Matrix Operator of Musielak-φ Function Sequence Space

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Keywords: Matrix Operator, Musielak-φ Function, Banach Space.

Abstract: Let X be a Banach space and $\Phi = \{\varphi_i\}$ be a sequence of φ-function called Musielak-φ-function. In this work, we introduce a vector valued sequence space generated by Musielak-φ function, $\ell^{\exists}(X,\Phi)$, and study the matrix operator from the space $\ell_1(X)$ into the space $\ell^{\exists}(X,\Phi)$.

1 INTRODUCTION

Let $X = (X, \|\cdot\|_X)$ be a Banach space. We denote space $\Omega(X)$ as a colection of all X-valued sequences. For every natural numbers i, a sequence $x = (x(i)) \in \Omega(X)$ means $x(i)$ in Banach space X. Any linear subspace $E \subset \Omega(X)$ is called *X-valued sequence space*.

A function φ that defined from $\mathbb{R}$ into $\mathbb{R}^+ \cup \{0\}$ is called a *φ-function* if φ is even, continuous, vanishing at zero, and increasing (Rao and Ren, 2002). For any φ-function and for every real number x, if there exists a constant $K > 0$ such that $\varphi(2x) \leq K\varphi(x)$, then φ is called satisfy Δ_2-*condition*. For a φ-function, φ, satisfied Δ_2-*condition*, the following space, denoted by $\ell^{\exists}(\varphi)$ is a generalization of Orlicz sequence space (Kolk, 2015) i. e.

$$\begin{aligned}\ell^{\exists}_{\varphi} &= \left\{x = (x_i) : x_i \in \mathbb{R} \text{ and } \varphi\left(\frac{x}{\rho}\right) \in \ell_1 \right. \\ &\qquad \left. \text{for some } \rho > 0\right\} \\ &= \left\{x = (x_i) : x_i \in \mathbb{R} \text{ and } \sum_{i=1}^{\infty} \varphi\left(\frac{x_i}{\rho}\right) < \infty \right. \\ &\qquad \left. \text{for some } \rho > 0\right\}.\end{aligned}$$

respected to the following norm

$$||x||_{\varphi} = \inf\left\{\rho > 0 : \varphi\left(\frac{x}{\rho}\right) \leq 1\right\}.$$

The space $\ell^{\exists}(\varphi)$ is a Banach space.

An X-valued sequence space defined by arithmetic mean of φ-function has been introduced (Gultom and Herawati, 2018). They studied some topological properties using paranorm and inclusion relations of this space. For E be a Riesz space (Herawati et al, 2016) introduced E-valued sequence space defined by an order φ-function and proved that spaces are ideal Banach lattices using lattice norm. In (Mursaleen, 2013) introduced some sequence spaces defined by a Musielak-Orlicz function and studied some topological properties respected to n-norm and proved some inclusion relations between these spaces. Using Musielak-Orlicz function to generated sequence spaces equipped with the Luxemburg norm, packing constant of these space has been studied (Hudzik et al, 1994). (Suantai, 2003) considered the characterization problem of infinite real matrices operators $\ell(X,p)$ for $p = (p_i)$ is a bounded sequence with $0 \leq p_i \leq 1$ for all natural number i and two others sequence space into the Orlicz sequence space, ℓ_M, for M is an Orlicz function.

A matrix operator is an operator from any sequence space X to another sequence space Y by using infinite real matrix $A = (a_{nk})$. i.e. an operator $A : X \to Y$ with

$$Ax = (A_n(x)) \in Y, \text{ for any } x \in X$$

for every $x \in X$ and for every natural number n

$$A_n(x) = \sum_{k\geq 1} a_{nk}x_k < \infty.$$

Collection of matrix operator $A : X \to Y$ denoted by (X,Y). The sequence $e^{(k)}$ is a sequence which only non zero-term is 1 in k^{th} entry for every natural number k.

Let $\ell^{\exists}(X,\Phi)$ for X and $\Phi = \{\varphi_i\}$ be a Banach space and a Musielak-φ function, respectively, studied with luxemburg norm (Ofie and Herawati, 2018),

Ofie, M. and Herawati, E.
Matrix Operator of Musielak-j Function Sequence Space.
DOI: 10.5220/0010080309790981
In *Proceedings of the International Conference of Science, Technology, Engineering, Environmental and Ramification Researches (ICOSTEERR 2018) - Research in Industry 4.0*, pages 979-981
ISBN: 978-989-758-449-7
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i.e.

$$E^{\exists}(X,\Phi) = \left\{x=\big(x(i)\big)\in\Omega(X): \Phi\left(\frac{x}{\rho}\right)\in E(X) \text{ for some } \rho>0\right\}$$

where $E=\ell_1$ and

$$\Phi\left(\frac{x}{\rho}\right)=\left(\varphi_i\left(\frac{||x(i)||_X}{\rho}\right)\right).$$

Using the space $\ell_1(X)$ studied by (Maddox, 1988) i.e.

$$\ell_1(X) = \left\{x=\big(x(i)\big)\in\Omega(X): x(i)\in X \text{ and } \sum_{i=1}^{\infty}||x(i)||_X=||x(i)||_1<\infty\right\}$$

we give the sufficient and necessary condition a matrix operator from the space $\ell_1(X)$ into the space $\ell^{\exists}(X,\Phi)$ in the present paper.

2 MAIN RESULTS

Firstly, in this work we introduce the space $E^{\exists}(X,\Phi)$ for $E=\ell_1$, i.e.

$$\begin{aligned}\ell_1^{\exists}(X,\Phi) &= \left\{x=\big(x(i)\big)\in\Omega(X): \Phi\left(\frac{x}{\rho}\right)\in\ell_1(X) \text{ for some } \rho>0\right\}\\ &= \left\{x=\big(x(i)\big)\in\Omega(X): \left(\varphi_i\left(\frac{||x(i)||_X}{\rho}\right)\right)\in\ell_1(X) \text{ for some } \rho>0\right\}\\ &= \left\{x=\big(x(i)\big)\in\Omega(X): \sum_{i=1}^{\infty}\varphi_i\left(\frac{||x(i)||_X}{\rho}\right)<\infty \text{ for some } \rho>0\right\}.\end{aligned}$$

Theorem 2.1. *If φ_i satisfy Δ_2-condition for every natural numbers i, then the following set $\ell_1^{\exists}(X,\Phi)$ becomes a linear space.*

Proof.

(Ofie and Herawati, 2018).

Theorem 2.2. *If φ_i is a convex for every natural numbers i, then the space $\ell_1^{\exists}(X,\Phi)$ is a normed space with the following norm.*

$$||x||_\varphi=\inf\left\{\rho>0:\Phi\left(\frac{x}{\rho}\right)\le 1\right\}.$$

Proof.

Obviously $x=0$ in $\ell_1^{\exists}(X,\Phi)$ implies $||x||_\varphi=0$. Conversely, $||x||_\varphi=0$. Since Musielak-φ function, Φ, satisfies convex property, we obtain

$$\varphi(x)=\varphi\left(n\frac{x}{n}\right)\le\frac{1}{n}\varphi\left(\frac{x}{1/n}\right)\le\frac{1}{n}$$

for every $n\in\mathbb{N}$. Therefore $\Phi(x)=0$, which implies $x=0$. The following step we will show the homogeneous property. It is clearly for real numbers $\alpha=0$, then

$$\Phi(\alpha x)=0.$$

Assume for $\alpha\ne 0$. Since $||x||_\varphi\le\rho$, we have

$$\Phi\left(\frac{\alpha x}{\rho|\alpha|}\right)\le 1.$$

Thus $||\alpha x||_\varphi\le\rho|\alpha|$. Then $||\alpha x||_\varphi\le|\alpha|||x||_\varphi$, which implies

$$||x||_\varphi=\left|\left|\alpha\frac{x}{|\alpha|}\right|\right|_\varphi\le\frac{1}{\alpha}||\alpha x||_\varphi.$$

Therefore $|\alpha|||x||_\varphi\le||\alpha x||_\varphi$ for all real numbers α. Thus, $|\alpha|||x||_\varphi=||\alpha x||_\varphi$.

For next, we will show the triangle inequallity. Since for every $x,y\in\ell_1^{\exists}(X,\Phi)$ and φ_i satisfy convex property for every i, for non-negative reall numbers α,β with $\alpha+\beta=1$ we have

$$||x||_\varphi<\alpha \text{ and } ||y||_\varphi<\beta,$$

and

$$\begin{aligned}\Phi\left(\frac{x+y}{\alpha+\beta}\right) &= \Phi\left(\frac{\alpha}{\alpha+\beta}\frac{x}{\alpha}+\frac{\beta}{\alpha+\beta}\frac{y}{\alpha+\beta}\right)\\ &\le \frac{\alpha}{\alpha+\beta}\Phi\left(\frac{x}{\alpha}\right)+\frac{\beta}{\alpha+\beta}\Phi\left(\frac{y}{\beta}\right)\\ &\le \frac{\alpha}{\alpha+\beta}+\frac{\beta}{\alpha+\beta}=1.\end{aligned}$$

Consequently

$$||x+y||_\varphi\le\alpha+\beta$$

by definition of norm, then

$$||x+y||\le||x||_\varphi+||y||_\varphi.$$

Preposition 2.1. Let $x\in{\ell_1}^{\exists}(X,\Phi)$ for every Musielak-φ function $\Phi=(\varphi_i)$. If $||x||_\varphi\le 1$, then

$$\Phi(x)\le||x||_\varphi.$$

Proof.

Assume that for any vector $x\in\ell^{\exists}(X,\Phi)$, $x\ne 0$. By the definition of the norm, we get

$$\sum_{i\in\mathbb{N}}\varphi_i\left(\frac{||x(i)||_X}{||x||_\varphi}\right)\le 1. \tag{2.1}$$

For $||x||_\varphi = 1$, we get

$$\sum_{i\in\mathbb{N}} \varphi_i\big(||x(i)||_X\big) \leq ||x||_\varphi.$$

Now, assume $0 < ||x||_\varphi < 1$. Because φ_i is an increasing function for all $i \in \mathbb{N}$ and by Δ_2-condition, from (1) we conclude

$$\begin{aligned}\sum_{i\in\mathbb{N}} \varphi_i\big(||x(i)||_X\big) &< \sum_{i\in\mathbb{N}} \varphi_i\left(\frac{2||x(i)||_X}{||x||_\varphi}\right)\\ &\leq ||x||_\varphi \sum_{i\in\mathbb{N}} \varphi_i\left(\frac{||x(i)||_X}{||x||_\varphi}\right) \leq ||x||_\varphi.\end{aligned}$$

By using this preposition, we proof this Theorem as given below.

Theorem 2.3. *Let $A = (a_{ni})$ be an infinite real matrix. Then $A \in (\ell_1(X), \ell^{\exists}(X,\Phi))$ if and only if for all $x = (x(i)) \in \ell_1(X)$ there exists positive integer number m_0 such that*

$$\sup_{m_0||x||_1\leq 1} \sum_{n=1}^{\infty} \varphi_n\left(\frac{||A_n(x)||_X}{m_0}\right) \leq 1.$$

Proof.
($\Rightarrow$) *Using Zeller's theorem, we get $A \in (\ell_1(X), \ell^{\exists}(X,\Phi))$ is a continuous operator. Thus, there exists a natural number m_0 that if*

$$||x||_1 \leq \frac{1}{m_0}$$

implies

$$||Ax||_\varphi \leq 1 \tag{2.2}$$

for all $x = (x(i)) \in \ell_1(X)$. By using preposition 2.2 we have

$$\sum_{n=1}^{\infty} \varphi_n\left(\left|\left|\frac{A_n(x)}{m_0}\right|\right|_X\right) \leq 1.$$

Thus, we get

$$\sup_{m_0||x||_1\leq 1} \sum_{n=1}^{\infty} \varphi_n\left(\left|\left|\frac{A_n(x)}{m_0}\right|\right|_X\right) = \sup_{m_0||x||_1\leq 1} \sum_{n=1}^{\infty} \varphi_n\left(\frac{||A_n(x)||_X}{m_0}\right) \leq 1.$$

($\Leftarrow$) *Let $A = (a_{ni})$ an infinite matrix and $x = (x(i)) \in \ell_1(X)$. We will show that $Ax \in \ell^{\exists}(X,\Phi)$. There exists $m_0 \in \mathbb{N}$ such that*

$$\sup_{m_0||x||_1\leq 1} \sum_{n=1}^{\infty} \varphi_n\left(\frac{||A_n(x)||_X}{m_0}\right) \leq 1.$$

Because for every $x \in \ell_1(X)$ with $||x||_1 \leq \frac{1}{m_0}$, we get

$$\sum_{n=1}^{\infty} \varphi_n\left(\frac{||A_n(x)||_X}{m_0}\right) \leq 1 < \infty.$$

Since $m_0 > 0$, we have $Ax \in \ell^{\exists}(X,\Phi)$.

3 CONCLUSION

According to the main result, we conclude the sufficient and necessary condition for a matrix operator acting from the space $\ell_1(X)$ into the space $\ell^{\exists}(X,\Phi)$.

ACKNOWLEDGEMENT

Special thanks to TALENTA USU 2018 for funded and supported this paper.

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