The Shape of an Image
A Study of Mapper on Images

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Abstract: We study the topological construction called *Mapper* in the context of simply connected domains, in particular on images. The Mapper construction can be considered as a generalization for contour, split, and joint trees on simply connected domains. A contour tree on an image domain assumes the height function to be a piecewise linear Morse function. This is a rather restrictive class of functions and does not allow us to explore the topology for most real world images. The Mapper construction avoids this limitation by assuming only continuity on the height function allowing this construction to robustly deal with a significantly larger set of images. We provide a customized construction for Mapper on images, give a fast algorithm to compute it, and show how to simplify the Mapper structure in this case. Finally, we provide a simple procedure that guarantees the equivalence of Mapper to contour, join, and split trees on a simply connected domain.

1 INTRODUCTION

Recently, the study of data has benefited from the introduction of topological concepts (Carlsson, 2009; Carlsson et al., 2008; Carlsson et al., 2005; Collins et al., 2004), in a process known as Topological Data Analysis (TDA).

One of the most successful topological tools for shape analysis is the *contour tree* (Boyell and Ruston, 1963). The contour tree of a scalar function, defined on a simply connected domain, can be thought of as an efficient topological summary of that domain. This structure is obtained by encoding the evolution of the connectivity of the level sets induced by a scalar function defined on the domain. Contour trees are of fundamental importance in computational topology, geometric processing, image processing and computer graphics.

Contour trees are particularly useful for processing massive data. Contour trees, and their more general version *Reeb graphs* (Reeb, 1946), have been used in numerous applications including shape understanding (Attene et al., 2003), visualization of isosurfaces (Bajaj et al., 1997), contour indexing (Boyell and Ruston, 1963), image processing (Kweon and Kanade, 1994), data simplification (Carr et al., 2004; Rosen et al., 2017b), and many other applications. Contour tree algorithms can be found in many papers such as (Takahashi et al., 2009; Rosen et al., 2017a) and Reeb graphs algorithms are studied in (Shinagawa and Kunii, 1991; Doraiswamy and Natarajan, 2009).

Singh et al. proposed a method to understand the shape of data using a topology-inspired construction called *Mapper* (Singh et al., 2007). Since then, Mapper has became one of the most popular tools used in TDA. It has been applied successfully for various data related problems (Lum et al., 2013; Nicolau et al., 2011) and studied from multiple points of view (Carrière and Oudot, 2015; Dey et al., 2017).

The construction of Mapper is closely related to Reeb graphs and contour trees (Singh et al., 2007). Indeed this construction can be considered as a generalization of Reeb graph under some technical conditions (Munch and Wang, 2015). The relation between Reeb graph and Mapper has recently been made precise in (Carrière and Oudot, 2015).

The true power of Mapper lies in its general description in terms of topological spaces and maps on them. This abstract version of the construction is usually called *topological Mapper*. In the original work where Mapper was introduced (Singh et al., 2007), Mapper was applied to study the shape of point clouds. This version of Mapper is now referred to as *statistical Mapper* (Stovner, 2012). While topological Mapper allows one to introduce the main ideas of Mapper in general terms, statistical Mapper deals with aspects

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related to point clouds, such as clustering and noise. Similar technical aspects arise when trying to apply Mappers on other domains, such as images.

The purpose of this article is to study Mapper on specific domains, namely simply connected domains and apply this study to images. While the focus of this article is Mapper on images, we state the results whenever possible on a general simply connected domain.

1.1 Contribution

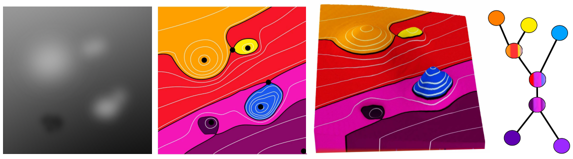
Mapper construction on images operates on a height function defined on the image domain. The height function can be a color channel or luminance of the input image itself or the gradient magnitude of the image, which is typically a compact and connected region in $\mathbb{R}^2$. After discussing the topological and statistical versions of Mapper construction on image domains, we relate this construction to the contour tree algorithm that enables Mapper to realize contour, merge, and split trees.

The method we propose here has multiple advantages. Beside being theoretically justified, the construction of Mapper is flexible and applicable to continuous scalar function defined on a simply connected domain in any dimension. Contour tree algorithms on simply connected domains assume the height function on the domain to be piecewise linear Morse function. While this class of function is useful for a wide variety of applications, it is rather restrictive for images and it does not allow us to explore the topology for most real world images without heavy preprocessing of the image height function. Mapper construction avoids this limitation by assuming only continuity on the height function allowing this construction to robustly deal with a significantly larger class of images. Moreover, Mapper naturally gives a multi-resolution hierarchical understanding of topology of the underlying domain.

The approach we take to Mapper here is geared for simply connected domains and, in particular, for images. Using the properties of such domains, we provide a fast construction algorithm. Finally, we provide a simple algorithm that guarantees the equivalence of Mapper construction to contour, join, and split trees on a simply connected domain.

2 PRELIMINARIES AND MOTIVATION

As mentioned in the introduction, Mapper is closely related to the contour tree. This related structure motivates the construction of Mapper.



(a) (b) (c) (d)

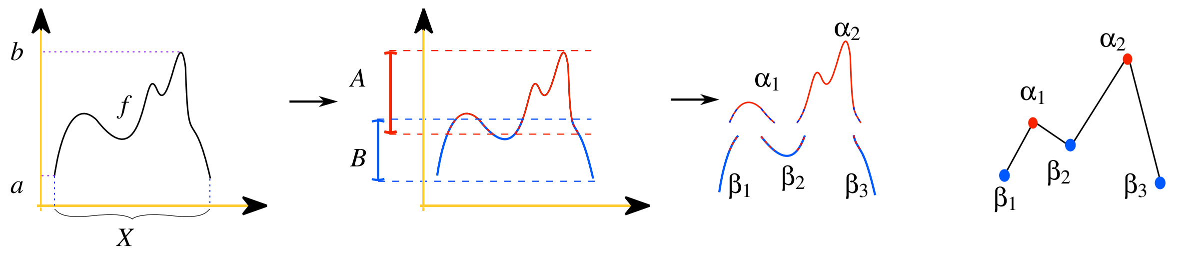
Figure 1: (a) Scalar function is segmented into (b) topological regions by converting that scalar field into a (c) landscape, using the intensity value for height. The connection of those regions can be converted into a contour tree (d) that describes the topology.

Contour Trees. The contour tree of a scalar field, defined on a simply connected domain, tracks the evolution of contours in that field and stores this information in a tree structure. Each node in the tree represents a critical point where contours appear, disappear, merge, or split. Each edge corresponds to adjacent and topologically equivalent contours. In essence, the contour tree forms a topological skeleton that connects critical points (i.e. local minima, maxima, and saddle points). Figure 1 shows an example of the contour tree of a scalar field defined on a 2d domain.

In practice, we usually want to compute contour trees on a piecewise linear Morse function defined on a simplicial complex. The mathematical framework specified for contour tree does not apply directly on such domains. The difficulty rises when one tries to extract isosurfaces for a scalar value as the preimages of an scalar values may not be an isosurface (Szymczak, 2005). Nonetheless several contour tree algorithms have been proposed, but they all depend some method of isosurface extraction. Hence two different methods of isosurface extraction might lead to two different contour trees.

Mapper. The construction of Mapper avoids the problem of dealing with isosurfaces all together by focusing on portions of the range of the scalar field. To illustrate this, consider the simple scalar function $f : X \longrightarrow [a,b]$ example given in Figure 2. Cover the range $[a,b]$ by two overlapping intervals $A := (a-\varepsilon, c+\varepsilon)$ and $B =: (c-\varepsilon, b+\varepsilon)$ such that $c \in [a,b]$ and $\varepsilon > 0$. Note the interval A and B cover the interval $[a,b]$ in the sense : $[a,b] \subset A \cup B$.

Now, consider the inverse images $f^{-1}(A)$ and $f^{-1}(B)$. Figure 2 (c) illustrates that $f^{-1}(A)$ consists of two connected components α_1 and α_2 and $f^{-1}(B)$ consists of a three connected components β_1, β_2 and β_3. Moreover, there are some overlaps between these connected components. Namely, the intersections $\alpha_1 \cap \beta_1$, $\alpha_1 \cap \beta_2$, $\alpha_1 \cap \beta_2$ and $\alpha_2 \cap \beta_3$ are non-empty. We record the information of the connected components and their



(a) (b) (c) (d)

Figure 2: The construction of Mapper on a 1d function. (a) A scalar function $f : X \longrightarrow [a,b]$. (b) The range $[a,b]$ is covered by the two intervals A, B. (c) This gives a decomposition of the domain X. (d) Mapper is constructed.

non-empty overlap by a graph structure. The nodes of this graph represent the connected components and the edges represent the non-empty intersection between these components. The Mapper construction is the graph associated to the function f and the cover A, B in this manner.

Mapper's Relationship to Contour Trees. One can notice that this graph is very related to the contour tree. Both the contour tree and Mapper essentially track the same topological information in the scalar field, but the way this information is encoded in each one of them is different. The nodes of the contour tree of a scalar field are represented by the critical points the field and the edges represent the regions in the domain where the are no topological change in the contours. On the other hand the nodes in Mapper represent connected regions in the domain and the edges represent the connection between two different connected components. Hence, the combination stores when a topological change occur to the contour.

3 TOPOLOGICAL MAPPER

We now give the general definition of Mapper for a continuous scalar function defined on a simply connected domain.

Let X be a simply connected domain in $\mathbb{R}^n$. We will assume that X is compact and connected. A *cover* of X is a collection of open sets $\mathcal{U} = \{U_i\}_{i\in I}$ such that $X \subset \cup_{i\in I} U_i$. I here is any indexing set. The compactness condition implies that we can always find a finite cover for X. In the case of an image, X is a compact simply connected subset of $\mathbb{R}^2$. The 1-**nerve** of $N_1(\mathcal{U}, X)$ of X induced by the covering of $\mathcal{U}$ is a graph with nodes are represented by the elements of $\mathcal{U}$ and edges represented by the pairs A, B of $\mathcal{U}$ such that $A \cap B \neq \emptyset$. The nerve of a space X can be thought of as a topological skeleton of the underlying space. The main idea of Mapper lies in the way of constructing this cover using the *range* of a function f defined on X. More precisely, a continuous scalar function $f : X \longrightarrow [a,b]$ on X and a cover for the range of f give rise to a natural cover of X in the following way. A cover for an interval $[a,b]$ is finite collection of open intervals $\mathcal{U} = \{(a_1,b_1),...,(a_n,b_n)\}$ that cover $[a,b]$, i.e. $[a,b] \subset \cup_{i=1}^{n}(a_i,b_i)$. Now take the inverse images of each open set in $\mathcal{U}$ under the function f. The result is $\mathcal{U}(f) := \{f^{-1}((a_1,b_1)),...,f^{-1}((a_n,b_n))\}$ is an open cover for the space X. The open cover $\mathcal{U}(f)$ can now be used to obtain the 1-nerve graph $M(X, f, \mathcal{U}) := N_1(X, \mathcal{U}(f))$. The Mapper construction is by definition the graph $M(X, f, \mathcal{U})$.

3.1 Cover Resolution

For a fixed function f the graph $M(X, f, \mathcal{U})$ depends on the choice of the cover $\mathcal{U}$ of the interval $[a,b]$. This idea of Mapper resolution can be made precise via the notion of *cover refinement* (Munkres, 2000). Let X be a space and let $\mathcal{A}$ and $\mathcal{B}$ be two covers of X. The cover $\mathcal{B}$ is a *refinement* a cover $\mathcal{A}$ if for each element of B of $\mathcal{B}$ there is at least one element A of $\mathcal{A}$ such that $B \subseteq A$. If $\mathcal{B}$ is a refinement a cover $\mathcal{A}$, there is a embedding of the graph $N_1(X, \mathcal{A})$ inside the graph $N_1(X, \mathcal{B})$. That is there is one-to-one function ϕ that maps between the vertices sets $N_1(X, \mathcal{A})$ and $N_1(X, \mathcal{B})$ together with an assignment that assigns to every edge $e = (u, v)$ in $N_1(X, \mathcal{A})$ a path in $N_1(X, \mathcal{B})$ between $\phi(u)$ and $\phi(v)$. See (Munkres, 2000). Figure 8 show examples 4 nested sequences of cover refinement along with their corresponding graphs. This simple, effective, way to give a multi-resolution Mapper is one of its main advantages over contour tree.

4 TOPOLOGICAL MAPPER ON IMAGES

In this section, we discuss the details of *topological Mapper on images* that will be used in our algo-

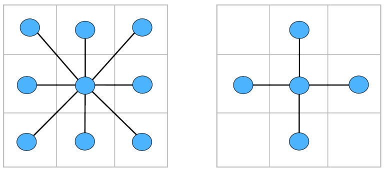


Figure 3: The two types of pixel adjacency relation.

rithm for *the statistical Mapper on images* discussed in section 5. We assume that $f : X \longrightarrow [a,b]$ is continuous height function defined on the image domain $X \subset \mathbb{R}^2$.

4.1 Choosing the Cover

While the choice of cover for the Mapper construction is flexible, certain covers that give rise to a non-desirable tree structure. We describe an effective way to construct the cover for the domain that will help in computing Mapper efficiently.

Start by splitting the interval $[a,b]$ into n subintervals $[c_1,c_2],[c_2,c_3],...,[c_{n-1},c_n]$ such that $c_1 = a$ and $c_n = b$. Choose $\varepsilon > 0$ and construct a cover $\mathcal{U}(\varepsilon,n) = \{U_i = (c_i - \varepsilon, c_{i+1} - \varepsilon)\}_{i=1}^{n-1}$ for the interval $[a,b]$. We want to choose ε so that only adjacent intervals intersect. The choice of ε should satisfies the following conditions:

1. The intersection $U_i \cap U_j = \emptyset$ unless $j \in \{i-1,i,i+1\}$ for $2 \leq i,j \leq n-2$.
2. $U_1 \cap U_j = \emptyset$ unless $j \in \{1,2\}$ and finally $U_{n-1} \cap U_j = \emptyset$ unless $j \in \{n-2,n-1\}$.

This choice of ε ensures that only adjacent intervals intersect with each other. We denote $\mathcal{U}_{odd}$ to the subset of $\mathcal{U}$ consisting of intervals with odd indices. Similarly we define $\mathcal{U}_{even}$ to be the collection of open sets $U_i \in \mathcal{U}$ such that index i is even. These cover choices minimize the number of overlaps between the cover elements.

4.2 Determining the Nodes

A node in Mapper is a connected component of $f^{-1}((c,d))$, where (c,d) is an open interval in the cover $\mathcal{U}$ of the range of f. Given a range (c,d), in the case of an image X, we want to find the those pixels in X whose pixel value lie in (c,d). Given a region R in an image X consisting of a collection of pixels whose pixel value lie within the range (a,b), we want to determine the connected components in the R. Here one needs to specify what exactly is meant by a connected component in this context. The image X induces a graph structure with nodes being the pixels and the edges are determined by the local pixel adjacency relation. There are two common types of pixel adjacency relations shown in Figure 3.

Using the graph on an image with either one of the pixel adjacency relation conventions, we can now consider the connected components of subgraph consists of the pixels in a region R. A *walk* on a graph G is a sequence of vertices and edges $(v_0,e_0,v_1,e_1,\cdots,e_{l-1},v_l)$ such that $e_i = [v_{i-1},v_i] \in E(G)$. A graph is said to be *connected* if there is a walk between any two vertices. A connected component in a graph is a maximal connected subgraph. Finding connected components of a graph is well-studied in graph theory and it can be found by in linear time using either breadth-first search or depth-first search (Hopcroft and Tarjan, 1971).

4.3 Determining the Edges

An edge in Mapper is created whenever two connected components have non-trivial intersection. The cover that we described for the range $[a,b]$ in section 4.1 was chosen to minimize the number of sets we check for intersection. Namely the condition that we impose on the cover of $[a,b]$ ensures that only adjacent open interval overlap. In other words, if U_i and U_j are two open sets in the cover of $\mathcal{U}(\varepsilon,n)$ of the interval $[a,b]$, then by the choice of the cover specified in section 4.1, we check if the connected components of $f^{-1}(U_i)$ and $f^{-1}(U_j)$ intersect only when we know that U_i and U_j are adjacent to each other. This significantly reduces the number of set intersections checked.

5 ALGORITHM

The creation of the Mapper graph is done in three stages. First, all pixels in the image are labeled by the cover they map to. Pixels with the same label are then grouped by searching for all connected components with the same label. This provides the nodes for the Mapper graph. Next, the connected component regions are scanned for overlaps. Every pair of overlapping regions in the image corresponds to an edge connecting the nodes in the Mapper graph. Finally, the third stage simplifies the Mapper graph by removing nodes with two valencies.

5.1 Node Finding

In our approach, pixel labeling is done using a pair of lookup tables, one for the even cover $\mathcal{U}_{even}$ and one for the odd cover $\mathcal{U}_{odd}$. When a lookup table maps outside of its set of covers, it returns a value that signifies that

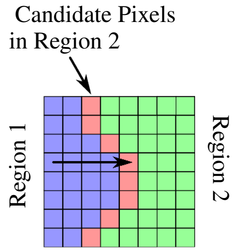


Figure 4: Line scanning for candidate pixels.

the pixel does not map to a cover in this table. This even/odd separation has an important benefit that when one lookup table is applied to the image, none of the resulting regions overlap. This means that instead of processing the image for each covering one-by-one, the image only needs to be processed twice, once for $\mathcal{U}_{even}$ and once for $\mathcal{U}_{odd}$, to find all the connected regions.

Breadth-first search (BFS) is used to find connected regions once the pixels have been labeled. By taking advantage of the queue structure of BFS, every pixel in a connected region can be traversed before moving onto the next region as long as only the top of the queue is being modified. This continuity of the search allows us to add pixels in other regions to the same queue, thus allowing processing many regions with one search. As a region is traversed, pixels are marked with an identification unique to that region. In our implementation, this identification is created using the position of the first pixel in the region touched during the search.

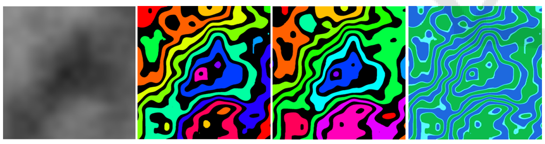


Figure 5: Region search is done twice, once for even and once for odd covers. Here, each region identified during the search is given a unique color. If a pixel is not found to map to a cover during the search the, pixel is not colored (these are the pixels colored in black in the middle two images). This shows how splitting the covers gives a pair of images which do not contain overlapping regions. Regions in one image will, however, overlap with regions in the other image, as shown in the image on the bottom right.

Our approach initializes the BFS queue with *candidate pixels* which are pixels found by scanning each row in the image from left to right until a pixel, which differs in label from the previous pixel is found (see Figure 4). This gives the pixels that start a region along every line in the image. Since a region needs at least one pixel to be in the queue at the start of the search, the use of candidate pixels ensures each region in the image will be traversed, while reducing the number of pixels in the queue at the start of the search.

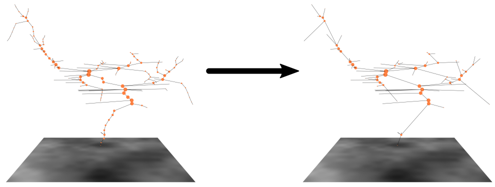


Figure 6: Simplification of the mapper graph.

At the end of the search, every pixel will have an associated identification that represents the connected component region it belongs to. Finally, these regions define the nodes in the Mapper graph. See Figure 5 for illustration of the process of node finding done on an example image.

5.2 Edge Finding

Once the regions in the image have been identified for both the even and odd covers, overlaps between regions need to be found. A naive approach would be to create a set of pixel locations for every region in both sets of coverings, and check whether pairs of sets are disjoint. This type of approach, however, requires every pair of sets to be tested for disjointness, making it inefficient.

To determine region overlap, we take advantage of the candidate pixels found during node finding, see Figure 4. Since these pixels signifies the entrance of a region with a different labeling, this means that there are two different regions from the two opposing covers overlap. Notice that this method takes advantage of the way we construct the cover in section 4.1.

5.3 Graph Simplification

The resulting Mapper graph can contain thousands of nodes. Many of these nodes can be removed as they do not indicate topological events. In the Mapper graph, a node with valency equal to 2 corresponds to a region where no topological event occur. In other words, such a node is not a merge, split, creation, or termination of a region. These nodes are analogous to regular points in the contour tree. Hence, these nodes can be safely removed to obtain a simplified graph, such as in Figure 6.

6 REALIZING THE CONTOUR TREE

The Mapper construction can be used to realize the contour tree. Here we give a choice of covering that guarantees that Mapper gives rise to all the topological information encoded in the contour tree. We need to assume that the given function is a piecewise linear Morse function $f : X \longrightarrow [a,b]$ on a simply connected domain X. The assumption of piecewise linear Morse is necessary in order to work with a contour tree. For precise definitions related to Morse theory on simplicial complex see (Pascucci et al., 2004).

Recall that every node in the contour tree corresponds to a critical point. The critical point of a function signifies a topological change in the space X with respect the scalar function. Moreover, if t_1 and t_2 are two consecutive critical values f then for any two values $c_1, c_2 \in (t_1, t_2)$ the number of connected components of both $f^{-1}(c_1)$ and $f^{-1}(c_2)$ are the same. In other words, a topological change that occur to the space only when as we sweep though a critical value. Hence, in order for Mapper to give us the information encoded in the contour tree, it is sufficient to make a choice of the covering on $[a,b]$, so that we store the following information:

1. The number of connected components between every two consecutive critical values of f.
2. The way the connected components merge, split, appear, and disappear when passing through a critical point.

The following procedure gives a choice of covering for $[a,b]$ that satisfies the previous two criteria:

1. Let $t_1, t_2, ..., t_n$ be the critical values for f ordered in an ascending order. Let $p_1, p_2, ..., p_n$ be the corresponding critical points of f.
2. For each $1 \leq i \leq n-1$, we choose four numbers a_i b_i, c_i and c_i in the interval $[t_i, t_{i+1}]$ such that $a_i < d_i < c_i < b_i$.
3. Let $c_0 = a - \varepsilon$ and let $d_n = b + \varepsilon$ for some $\varepsilon > 0$.
4. Let $\mathcal{U}$ be the covering of $[a,b]$ consisting of the intervals $(a_1, b_1), ..., (a_{n-1}, b_{n-1})$ as well as (c_0, d_1),(c_1, d_2),...,(c_{n-1}, d_n).

Notice that the Mapper construction obtained using the covering $\mathcal{U}$ stores all the topological information encoded in the function f. Hence, any further refinement of the covering $\mathcal{U}$ will not produce any further details in the Mapper construction as far as the topology of the original domain is concerned. In other words, the above construction gives the highest Mapper resolution that one could obtain on a piecewise linear Morse function.

7 JOIN AND SPLIT TREES

The previous sections describe how Mapper can be used to obtain a contour tree. The Mapper construction is general and can be used to realize other structures such as the join and split trees (Carr et al., 2003). The only change one needs to make to the previous setup is making a different choice for the shape of the open interval that makes covering of the range. These choices will be justified after we illustrate the basic ideas of join/split trees.

For a continuous scalar function $f : X \longrightarrow [a,b]$ defined on a simply connected domain X the split tree $ST(f,X)$ of f on X tracks the topological changes occur of the set $\{p \in X | f(p) \geq c\}$ of a value c as this value is swept from ∞ to $-\infty$. Similarly, the join tree $JT(f,X)$ of f on X tracks the topological changes occur to the topology of the set $\{p \in X | f(p) \leq c\}$ as the value c goes from $-\infty$ to ∞. The Mapper construction can be used to compute both split and join trees on any simply connected domain. The only thing that must be chosen to obtain these structures is the shape of the open intervals for the covering $\mathcal{U}$ of range $[a,b]$.

The choice of covering for a join tree should be of a collection of open intervals of the form $(-\infty, c)$ that covers the interval $[a,b]$. That is, the cover must be a finite set $\{(-\infty, c_1), ..., (-\infty, c_n)\}$ such that $[a,b] \subset \cup_{i=1}^{n}(-\infty, c_i)$. As the values to c_i increases, only merging events occur in the set $\{p \in X | f(p) \geq c\}$, which is reflected in the resulting Mapper graph. On the other hand, the choice of covering needed to construct the split tree is a collection of open intervals of the form (c, ∞).

8 RESULTS

To demonstrate how our work performs we run a few experiments on some images with various complexities. Figure 7 shows the illustrative examples on some images. The height functions chosen on these images are the input images themselves. The figure shows the images along with the Mapper graph on drawn on the top of them. The vertical position of the node is chosen to be the average of the pixel values of the region that corresponds to that node. On the other hand the (x,y) position of a node is the center mass of the pixel positions of the pixels in the region. The size of the node is proportional to the number of pixels in the corresponding connected component.

In Figure 8 we show how multiple refinement of covering give rise to a hierarchy of Mapper on the same image. The graphs in the figure, shown from left

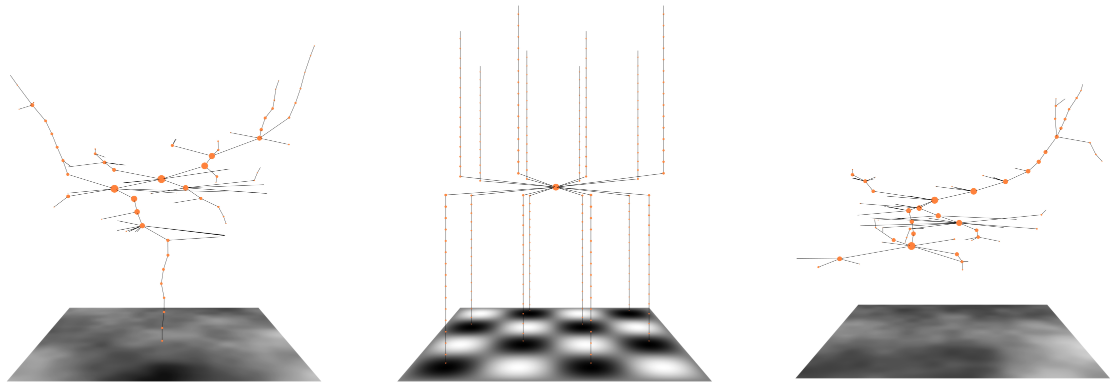


Figure 7: Examples of Mapper on images using pixel values as the height function. The range of these images was covered by a cover of 32 open sets.

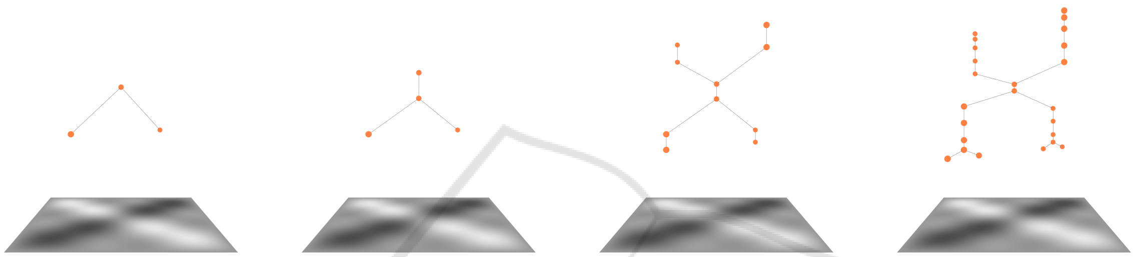


Figure 8: Multi-resolution of Mapper using different covering resolutions. The graphs are constructed from left to right by using $2,4,8,16$ slices of the range covering.

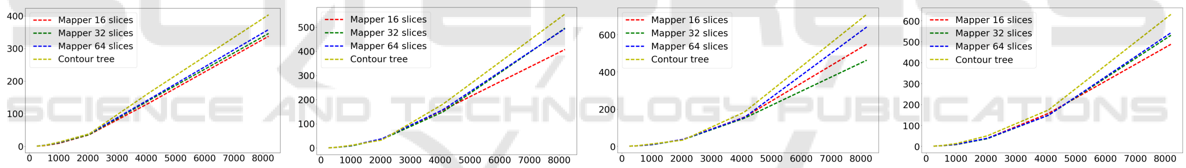


Figure 9: Performance analysis of Mapper in comparison with contour tree on procedurally and non-procedural images. Each image was done with the resolutions : $256^2, 512^2, 1024^2, 2084^2, 4096^2$, and 8192^2. Each resolution was tested against Mapper and contour tree. Mapper was tested using 16,32, and 64 cover slices. The x-axis represents the square root of the resolution of the image. The y-axis represents the running time in milliseconds.

to right, are generated by using $2,4,8,16$ slices of the covering. The figure shows immediately the effect of cover refinement of the resolution and level of details.

8.1 Running Time

We tested our algorithm on a 3.7 GHs AMD with a 16 GB of memory. We implemented the results shown in Figures in Java and tested them on the Windows platform. We tested the running time of the algorithm against two parameters : changing number of slices in the covers and increasing the resolution of the image. We performed the tests on procedural and non-procedural images. See Figure 9 for the performance analysis. We also ran a comparison between the Mapper algorithm we present here and a contour tree algorithm. The contour tree algorithm we used is a version of algorithm given in (Carr et al., 2003).

While both contour tree and Mapper give almost identical performance for images with small resolutions, Mapper outperforms contour tree as we increase the resolution of on the image. See Figure 9.

One can notice here that the performance computation time of Mapper increase linearly with the increase of number of slices in the cover. Moreover, observe in Figure 9 that Mapper computes faster than contour tree even when we choose to calculate it on the highest resolution.

9 LIMITATIONS

Mapper assumes the underlying height function to be continuous. If the provided function is not continuous

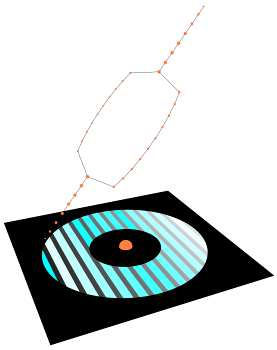


Figure 10: Mapper on an image with a discontinuous height function is not guaranteed to produce a tree.

Mapper still produces a graph, but it is no longer guaranteed that this graph is a tree. Figure 10 an example of an image whose height function is discontinuous.

Depending on the application at a hand this limitation of Mapper could potentially be used for image understanding. As illustrated in Figure 10 the graph captures the ”shape” in the underlying image.

10 CONCLUSIONS

We introduce the study of Mapper on simply connected domains, in particular 2d images. On simply connected domains, the Mapper construction generalizes contour, split, and join trees. Our work here uses the properties of the image domain to obtain a customized algorithm for Mapper on images, which we show to have advantages in making the graph calculation more efficient. The algorithmic aspects to deal with additional domains have also been addressed in this work. We plan to investigate such directions more in the future.

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