}(u_k - u_k^*),$$

where w_k^*, u_k^* are defined in (7b,c) respectively. Note also that the terms that are not accessed by the controller (i.e the terms with x_k), are exclude from u_k^*. Considering the above $\bar{w}_k$ and $\bar{u}_k$ we seek a controller of the form

$$\bar{u}_k = -\hat{C}_{1,k} \hat{x}_k.$$

We, therefore, re-formulate the state equation of (2a) adding the additional terms to recover the original equation of (2a). Considering the above, we obtain the following new state equation:

$$x_{k+1} = \hat{A}_k x_k + \bar{B}_{1,k} \bar{w}_k + \bar{B}_{2,k} \bar{u}_k + \bar{B}_{3,k} r_k + \bar{B}_{4,k} \theta_{k+1}, \qquad (18)$$

where

$$\begin{aligned}&\hat{A}_k = Q_{k+1}^{-1} M_{k+1} A_k, \quad \bar{B}_{1,k} = \gamma B_{1,k} R_{k+1}^{-1/2},\\ &\bar{B}_{2,k} = Q_{k+1}^{-1} M_{k+1} B_{2,k} \Phi_{k+1}^{-1/2},\\ &\bar{B}_{3,k} = B_{3,k} + B_{1,k} R_{k+1}^{-1} B_{1,k}^T Q_{k+1} B_{3,k}\\ &-\bar{B}_{2,k} \Phi_{k+1}^{-1/2} B_{2,k}^T M_{k+1} B_{3,k},\\ &\bar{B}_{4,k} = \bar{B}_{1,k} R_{k+1}^{-1} \bar{B}_{1,k}^T - \bar{B}_{2,k} \Phi_{k+1}^{-1/2} B_{2,k}^T M_{k+1} Q_{k+1}^{-1}.\end{aligned} \tag{19a-e}$$

Replacing for $\bar{w}_k$ and $\bar{u}_k$ we obtain:

$$\begin{aligned}x_{k+1} = (Q_{k+1}^{-1} M_{k+1} A_k) x_k + \bar{B}_{1,k} (\gamma^{-1} R_{k+1}^{1/2}\\ (w_k - w_k^*)) + \bar{B}_{2,k} (\Phi_{k+1}^{1/2} (u_k - u_k^*)) + \bar{B}_{3,k} r_k + \bar{B}_{4,k} \theta_{k+1}.\end{aligned} \tag{20}$$

We consider the following Luenberger-type state observer:

$$\begin{aligned}\hat{x}_{k+1} = \hat{A}_k \hat{x}_k + K_{o,k} (y_k - C_{2,k} \hat{x}_k) + d_k, \quad \hat{x}_0 = 0,\\ \hat{z}_k = \hat{C}_{1,k} \hat{x}_k,\end{aligned} \tag{21}$$

where

$$d_k = \bar{B}_{2,k} \bar{u}_k + \bar{B}_{3,k} r_k + \bar{B}_{4,k} \theta_{k+1}.$$

Denoting $e_k = x_k - \hat{x}_k$ and using the latter we obtain:

$$e_{k+1} = (\hat{A}_k - K_{o,k} C_{2,k}) e_k + \hat{B}_{1,k} \hat{w}_k,$$

where we define

$$\hat{w}_k \stackrel{\Delta}{=} [\bar{w}_k^T \quad n_k^T]^T, \quad \hat{B}_{1,k} = [\bar{B}_{1,k} \ \ -K_{o,k} D_{21,k}].$$

Defining also $\xi_k = [x_k^T \quad e_k^T]^T$, $\bar{r}_k \stackrel{\Delta}{=} [r_k^T \quad \theta_{k+1}^T]^T$, we obtain

$$\begin{aligned}\xi_{k+1} = \tilde{A}_k \xi_k + \tilde{B}_{1,k} \hat{w}_k + \hat{B}_{3,k} \bar{r}_k,\\ \tilde{z}_k = \tilde{C}_{1,k} \xi_k,\end{aligned} \tag{22a-b}$$

where

$$\begin{aligned}&\tilde{A}_k = \begin{bmatrix} \tilde{A_{11,k}} & \bar{B}_{2,k} \hat{C}_{1,k} \\ 0 & \tilde{A_{22,k}} \end{bmatrix}, \tilde{B}_{1,k} = \begin{bmatrix} \bar{B}_{1,k} & 0 \\ \bar{B}_{1,k} & -K_{o,k} D_{21,k} \end{bmatrix},\\ &\hat{B}_{3,k} = \begin{bmatrix} \bar{B}_{3,k} & \bar{B}_{4,k} \\ 0 & 0 \end{bmatrix},\\ &\tilde{A_{11,k}} = \hat{A}_k - \bar{B}_{2,k} \hat{C}_{1,k},\ \tilde{A_{22,k}} = \hat{A}_k - K_{o,k} C_{2,k}\\ &\hat{C}_{1,k} = \Phi_{k+1}^{-1/2} B_{2,k}^T M_{k+1} A_k, \qquad \tilde{C}_{1,k} = [0\ \hat{C}_{1,k}].\end{aligned} \tag{23a-g}$$

Applying the results of Theorem 2 to the system of (22) we obtain the following Riccati-type equation:

$$\hat{Q}_k = \tilde{A}_k^T [\hat{Q}_{k+1}^{-1} - \gamma^{-2} \tilde{B}_{1,k} \tilde{B}_{1,k}^T]^{-1} \tilde{A}_k + \tilde{C}_k^T \tilde{C}_k, \tag{24}$$

where $\hat{Q}_0$ is given in (15b). Denoting $\hat{P}_k = \hat{Q}_k^{-1}$ and using Schur complement we obtain the DLMI of (15a). The latter DLMI is initiated with the initial condition of (15b) which corresponds to the case where a weighting $\gamma^2 \varepsilon^{-1} I_n$ is applied to $\hat{x}_0$ in order to force nature to select $\hat{x}_0 = 0$ in the corresponding differential game (see (Gershon and U. Shaked, 2005), Chapter 9, for details).

In the case where $\{r_k\}$ is measured on line, or with preview $h > 0$, we note that nature strategy which is not restricted to causality constraints, will be the same as in the case of full preview of $\{r_k\}$, meaning that $\bar{w}_k$ of (17) is unchanged. We obtain the following:

Lemma 2: *H_∞ Output-feedback Tracking with full preview of $\{r_k\}$: We obtain*

$$\bar{u}_k = \Phi_{k+1}^{1/2} u_k + \Phi_{k+1}^{-1/2} B_{2,k}^T M_{k+1} (B_{3,k} r_k + Q_{k+1}^{-1} [\theta_{k+1}]_+)$$

where we note that in this case $[\theta_{k+1}]_+ = \theta_{k+1}$. Solving (9) we obtain:

$$[\theta_{k+1}]_+ = \hat{\Phi}_{k+1} \theta_N + \textstyle\sum_{j=1}^{N-k-1} \Psi_{k+1,j} \bar{B}_{N-j} r_{N-j}$$

where

$$\begin{aligned}\hat{\Phi}_{k+1} &\stackrel{\Delta}{=} \bar{A}_{k+1}^T \bar{A}_{k+2}^T \ldots \bar{A}_{N-1}^T\\ \Psi_{k+1,j} &\stackrel{\Delta}{=} \left\{ \begin{array}{ll} \bar{A}_{k+1}^T \bar{A}_{k+2}^T \ldots \bar{A}_{N-j-1}^T & j < N-k-1 \\ I & j = N-k-1 \end{array} \right\}\end{aligned} \tag{25a-b}$$

Proof: Considering (9) and taking $k+1 = N$ we obtain:

$$\theta_{N-1} = \bar{A}_{N-1}^T \theta_N + \bar{B}_{N-1} r_{N-1},$$

where θ_N is given in (9). Similarly we obtain for $N-2$

$$\begin{aligned}\theta_{N-2} &= \bar{A}_{N-2}^T \theta_{N-1} + \bar{B}_{N-2} r_{N-2}\\ &= \bar{A}_{N-2}^T [\bar{A}_{N-1}^T \theta_N + \bar{B}_{N-1} r_{N-1}] + \bar{B}_{N-2} r_{N-2}\\ &= \bar{B}_{N-2} r_{N-2} + \bar{A}_{N-2}^T \bar{A}_{N-1}^T \theta_N + \bar{A}_{N-2}^T \bar{B}_{N-1} r_{N-1}.\end{aligned}$$

The above procedure is thus easily iterated to yield (25a,b). Taking, for example $N = 3$ one obtains from (9) the following equation for θ_1 :

$$\theta_1 = \bar{A}_1^T \bar{A}_2^T \theta_3 + \bar{A}_1^T \bar{B}_2 r_2 + \bar{B}_1 r_1.$$

The same result is recovered by taking $k = 0$ in (25a,b) where $j = 1, 2$.

Lemma 3: *H_∞ Output-feedback Tracking with no preview of $\{r_k\}$: In this case $[\theta_{k+1}]_+ = 0$ since at time k, r_i is known only for $i \le k$. We obtain:*

$$\bar{u}_k = \Phi_{k+1}^{1/2} u_k + \Phi_{k+1}^{-1/2} B_{2,k}^T M_{k+1} B_{3,k} r_k.$$

Lemma 4: H_∞ Output-feedback tracking with fixed-finite preview of $\{r_k\}$: In this case we obtain:

$$\bar{u}_k = \Phi_{k+1}^{1/2} u_k + \Phi_{k+1}^{-1/2} B_{2,k}^T M_{k+1} (B_{3,k} r_k + Q_{k+1}^{-1} [\theta_{k+1}]_+)$$

and

$$d_k = \bar{B}_{2,k} \bar{u} + \bar{B}_{3,k} r_k + \bar{B}_{4,k} [\theta_{k+1}]_+.$$

where $[\theta_{k+1}]_+$ satisfies :

$$[\theta_{k+1}]_+ = \left\{ \begin{array}{ll} \sum_{j=1}^{h} \bar{\Psi}_{k+1,j} \bar{B}_{k+h+1-j} r_{k+h+1-j} & k+h \le N-1 \\ \hat{\Phi}_{k+1} \theta_N + \sum_{j=1}^{h-1} \bar{\Psi}_{k+1,j} \bar{B}_{N-j} r_{N-j} & k+h = N \end{array} \right\}.$$

where $\bar{\Psi}_{k+1,j}$ is obtained from (25b) by replacing N by $k+h+1$.

4 STATIONARY OUTPUT-FEEDBACK TRACKING CONTROL

We treat the case where the matrices of the system in (2) and (3) are all time-invariant, N tends to infinity and the system of (2) is exponential l^2 stable. In this case, the solution $\tilde{Q}_k$ of (13a,b), if it exists, will tend to the mean square stabilizing solution of the following equation:

$$\begin{array}{ll}\tilde{Q} = A^T\bar{M}A + C^TC, & \gamma^2 I - B_1^T\tilde{Q}B_1, \\ \tilde{Q}(0) = \gamma^2 R^{-1} - \varepsilon I. & \end{array} \tag{26a,c}$$

We Introduce the following Lyapunov function:

$$V_k = \xi_k^T\tilde{Q}\xi_k, \text{ with } \tilde{Q} = \begin{bmatrix} Q & \alpha\hat{Q} \\ \alpha\hat{Q} & \hat{Q} \end{bmatrix}, \tag{27}$$

where ξ_k is the state vector of (22), Q and $\hat{Q}$ are $n\times n$ matrices and α ia a scalar. Considering (26) and the above result, we obtain the following theorem:

Theorem 3: *Consider the system of (2), (3) where the matrices $A, B_1, B_2, B_3, C_2, D_{21}, C, D_2$ and D_3 are all constant and $T \to \infty$. Given $\gamma > 0$, there exists a controller that minimizes $\max\tilde{J}_E$ of (4) if there exist $Q = Q^T \in \mathcal{R}^{n\times n}$, $\hat{Q} = \hat{Q}^T \in \mathcal{R}^{n\times n}$, $Y \in \mathcal{R}^{n\times q}$ and a tuning scalar parameter α that satisfy $\tilde{\Upsilon} > 0$ where $\tilde{\Upsilon}$ is the following LMI:*

$$\begin{bmatrix} Q & \alpha\hat{Q} & \Upsilon(1,3) & \Upsilon(1,4) & 0 & 0 & 0 \\ * & \hat{Q} & \Upsilon(2,3) & \tilde{\Upsilon}(2,4) & 0 & 0 & \hat{C}_1^T \\ * & * & Q & \alpha\hat{Q} & \Upsilon(3,5) & \Upsilon(3,6) & 0 \\ * & * & * & \hat{Q} & \Upsilon(4,5) & 0 & 0 \\ * & * & * & * & I & 0 & 0 \\ * & * & * & * & * & I & 0 \\ * & * & * & * & * & * & I \end{bmatrix} \tag{28}$$

where

$$\begin{array}{l} Y \triangleq K_o^T\hat{Q}, \Upsilon(1,3) = \hat{A}^TQ - \hat{C}_1^T\bar{B}_2^TQ, \\ \Upsilon(1,4) = (\hat{A}^T - \hat{C}_1^T\bar{B}_2^T)\alpha\hat{Q}, \\ \Upsilon(2,3) = \hat{C}_1^T\bar{B}_2^TQ + \alpha\hat{A}^T\hat{Q} - \alpha C_2^TY, \\ \Upsilon(2,4) = (\alpha\hat{C}_1^T\bar{B}_2^T + \hat{A}^T)\hat{Q} - C_2^TY, \\ \Upsilon(3,5) = \gamma^{-1}(Q + \alpha\hat{Q})\bar{B}_1,\ \ \Upsilon(3,6) = -\gamma^{-1}\alpha Y^TD_{21}, \\ \Upsilon(4,5) = \gamma^{-1}\hat{Q}\bar{B}_1(1+\alpha). \end{array}$$

Proof: The proof outline for the above stationary case resembles the one of the finite-horizon case. Considering the stationary version of (2), (3) the stationary state-feedback control problem is solved to obtain the optimal stationary strategies of both $w_{s,k}^*$ and $u_{s,k}^*$ (Gershon and U. Shaked, 2010). Thus we obtain:

$$\begin{array}{c} w_{s,k}^* = R_{k+1}^{-1}B_1^T[\theta_{k+1} + P(Ax_k + B_2u_k + B_3r_k)], \\ u_{s,k}^* = -\Phi_k^{-1}\{B_2^T\hat{M}[Ax_k + B_3r_k + P\theta_{k+1}^c]\}, \\ \hat{M} \triangleq P^{-1}[I - \gamma^{-2}B_1B_1^TP^{-1}]^{-1}, \end{array}$$

where P^{-1} is the stationry version of the solution Q_k of (5).

Using the above optimal strategies we transform the problem to an estimation one, thus arriving to the stationary counterpart of the augmented system of (22). Applying the result of (26) to the latter system the algebraic counterpart of (24) is obtained which, similarly to the finite-horizon horizon case, becomes the following stationary version of (15):

$$\begin{bmatrix} \tilde{P}^{-1} & \tilde{A}^T & 0 & \tilde{C}_1^T \\ \tilde{A} & \tilde{P} & \gamma^{-1}\tilde{B}_1 & 0 \\ 0 & \gamma^{-1}\tilde{B}_1^T & I & 0 \\ \tilde{C}_1 & 0 & 0 & I \end{bmatrix} \geq 0. \tag{29}$$

Multiplying the above LMI by $diag\{I_n, \tilde{P}^{-1}, I_{2p}, I_l\}$ from the left and the right, denoting $\tilde{Q} = \tilde{P}^{-1}$ and using the matrix partition of $\tilde{Q}$ of (27), the result of (28) is obtained.

5 CONCLUSIONS

In this paper we solve the problem of deterministic output-feedback tracking control with preview. Unlike the state-feedback case the solution is not obtained by applying a game theory approach, (where a saddle-point tracking strategy is derived) but rather as a min-max optimization problem. The output-feedback tracking problem is solved by applying a special form of the BRL to a filtering problem which is formulated once the state-feedback solution is obtained. The parameters of the *a priori* type state observer used in our proof are recovered by iterative solution of the DLMI of Theorem 2 which can be easily implemented. The result of the finite horizon case were extended to the stationary case, where a simple LMI condition is formulated for all the three preview patterns. The theory is demonstrated by a numerical example.

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Appendix

Proof Sketch of Theorem 1: The proof of Theorem 1 is based on the proof of Lemma 1 which concerns the state-feedback control problem (for a detailed proof see (Gershon and U. Shaked, 2010), (Gershon and U. Shaked, 2005)). We first bring in Part I, a proof sketch of Lemma 1 which is also needed for the solution of the output-feedback control. We then bring, in Part II, the derivation of the sufficiency part of Theorem 1, which is derived from Lemma 1, and the necessity part of the proof of Theorem 1.

Part I: The proof of Lemma 1 follows the standard line of applying a Lyapunov-type quadratic function in order to comply with the index of performance. This is usually done by using two successive completing to squares operations however, since the reference signal of r_k is introduced in the dynamics of (2a), we apply a third completing to squares operation with the aid of the fictitious signal of θ_{k+1}. This latter signal finally affects the controller design through it's causal part $[\theta_{k+1}]_+$ (for a detailed proof see (Gershon and U. Shaked, 2010), (Gershon and U. Shaked, 2005)).

Part II: The sufficient part of the proof of Theorem 1 stems from the above proof of Lemma 1 where $B_{2,k} = 0$, $D_{2,k} = 0$. Analogously to the proof of Lemma 1, we obtain the following: $J_B(r_k, u_k, w_k, x_0) =$

$$-\sum_{k=0}^{N-1} ||\hat{w}_k - R_{k+1}^{-1} B_{1,k}^T \tilde{\theta}_{k+1}||^2_{R_{k+1}}$$

$$-\gamma^2 ||x_0 - (\gamma^2 R^{-1} - \tilde{Q}_0)^{-1} \tilde{\theta}_0||^2_{R^{-1} - \gamma^{-2}\tilde{Q}_0}.$$

where we replace θ_{k+1}, Q_k by $\tilde{\theta}_{k+1}$ and $\tilde{Q}_k$, respectively and where

$$\tilde{P}_0 = [R^{-1} - \gamma^{-2}\tilde{Q}_0]^{-1},\ x_0 = \gamma^{-2}\tilde{P}_0\tilde{\theta}_0 = [\gamma^2 R^{-1} - \tilde{Q}_0]^{-1}\tilde{\theta}_0$$

The necessity follows from the fact that for $r_k \equiv 0$, one gets $\tilde{J}(r, \varepsilon) = 0$ (noting that in this case $\tilde{\theta}_k \equiv 0$ in (14) and therefore the last 3 terms in $\tilde{J}(r, \varepsilon)$ of Theorem 1 are set to zero) and $J_B < 0$. Thus the existence of $\tilde{Q} > 0$ that solves (13) is the necessary condition in the BRL.