# An Optimal Control Problem Formulation for a State Dependent Resource Allocation Strategy

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Abstract: In this paper, the problem of optimal resource allocation depending on the system evolution is faced. A preliminary analysis defines the global effort required in any subset of the system state space according to needed or desired goals. Then, in the definition of the cost function, the control action is weighted by a piecewise constant function of the state, whose different constant values are defined for each subset previously defined. The aim is to weight the control according to the distinct conditions, so getting different solutions in each state space region so to optimize the planned resources according to the global goal. A constructive algorithm to compute iteratively the final control law is outlined. The effectiveness of the proposed approach is tested on a typical model of human immunodeficiency virus (HIV) present in literature.

## 1 INTRODUCTION

In a control problem formulation, the main attention is given on the global performances of the system according to the desired state or output behavior. There are several cases in which the effort for the control goal achievements must be taken into consideration for a suitable, realistic and physically acceptable result, especially when optimality is also desired for the time length of the control action. In fact, in such cases, the containment of the control strength and the problem of the resource limitations can be considered together in the same way; this is usually performed both by introducing constraints in the control and in the ad hoc choice of the cost index in which the cost of the control is suitable defined. This is a classical problem that can be easily solved by means of the Pontryagin minimum principle; in the obtained solution the optimal control can present discontinuities (Hartl et al., 1995) at unknown instants, due to the presence of the constraints on the input amplitude.

Applications of optimal control techniques range from economics to biology, mechanics, telecommunications and so on (Jun, 2004), (C.Liu et al., 2008), (Nguyen and Sorenson, 2009). For the minimum time problems with linear steady state system, the optimal solution is bang-bang type with a limited number of switching points (M.Athans and Falb, 1996). In (Pasamontes et al., 2011) it is proposed to control a solar collector making use of a switching control strategy, showing that also changes in the dynamics can be handled in the contest of optimal control. Impulsive switching systems are another class of hybrid dynamical systems in which global optimal control strategies are proposed (R.Gao et al., 2010); they are characterized by abrupt changes at the switching instants.

The problem of optimal resources allocation may arise when dealing with competing alternative projects which share common resources; this is the so-called multi-armed bandit problem that has received much attention especially in economics, (Asawa and Teneketzis, 1996). In this case, the problem relies in determining the best strategy, among a set of possible ones, knowing the state of each phase. The decision is made on the basis of a payoff, i.e. a cost, associated to the action.

In general, when dealing with the optimal control of switched systems, like for example the optimal timing control problem, switching cost index can be introduced to take into account the changes in the controlled dynamics. One of the characteristic of these problems is that the systems involved present continuous dynamics subject to external discontinuous input actuated by a switching signal generator. Different schemes can be proposed and the optimal control theory applied to hybrid systems allows to determine the control input that optimizes a chosen performance



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index defined on the state trajectory of the system. This leads to two possible sub-problems, the time optimization problem and the optimal mode-scheduling problem (Ding, 2009). The former relies in finding the optimal placements of switching times assuming a fixed switching sequence; the latter is the problem of determining the optimal switching sequence of a switched system.

The presence of (white) noise perturbations can be also considered, as in (Liu et al., 2005); an interesting aspect is that the control weights are indefinite and the switching regime is described via a continuous-time Markov chain. It is proposed a near-optimal control strategy aiming at a reduction of complexity.

Numerical problems arising when dealing with optimal switching control are considered in (Luus and Chen, 2004) where a direct search optimization procedure is discussed.

In this paper the problem of optimal resource allocation is related to the real time system behavior considering the total amount of resources, i.e. the input constraint, fixed, and acting on the cost function, in particular on the weight of the input, in order to change the total cost according to the operative conditions. The idea is to replicate a planning scheme in which the designer fixes the relevance of the control action according to the conditions and, consequently, changes the politics of intervention making the control effort more or less relevant. For example, in a economic contest, within a prefixed total amount of resources (input constraint), the investment of more or less budget for the solution of some problem can be driven by some social indicator indexes, like unemployment below or over a prefixed critical percentages, or the national PIL lower or higher a prefixed threshold which guarantees economic grown, or the level taxation, and so on.

Then, a cost index in which the control action is weighted by a spatial piecewise constant function of the state is introduced, so that its value changes depending on the current state. The effect is to get different cost functions, defined over each state space region, which weight the control differently depending on the region in which the system operates, in order to implement, in the context of the classical optimal control formulation, a state dependent strategy. Changing the weight for the control for each distinct state space region corresponds to give a different relevance to the control amplitude action with respect to the other contributions, mainly errors, in the cost function. The result is that planning the different constant weights for the control reflects in allowing the control to use different amplitudes, clearly higher in correspondence to lower costs and lower for higher costs.

While the system evolves remaining in the same state space region, the solution of the optimal control problem gives an optimal solution for the control action. When, during the state evolution, the trajectory crosses from one region to another, a switch of the cost function occurs at the time instant in which the state reaches the regions separation boundary. From that time on, a different optimal control problem is formulated, equal to the previous one except for the input weight in the cost function.

This procedure is iterated until the final state conditions are reached. The overall control results to be a switching one, whose switching time instants are not known in advance but are part of the solution of the optimal control problem, depending on the optimal state evolution within each region. This kind of approach is different from the others previously recalled; here, the discontinuous switching solution does not arise either for the presence of switching dynamics, or for control saturation, but comes from the particular choice of the cost index. The control strategy changes since in the cost index it is assumed that the control needs to be weighted differently bringing to different strategies depending on the actual state value. It can be referred as a real time state dependent weight.

A first use of a switching formulation for an optimal control problem is proposed in (Di Giamberardino and Iacoviello, 2017), applied to a classical SIR epidemic diffusion. The effectiveness of the proposed approach is then here shown making use of a biomedical example, the control of an epidemic disease, the immunodeficiency virus (HIV). The HIV model proposed in (Wodarz, 2001) and modified in (Chang and Astolfi, 2009) is adopted. The choice of this example comes from the consideration that usually the medical and social interest for the presence of an epidemic spread depends on the level of diffusion of the infection, being considered in some sense natural if it is lower than a physiologic level and becoming more and more relevant as the intensity of the infection increases. Then, according to the present approach, a state dependent coefficient that weights differently the control depending on the number of the infected cells is introduced, taking as state space region division the sets that correspond to a physiological level, a high but not serious level and a very high risk level. This corresponds to change the intervention strategy depending on the varied conditions; as already noted, the possible switching instants are not known in advance but are determined on the basis of the dynamic variables evolution and on the optimization process.

In general, the introduction of a continuous state function as a weight in the cost index can be found in (Behncke, 2000) for the case of SIR dynamics. There, the feasibility of different control actions is investigated along with the possibility of introducing a weight of the vaccination control depending on the number of susceptible subjects; it is assumed the hypothesis that vaccination at higher densities may be less expensive and logistically easier. The continuous weight tate space function brings to some additional conditions, more than the usual ones of an optimal control problem formulation, to be fulfilled.

The introduction of a spatial piecewise constant function instead of a generic one as weight function for the control brings back the problem formulation, and then the problem solution, to a classical formulation, except for the fact that the whole solution is obtained composing the different local solutions computed in each region in which the state trajectory evolves.

The paper is organized as follows: Section 2 is devoted to the description of the proposed approach, based on an iterative optimal control computation driven by the state values. In Section 3 some recalls on the HIV model and the control described in (Chang and Astolfi, 2009) are given. Then, in Section 4 the proposed control strategy with the state dependent cost index is applied to the HIV model. In Section 5 the numerical results obtained for the case study here considered are presented and discussed. Conclusions and future work are outlined in Section 6.

## 2 PROBLEM FORMULATION

Starting from some brief recalls on the classical optimal control formulation, the proposed approach is introuced and described.

### 2.1 Recalls on Optimal Control Problem Formulation

In the optimal control theory, the following classical minimum time problem is considered.

Given a generic dynamical system

$$\dot{x}(t) = f\left(x(t), u(t)\right) \tag{1}$$

with $x \in \Re^n$, $u \in \Re^m$, $x(t_0) = x_0$, where $f \in C^2$ with respect to its arguments, and the $q$-dimensional inequality constraints on the control action

$$q\left(u(t)\right) \leq 0 \tag{2}$$

and assumed the cost index

$$J\left(u(t), T\right) = \int_{t_0}^{T} L\left(x(t), u(t)\right) dt \tag{3}$$

in which the Lagrangian $L\left(x(t), u(t)\right) : \Re^{n+m} \to \Re$ depends on the state as well as on the control, find the optimal values for the control ($u^0(t)$) and the final time ($T^0$), under the state constraint (1) and the inequality constraint on the input (2), satisfying the final condition

$$\chi\left(x(T), T\right) = x(T) - x^f = 0 \tag{4}$$

for a given $x^f \in \Re^n$, with $\chi \in C^1$, such that $dim(\chi) = \sigma$, $1 \leq \sigma \leq n+1$.

As well known, once the constraints are given, the obtained solution is optimal for the specific choice of the cost function $J(u(t), T)$. Changing such a function, also the solution changes. This means that the choice of the function $J(u(t), T)$ or, equivalently, of the Lagrangian function $L(x(t), u(t))$ represents a crucial aspect of the whole design procedure. In addition, their structure strongly affects not only the result but also the design procedure. In fact, usually, a linear combination of linear or quadratic terms is adopted for $L$, with constant coefficients representing the weight of each term in the sum, i.e. how much it is important in the evaluation of the optimality of the solution.

Such a structure is justified by the simplicity in the problem formulation and in the computation of the solution as well.

There are authors proposing richer formulations, in which some of the weights for the input variables can be taken as nonlinear functions of the state variables, in order to assign different relevance to the control action depending on the operative conditions, (Behncke, 2000). Clearly, this kind of formulation introduces some additional conditions to be fulfilled and the complexity in the computation of the optimal control grows significantly, requiring additional specific hypothesis on the system behavior.

The idea developed in the present work, and illustrated in the next Subsection 2.2, is to maintain the richness of the nonlinear state dependent weights and, at the same time, to preserve the simplicity coming from the use of linear/quadratic terms in the problem formulation and solution.

### 2.2 The Proposed Approach

A generic quadratic function of $u(t)$ in $L\left(x(t), u(t)\right)$ depending on the state, can be written as $u^T P(x) u$, where $P(x)$ represents the different weight of the input as a function of the state and then of the operative conditions.

In the proposed approach, which aims at simplifying the optimal control formulation preserving the richness of a state space dependent weight, the state space is divided into $N$ subsets $I^i$, such that $\cup_{i=1}^{N} I^i = \Re^n$, each of them corresponding to different strategies to be adopted. Therefore, the function $P(x)$ is defined as

$$P(x) = \Pi_i \quad \text{when } x \in I^i \tag{5}$$

with $\Pi_i \in \Re^{m\times m}$ positive defined $m \times m$ matrix, $i = 1,\dots,N$, where the entries of $\Pi_i$ are designed in order to manage the input cost as the state changes.

Then, while $x \in I^i$, the term $u^T \Pi_i u$ is used in the Lagrangian $L(x(t),u(t))$ which can be rewritten as $L_{\Pi_i}(x(t),u(t)$ to put in evidence such dependency.

Once that in the optimal control problem formulation no state dependent weight is esplicitely present, the solution can be found according to the well known approach which makes use of the Hamiltonian

$$H_{\Pi_i}(x,\lambda,u) = L_{\Pi_i}(x,u) + \lambda^T f(x,u) \tag{6}$$

where $\lambda : \Re \to \Re^n$, $\lambda(t) \in C^1$ almost everywhere, is the $n$–dimensional multiplier function for the differential constraint given by the dynamics. Clearly, such a formulation holds only when $x \in I^i$.

Under the constraints (2) and (4), the optimal solution can be obtained solving the necessary conditions given by

$$\dot{\lambda} = -\left.\frac{\partial H_{\Pi_i}(x,\lambda,u)}{\partial x}\right|^T \tag{7}$$

$$0 = \left.\frac{\partial H_{\Pi_i}(x,\lambda,u)}{\partial u}\right|^T + \left.\frac{\partial q(u)}{\partial u}\right|^T \eta \tag{8}$$

$$\eta^T q(u) = 0 \tag{9}$$

$$\eta \geq 0 \tag{10}$$

$$0 = H(x(T),\lambda(T),u(T)) \tag{11}$$

$$\lambda(T) = -\left.\frac{\partial \chi(x(T),T)}{\partial x(T)}\right|^T \zeta \tag{12}$$

where $\eta(t) \in \Re^p$, $\eta \in C^0$ almost everywhere, $\zeta \in \Re^\sigma$, and along with conditions (1), (2) and (4). The solution obtained holds until $x(t) \in I^i$ and it is optimal in such a region. If the solution is such that the computed trajectory goes outside the region $I^i$ entering a contiguous region $I^j$, then a new problem has to be formulated with initial condition for the state as the value on the boundary between the regions $I^i$ and $I^j$ reached by the previously computed control, and making use of the Lagrangian $L_{\Pi_j}(x,u)$ and, then, of $H_{\Pi_j}(x,\lambda,u)$ in the necessary conditions.

The final solution is obtained by concatenating all the partial solutions computed.

Clearly, such a solution cannot be defined as optimal since in this formulation it is not computed according to a unique cost index, but it is optimal if restricted to each state space region.

In order to better illustrate the proposed approach, an example in the epidemiological field is provided; in this kind of problems, the classical medical approach makes use of thresholds to classify the severity of the infection and then this can be used to modulate the control weight in the cost index.

In next Section 3 the mathematical model of one case study, the HIV infection, is briefly introduced, and the proposed procedure is used in Section 4.

## 3 THE MATHEMATICAL MODEL OF THE SAMPLE SYSTEM

Many different models have been proposed to describe the HIV (human immunodeficiency virus); the virus infects the CD4 T-cells in the blood of an HIV-positive subject; when the number of these cells is below 200 in each $mm^3$ the HIV patient has AIDS.

Models of the HIV generally consider the uninfected CD4 T-cells, the infected CD4 T-cells, the infectious virus, the noninfectious virus and the immune effectors, (Banks et al., 2006). In (Chang and Astolfi, 2009) also the effects of cytotoxic T lymphocyte (CTL) are taken into account aiming at determining a control that drives the patients into the long-term non progression (LTNP) status, instead to progress to the AIDS one. A simplified system is presented in (Joshi, 2002) where only the concentration of CD4 T-cells and the concentration of the HIV particles are analyzed; in this case two different treatments strategies are introduced in the differential equations. Among all the proposed strategies, the policy using two drug controls appears to be the best one, since it reduces the number of virus particles, beyond the rise of the number of uninfected CD4 T-cells, (Zhou et al., 2014). The problem of the fast mutation of the HIV is faced in (E.A.H. Vargas, 2014); this could cause resistance to specific drug therapies; the model predictive control shows the best performance among the ones based on a switched linear system to a nonlinear mutation model. In (Ding et al., 2012) it is suggested the use of the fractional-order HIV model as a description more realistic than traditional ones, thus obtaining very low levels dosage of anti-HIV drugs.

In this paper, the HIV model proposed in (Wodarz, 2001) and modified in (Chang and Astolfi, 2009) is used. It will be shortly recalled hereafter. In the complete model the state variables to be considered are:

- the uninfected CD4 T-cells, denoted by $x_1(t)$;
- the infected CD4 T-cells, denoted by $x_2(t)$;
- the helper–independent CTL, denoted by $z_1(t)$;

- the CTL precursor, denoted by $w(t)$: it provides long term memory for the antigen HIV;
- the helper-dependent CTL, denoted by $z_2(t)$: it destroys the infected cells $x_2(t)$.

The equations describing the relations among these variables are

$$\begin{aligned}
\dot{x}_1(t) &= \gamma - dx_1(t) - \beta(1-u(t))x_1(t)x_2(t) && (13)\\
\dot{x}_2(t) &= \beta(1-u(t))x_1(t)x_2(t) - \alpha x_2(t) + \\
&\quad -(p_1 z_1(t) + p_2 z_2(t))x_2(t) && (14)\\
\dot{z}_1(t) &= c_1 z_1(t)x_2(t) - b_1 z_1(t) && (15)\\
\dot{w}(t) &= c_2 x_1(t)x_2(t)w(t) - c_2 q x_2(t)w(t) + \\
&\quad -b_2 w(t) && (16)\\
\dot{z}_2(t) &= c_2 q x_2(t)w(t) - h z_1(t) && (17)
\end{aligned}$$

where $\gamma$, $d$, $\beta$, $\alpha$, $p_1$, $p_2$, $c_1$, $c_2$, $b_1$, $b_2$ and $h$ are the models parameters whose numerical values are discussed in (Wodarz, 2001) and the control $u(t)$ is assumed bounded.

In (Chang and Astolfi, 2009) the Authors aim to determine a control making use of equations (13) and (14) only, through a simplified representation in which the contribution of (15), (16) and (17) to the (13)–(14) dynamics is reduced to an approximated near-equilibrium polynomial term. The proposed modified model is

$$\begin{aligned}
\dot{x}_1(t) &= \gamma - dx_1(t) - \beta(1-u(t))x_1(t)x_2(t) && (18)\\
\dot{x}_2(t) &= -\beta x_1(t)x_2(t)ut + \pi(x_2(t)) && (19)
\end{aligned}$$

with

$$\pi(x_2(t)) = a + Bx_2(t) + Cx_2^2(t) + Dx_2^3(t) \qquad (20)$$

For sake of simplicity, in the sequel the model (18)–(19) with position (20) will be assumed, with the initial conditions denoted by $x_1(t_0) = x_{1,0}$ and $x_2(t_0) = x_{2,0}$.

As well known, in optimal control the central aspect is the definition of the cost index, that is what is required to be minimized; in this case, the control effort and the number of infected subjects seem to be a good choice.

Another aspect to be considered is the problem of resources allocation especially when they are particularly limited. For example, in (Yuan et al., 2015) this problem is faced when a limited quantity of vaccine has to be distributed between two non-interactive populations; in that case, a stochastic epidemic model is assumed.

Hereafter, the resource limitation is introduced by a constraint as (2).

# 4 IMPLEMENTATION OF THE PROPOSED APPROACH

The example introduced in previous Section 3 can be effectively used to describe the proposed approach. In fact, it is possible to define different strategies in terms of control effort to be applied according to the severity of the infection, measured by the number $x_2(t)$ of infected cells. In other words, for sake of simplicity, it is possible to find three levels of necessity of intervention; if $x_2(t)$ is below a certain threshold, say $\xi^1$, no actual infection is diagnosed and then no intervention is required; then, defined $\xi^2$ as the level of infected cells over which the infection presents severe effects, it is possible to choose two different effort in case of $x_2(t) \geq \xi^1$ is greater or lower than $\xi^2$: in the first case a stronger action is required than the one in the second case, and this requirement can be introduced in the control design setting a lower cost, i.e. weight, to the control if $x_2(t) \geq \xi^2$ and a higher weight when $\xi^1 \leq x_2(t) < \xi^2$.

So, according to the procedure described in Subsection 2.2, the state space $x = (x_1 \; x_2)^T$ is divided into three regions:

$$\begin{aligned}
I^1 &= \{x \in \Re^2 : x_2 < \xi^1\}\\
I^2 &= \{x \in \Re^2 : \xi^1 \leq x_2 < \xi^2\} && (21)\\
I^3 &= \{x \in \Re^2 : x_2 \geq \xi^2\}
\end{aligned}$$

$I^1$ is the region in which no control action is needed; $I^2$ is the region corresponding to the presence of the infection while $I^3$ corresponds to a severe stadium of infection.

Choosing the cost function

$$\begin{aligned}
J(u(t),T) = \int_{t_0}^{T} &[K_1 + K_2 x_1(t)x_2(t)u(t) + K_3 x_2(t) + \\
&+ P(x(t))u^2(t)]\,dt && (22)
\end{aligned}$$

with $K_i > 0$, $i = 1,2,3$, the state function $P(x(t))$ can be set as

$$\begin{aligned}
P(x(t)) &= \Pi_1, && x \in I^1\\
P(x(t)) &= \Pi_2, && x \in I^2 \qquad (23)\\
P(x(t)) &= \Pi_3, && x \in I^3
\end{aligned}$$

with $\Pi_3 < \Pi_2$, so that the control can assume higher values when the infection is severe ($x \in I^3$), being cheaper than in the case of $x \in I^2$. As far as $\Pi_1$ is concerned, its value is not relevant since when $x \in I^1$ no control action is required and then no control problem has to be formulated.

Assuming the nontrivial initial conditions $x_{1,0} \in \Re$ and $x_{2,0} \geq \xi^1$, $x_0 \in I^i$ for a certain $i > 1$, the constraint (4) can be rewritten as

$$\chi(x(T),T) = x_2(T) - \xi^1 = 0 \qquad (24)$$

while the resources limitation, i.e. the control constraint (2), can be explicitly written as

$$q(u(t)) = \begin{pmatrix} q_1(t) \\ q_2(t) \end{pmatrix} = \begin{pmatrix} -u(t) \\ u(t) - U \end{pmatrix} \leq 0 \qquad (25)$$

$U > 0$, where the first component represents the non negativity condition while the second one is the upper bound limitation.

To solve the problem the classical optimal control theory is applied; the Hamiltonian in each region $I^i$ is defined as

$$\begin{aligned} & H_{\Pi_i}(x_1(t), x_2(t), \lambda_1(t), \lambda_2(t), u(t)) = \\ & = K_1 + K_2 x_1(t) x_2(t) u(t) + K_3 x_2(t) + \Pi_i u^2(t) + \\ & + \lambda_1(t)\left(\gamma - d x_1(t) - \beta(1 - u(t)) x_1(t) x_2(t)\right) + \\ & \lambda_2(t)\left(-\beta x_1(t) x_2(t) u t + \pi(x_2(t))\right) \end{aligned} \qquad (26)$$

and then the necessary optimal conditions given in Subection 2.2 assume the explicit expressions

$$\begin{aligned} \dot{\lambda}_1(t) &= -\frac{\partial H_{\Pi_i}}{\partial x_1} = -K_2 x_2(t) u(t) + d\lambda_1(t) + \\ &\quad +\beta(1 - u(t)) x_2(t) \lambda_1(t) + \\ &\quad +\beta x_2(t) \lambda_2(t) u(t) \\ \dot{\lambda}_2(t) &= -\frac{\partial H_{\Pi_i}}{\partial x_2} = -K_2 x_1(t) u(t) - K_3 + \\ &\quad +\beta(1 - u(t)) x_1(t) \lambda_1(t) + \\ &\quad +\beta x_1(t) \lambda_2(t) u(t) + \\ &\quad -\lambda_2(t)\left[B + 2C x_2(t) + 3D x_2^2(t)\right] \\ 0 &= \frac{\partial H_{\Pi_i}}{\partial u} + \frac{\partial q_1}{\partial u}\eta_1 + \frac{\partial q_2}{\partial u}\eta_2 = 2\Pi_i u(t) \\ &\quad +K_2 x_1(t) x_2(t) + \beta x_1(t) x_2(t) \lambda_1(t) + \\ &\quad -\beta x_1(t) x_2(t) \lambda_2(t) - \eta_1(t) + \eta_2(t) \\ 0 &= \eta_1(t) q_1(u(t)) \\ 0 &= \eta_2(t) q_2(u(t)) \\ \eta_1(t) &\geq 0 \\ \eta_2(t) &\geq 0 \\ 0 &= H_{\Pi_i}(x(T), \lambda(T), u(T)) \\ \lambda_1(T) &= 0 \\ \lambda_2(T) &= -\zeta_2 \qquad \zeta_2 \in \Re \end{aligned}$$

with condition (24) too.

After some computations, defining the function $W(t)$ as

$$W(t) = x_1(t) x_2(t)\left(-K_2 - \beta\lambda_1 + \beta\lambda_2\right) \qquad (27)$$

the optimal control satisfying the necessary conditions previously introduced can be expressed as

$$u^1(t) = \begin{cases} 0 & \text{if } W(t) < 0 \\ \frac{W(t)}{2\Pi_i} & 0 < \frac{W(t)}{2\Pi_i} < U \\ U & \text{if } \frac{W(t)}{2\Pi_i} > U \end{cases} \qquad (28)$$

By integration, denoting with $\left(T^1, x_1^1(t), x_2^1(t), u^1(t)\right)$ the solution obtained over the time interval $\left[t_0, T^1\right]$, it is also the optimal solution as long as $x(t) \in I^i$.

If $x_0 \in I^i$ and $x(t) \in I^i$ $\forall t \in \left[t_0, T^1\right]$, one has that the solution is the whole optimal solution, which can be indicated with the superscript $^0$: $\left(T^0, x_1^0(t), x_2^0(t), u^0(t)\right) = \left(T^1, x_1^1(t), x_2^1(t), u^1(t)\right)$, and $x_2\left(T^0\right) = \xi^1$. Otherwise, there exists a time instant $t = t_1$ such that $x(t_1^-) \in I^i$ and $x(t_1^+) \in I^j$, $i \neq j$. Then, a new optimal control problem must be solved, with the same conditions as the previous ones after the substitutions $t_0 = t_1$, $x(t_0) = x(t_1)$, and the index $j$ instead of $i$.

In the present case, being two the effective regions, the optimal solution obtained in the first of the previous case necessarily means that $i = 2$. Otherwise, the switching condition does hold for $i = 2$ and $j = 3$ or vice versa.

The control computation ends at step $k \geq 1$ when, after a a priori unknown number $k - 1 \geq 0$ of switches, the solution $\left(T^k, x_1^k(t), x_2^k(t), u^k(t)\right)$ is such that $x(t) \in I^2$ $\forall t \in \left[t_{k-1}, T^k\right]$ and condition (24) is satisfied.

For $k > 1$, the whole solution is then given by concatenating the $k$ partial ones, so getting a switching solution with switching times $t_i$, $i = 1, 2, \ldots, k-1$ and optimal time $T^0 = t_k$.

It is important to stress that in the proposed approach the presence of switching instants depends on the evolution of the state: no information can be available, even on their existence. The state dependent switching conditions makes possible a different interpretation; the control law computed following this procedure can be regarded as a continuous time optimal control over a discrete time feedback update of the control parameters. The optimal control can be computed and applied until the state belongs to the given region $I^i$; crossing the regions boundary is equivalent to an event driven discrete state feedback which updates all the parameters, mainly the $\Pi_i$, and recompute a new optimal control over the new state space region $I^j$.

## 5 SIMULATION RESULTS

In this Section the results of some numerical simulations are presented, showing the behavior of the proposed control design approach making use of the HIV model presented in Section 3. In all the simulations performed, the parameters reported in Table 1, taken from (Chang and Astolfi, 2009), have been used for the model (18)–(19), along with the initial conditions

$x_{1,0} = 0.2$ and $x_{2,0} = 3$.

Table 1: Numerical values used for the HIV system parameters.

| | | | | |
|---|---|---|---|---|
| $\gamma$ | 1 | | $B$ | -3.1540 |
| d | 0.1 | | $C$ | 2.9402 |
| $\beta$ | 1 | | $D$ | -0.6 |
| $\alpha$ | 0.0668 | | | |

The choice of the HIV case study is quite meaningful, since a switching control form takes the form of a classical therapy strategy, being usually a piecewise constant control with the aforethought switching times: it consists of a full drug dose for a limited time and then a switch to zero, (Wodarz, 2001), sometimes putting in evidence the daily therapy, (Chang and Astolfi, 2009).

An optimal control approach demands to the cost function the ability to modulate the control according to all the variables involved, possibly increasing the performances of the control action. For a choice of the cost index as in (22), the solution depends on the values given to the weights assigned to each term. In fact, in a classical minimum time optimal control formulation, for the numerical choice of the constant weights $K_1 = 10$, $K_2 = 1$ and $K_3 = 20$, taking for example a constant weight $P(x(t)) = P = 1\ \forall x \in \Re^2$, for $U = 0.9$ as in (Chang and Astolfi, 2009) and $\xi^1 = 0.03$ in (4), the optimal control solution $u^0(t)$ obtained is depicted in Figure 1, while Figure 2 reports the optimal time evolution of the infected cells $x_2^0(t)$.

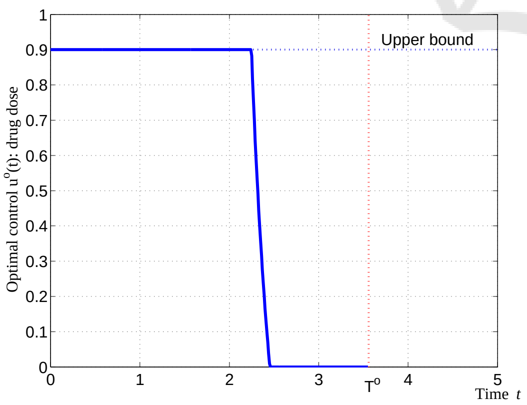


Figure 1: Optimal control for constant input weight $P = 1$.

As expected, the choice of the weight for the input $u(t)$ in the cost index lower or equal to the ones assigned to the terms containing the infected cells produces an optimal control behavior equal to the upper bound value from $t_0 = 0$ until the number of infected cells is reduced at a level in which a high control action is too expensive with respect to such a number,

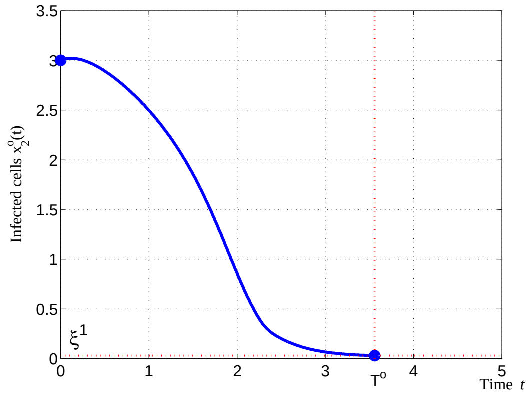


Figure 2: Infected cells evolution under optimal control.

and then it goes to zero as the $x_2(t)$ component decreases, so assuming a so called bang–bang behavior between the upper and the lower bounds.

If the approach proposed in this paper is adopted, the regions $I^1$, $I^2$ and $I^3$ as in (21) must be introduced, with their meaning discussed in Section 4, and with the corresponding weights $\Pi_i$ as in (23) for the control in the cost function (22).

The numerical values chosen are $\xi_2 = 2$, so that the initial condition lies in the dangerous region $I^3$ and the solution must cross the normal region $I^2$, $\Pi_2 = 100$ and $\Pi_3 = 1$, while $\Pi_1$ in this is not used due to the no action region $I^1$. The values for $\Pi_2$ and $\Pi_3$ with $\Pi_2 \gg \Pi_3$ have been chosen in order to significantly put in evidence the difference between a low cost, and then a higher margin for the control effort and a high cost, which should act against a high control effort.

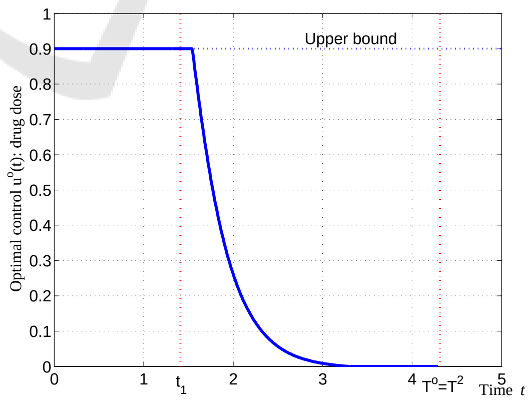


Figure 3: Full control for switched value of $P(x(t))$.

The solution obtained, depicted in Figure 3, is, confirming what planned, the concatenation of two segments; a first optimal segment over the region $I^3$ in the time interval $0 = t_0 \leq t < t_1 = 1.41$, computed with $P(x(t)) = \Pi_3$, and then, at $t = t_1$, the switch of $P(x(t))$ from $\Pi_3$ to $\Pi_2$ produces the second segment which brings to the final condition $x_2(T^0) = \xi^1 = 0.03$ at time $t = T^0 = 4.31$.

This composition of the whole control in the form of a switching solution can be well put in evidence plotting the solution obtained in the first step of the procedure, under the hypothesis that the state is contained in the set $I^3$, and marking the time instant $t = t_1$ in which the state trajectory reaches the boundary of $I^3$. This is done in Figure 4.

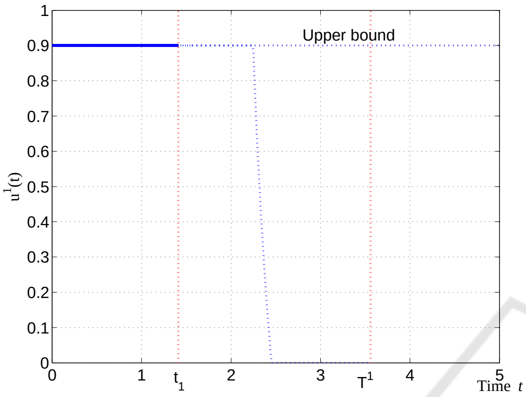


Figure 4: Optimal control obtained in the first step of the procedure, with the effective part from 0 to $t_1$ evidenced.

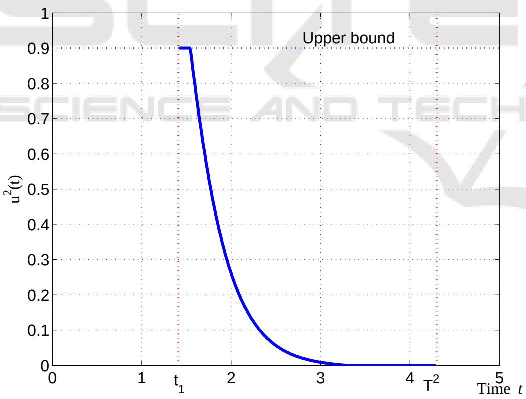


Figure 5: Optimal control obtained in the second step of the procedure, with the effective part from $t_1$ to $T^2$ evidenced.

Then, in Figure 5 the solution of the optimal control problem defined over $I^2$ starting from the initial condition on its boundary corresponding to the value reached in the previous phase is plotted. Comparing the two Figures 4 and 5, it is possible to understand the effect of the different weights of the input variable on the control law obtained; in the first case, with a lower cost, the upper bound, i.e. the maximum value, of the control is kept longer than in the second case, being cheaper. In the second case, the cost of the control forces the solution to reduce it as much as possible to guarantee that the state reaches the final condition balancing the cost of the error with the one of the control. The change of the control weight in the cost function at the boundary between $I^2$ and $I^3$ produces a new behavior, characterized by a shorted saturated action and a smoother decreasing shape, assuring, however, the convergence to the final state.

The concatenation of the effective part in Figure 4 with the one in Figure 5 yelds Figure 3. Note that the time instant in which the solution depicted in Figure 3 starts to decrease from the upper limit does not coincides with the switching instant $t_1$: after the switch, the control remains at its maximum but for less time than in the non switching case.

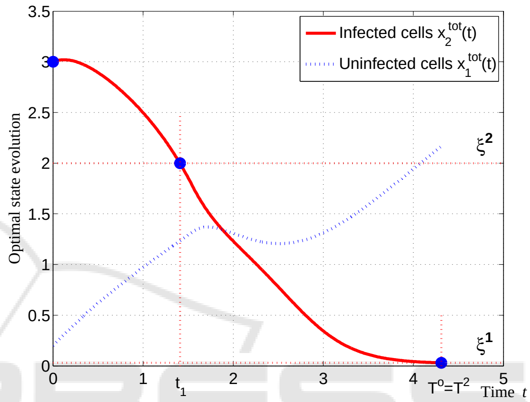


Figure 6: State evolution given by the full switched control.



The time history of the uninfected ($x_1(t)$) and infected ($x_2(t)$) cells is depicted in Figure 6 where the switching conditions and the corresponding time instants are evidenced.

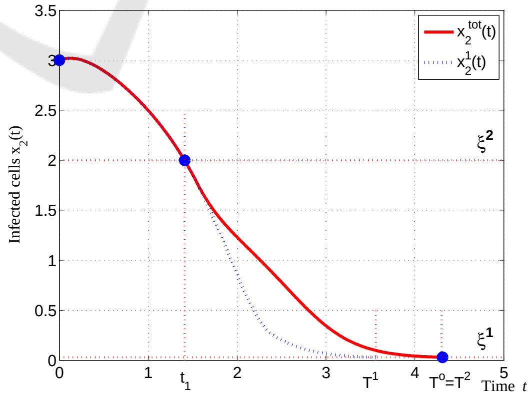


Figure 7: Time evolution of the infected cells in switched and non switched case: a comparison.

A comparison between the evolution of the infected cells obtained with the switching formulation and the classical one coming from the use of a unique constant value for the input weight is reported in Figure 7. Note that the non switching solution corresponds to keep $P(x(t)) = \Pi_3$ for all the state values, i.e. considering $I^2$ and $I^3$ as a unique region with a low cost for the input, like in a standard optimal control problem formulation. It can be noted that in the time interval corresponding to the evolution in the $I^2$ region, the solution, obtained using a low control weight only, makes the state reach the final condition faster and keeps the number of the infected cells lower than in the other case. Obviously, this is due to the fact that the higher cost for the control brings the optimal control formulation to save the control effort, still bringing to an effective solution as well.

Nevertheless, this apparent drawback is fully compensated by the fact that the control, over the whole time interval during which the drug is provided, requires a lower contribution. This can be shown computing and plotting the function $\int_0^t u(\tau)d\tau$ which give a measurement of the total drug to be used in the therapy.

Figure 8 is then obtained, showing that until both solutions require the full control action ($t = t_1$), up to its bound, the functions are obviously coincident; then, the decrement of the control in the switching case, starting when the classical one is still at maximum, produces a reduction of the total amount of input quantity, and then a reduced impact on the infected patient and, at the same time, on the cost related to the therapy, despite its longer time of application.

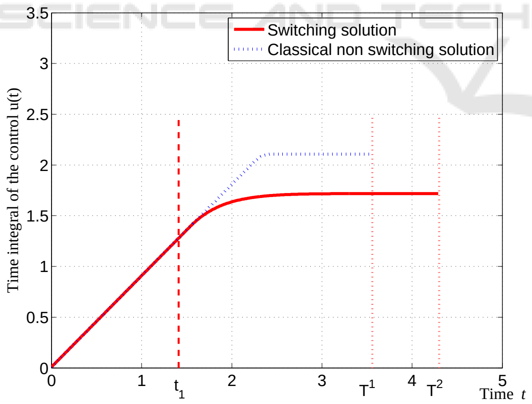


Figure 8: Integral cost of the control action for the switching solution and for the classical case: a comparison.

## 6 CONCLUSIONS

In this work a suitable non-linear cost index is assumed in a minimum time optimal control problem formulation, weighting the control by a state dependent locally constant function. This approach can deal with changes in the external conditions since it is based on the state evolutions; it can tackle practical applications in telecommunications, biology, mechanics, economics, just to mention a few. The effectiveness of the proposed approach is verified considering a model of human immunodeficiency virus (HIV) and proposing a cost index in which the control effort is weighted taking into account the number of infected cells, giving higher attention when they are dangerously over a fixed critical value and considering the infection not much severe below. Obviously the result can be easily generalized to the case of more than one critical value. The results obtained show that this approach provides an efficient resource allocation, so being more effective, for example from an economical point of view, than the classical theory with the constant weight choice.

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