Solution of a Singular H_∞ Control Problem: A Regularization Approach

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Abstract: We consider an infinite horizon H_∞ control problem for linear systems with additive uncertainties (disturbances). The case of a singular weight matrix for the control cost in the cost functional is treated. In such a case, a part of the control coordinates is singular, meaning that the H_∞ control problem itself is singular. We solve this problem by a regularization. Namely, we associate the original singular problem with a new H_∞ control problem for the same equation of dynamics. The cost functional in the new problem is the sum of the original cost functional and an infinite horizon integral of the squares of the singular control coordinates with a small positive weight. This new H_∞ control problem is regular, and it is a partial cheap control problem. Based on an asymptotic analysis of this H_∞ partial cheap control problem, a controller solving the original singular H_∞ control problem is designed. Illustrative example is presented.

1 INTRODUCTION

Controlled systems with uncertain dynamics are extensively studied in the literature (see e.g. ((Basar and Bernard, 1991); (Chang, 2014); (Doyle et al., 1990); (Fridman et al., 2014); (Glizer and Turetsky, 2012); (Petersen and Tempo, 2014); (Petersen et al., 2000)) and references therein). Two classes of uncertainties (disturbances) are usually distinguished: (1) disturbances belonging to a known bounded set of Euclidean space; (2) quadratically integrable disturbances. For controlled systems with quadratically integrable disturbances, the H_∞ control problem is frequently considered (see e.g. ((Basar and Bernard, 1991); (Chang, 2014); (Doyle et al., 1989); (Petersen et al., 2000)).

If the rank of the matrix of coefficients for the control variable in the output equation equals to the dimension of the control, then the solution of the H_∞ control problem can be reduced to a solution of a game-theoretic Riccati matrix algebraic equation. If the rank of the matrix of coefficients for the control in the output is smaller than the dimension of the control, then the weight matrix for the control cost in the cost functional of the H_∞ problem is singular meaning that the mentioned above Riccati equation does not exists. Such H_∞ control problems are called singular or nonstandard. Some cases of linear dynamics singular H_∞ control problems were studied in the literature, using different approaches. Thus, in (Petersen, 1987), the H_∞ problem with no control in the output was considered. For this problem, an "extended" game-theoretic Riccati matrix algebraic equation was constructed. Based on the assumption of the existence of a proper solution to this equation, the solution of the considered H_∞ problem was derived. In (Djouadi, 1998), an explicit expression for the optimal controller was derived using an operator theory approach and the Banach space duality. In (Stoorvogel, 2000), a Riccati matrix inequality approach was used to solve the problem. In (Chuang et al., 2011), the controller design was based on a physical model of the Atomic Force Microscope, studied in the paper.

In the present paper, we consider an infinite horizon singular H_∞ control problem. A regularization of this problem is proposed leading to a new H_∞ problem with a partial cheap control. The latter is solved by adapting a perturbation technique. Then, it is shown on how accurately the controller, solving this H_∞ partial cheap control problem, solves the original singular H_∞ control problem.

It should be noted that the regularization approach has been widely applied in the literature for analysis and solution of various control problem. Thus, in ((Bell and Jacobson, 1975); (Glizer, 2012c); (Glizer, 2012b); (Glizer, 2014); (Kurina, 1977)), different sin-

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gular optimal control problems were solved using this approach. In (Turetsky et al., 2014), this approach was applied for the design of a robust state-feedback control in some trajectory tracking problem for uncertain systems. In ((Shinar et al., 2014); (Glizer and Kelis, 2015a); (Glizer and Kelis, 2015b); (Glizer, 2016), (Glizer and Kelis, 2017)), some singular zero-sum and non zero-sum differential games were solved by application of the regularization approach. In a short conference paper ((Glizer, 2013)), a singular H_∞ control problem for linear time delay systems was studied in the case where the output equation of this problem is independent of the control. In the present paper, we consider the case where the output equation of a singular H_∞ control problem depends on the control. To the best of our knowledge, the rigorous analysis of such a case of singular H_∞ control problems by application of the regularization approach is carried out for the first time in the literature in this paper.

2 PROBLEM STATEMENT

We consider the following controlled system:

$$\frac{dZ(t)}{dt} = \mathcal{A}Z(t) + \mathcal{B}u(t) + \mathcal{F}w(t), \quad Z(0) = 0, \quad (1)$$

$$V(t) = \mathrm{col}\{\mathcal{C}Z(t),\ \mathcal{M}u(t)\}, \quad (2)$$

where $t \geq 0$, $Z(t) \in E^n$, $u(t) \in E^r$, $(n \geq r)$, ($u(t)$ is a control); $w(t) \in E^m$, ($w(t)$ is a disturbance); $V(t) \in E^p$, ($V(t)$ is an output); $\mathcal{A}$, $\mathcal{B}$, $\mathcal{F}$, $\mathcal{C}$, $\mathcal{M}$ are given constant matrices of dimensions $n \times n$, $n \times r$, $n \times s$, $p_1 \times n$, $p_2 \times r$, $(p_1 + p_2 = p,\ r \leq p_2)$, respectively.

Assuming that $w(t) \in L^2[0,+\infty; E^m]$, let us consider the following cost functional:

$$\mathcal{J}(u,w) = \Big(\|V(t)\|_{L^2}\Big)^2 - \gamma^2\Big(\|w(t)\|_{L^2}\Big)^2, \quad (3)$$

where $\gamma > 0$ is a given constant.

The H_∞ control problem with a performance level γ for the system (1)-(2) is to find a controller $u^0[Z(t)]$ that ensures the inequality

$$\mathcal{J}(u^0, w) \leq 0 \quad (4)$$

along trajectories of (1) for all $w(t) \in L^2[0,+\infty; E^m]$.

Now, we consider the following two matrices:

$$\mathcal{D} = \mathcal{C}^T\mathcal{C}, \quad \mathcal{N} = \mathcal{M}^T\mathcal{M}. \quad (5)$$

The matrices $\mathcal{D}$ and $\mathcal{N}$ are symmetric matrices of the dimensions $n \times n$ and $r \times r$, respectively. Furthermore, they are at least positive semi-definite. Moreover, if $\mathrm{rank}\mathcal{M} = r$, then the matrix $\mathcal{N}$ is positive definite. In the latter case, the matrix $\mathcal{N}$ is invertible. Thus, we can write down the following Riccati algebraic equation for the $n \times n$-matrix $\mathcal{P}$:

$$\mathcal{P}\mathcal{A} + \mathcal{A}^T\mathcal{P} + \mathcal{P}\mathcal{S}\mathcal{P} + \mathcal{D} = 0, \quad (6)$$

where $\mathcal{S} = \gamma^{-2}\mathcal{F}\mathcal{F}^T - \mathcal{B}\mathcal{N}^{-1}\mathcal{B}^T$.

Along with the equation (6), we consider the differential equation

$$\frac{dZ(t)}{dt} = \big(\mathcal{A} - \mathcal{B}\mathcal{N}^{-1}\mathcal{B}^T\mathcal{P}\big)Z(t), \quad t \geq 0. \quad (7)$$

As a particular case of the results of (Glizer, 2009b) (Lemma 2.1), we directly have the following assertion.

Proposition 1. *Let there exist a symmetric solution $\mathcal{P}$ of the equation (6) such that the trivial solution to the equation (7) is asymptotically stable. Then, the controller*

$$u^0[Z(t)] = -\mathcal{N}^{-1}\mathcal{B}^T\mathcal{P}Z(t) \quad (8)$$

solves the H_∞ control problem (1)-(3).

Remark 1. *Proposition 1 presents the solvability conditions of the H_∞ control problem (1)-(3) and the controller solving this problem. Due to the expressions for the matrix $\mathcal{S}$ and the controller (8), one can use this proposition only if the matrix $\mathcal{N}$ is invertible, i.e., in the case where the rank of the matrix $\mathcal{M}$ equals r (the dimension of the control vector). Otherwise, Proposition 1 is not applicable to solution of the H_∞ control problem (1)-(3).*

The objective of this paper is to develop a method of solution of the H_∞ control problem (1)-(3) in the case where $\mathrm{rank}\mathcal{M} = q < r$. More precisely, we assume that the matrix $\mathcal{M}$ has the block form

$$\mathcal{M} = \big(\mathcal{M}_1, O_{p_2 \times (r-q)}\big), \quad (9)$$

where the block $\mathcal{M}_1$ is of dimension $p_2 \times q$, $O_{n_1 \times n_2}$ is zero matrix of the dimension $n_1 \times n_2$, and

$$\Lambda \stackrel{\triangle}{=} \mathcal{M}_1^T\mathcal{M}_1 = \mathrm{diag}\big(\lambda_1, ..., \lambda_q\big), \quad \lambda_l > 0, \quad l = 1, ..., q. \quad (10)$$

The H_∞ control problem (1)-(3) subject to (9) is a singular (nonstandard) H_∞ control problem.

3 TRANSFORMATION OF THE H_∞ CONTROL PROBLEM (1)-(3),(9)-(10)

Let us partition the matrix $\mathcal{B}$ into the blocks as:

$$\mathcal{B} = \Big(\mathcal{B}_1, \mathcal{B}_2\Big), \quad (11)$$

where the matrices $\mathcal{B}_1$ and $\mathcal{B}_2$ are of dimensions $n \times q$ and $n \times (r-q)$, respectively.

In what follows, we assume:
(A1) the matrix $\mathcal{B}$ has full column rank r;
(A2) $\det \mathcal{B}_2^T \mathcal{D} \mathcal{B}_2 \neq 0$, where $\mathcal{D}$ is given in (5).

Let $\mathcal{B}_c$ be a complement matrix to the matrix $\mathcal{B}$, i.e., the dimension of $\mathcal{B}_c$ is $n \times (n-r)$, and the block matrix $(\mathcal{B}_c, \mathcal{B})$ is nonsingular. Thus

$$\widetilde{\mathcal{B}}_c = (\mathcal{B}_c, \mathcal{B}_1) \tag{12}$$

is a complement to $\mathcal{B}_2$.

Consider the following matrices:

$$\mathcal{H} = (\mathcal{B}_2^T \mathcal{D} \mathcal{B}_2)^{-1} \mathcal{B}_2^T \mathcal{D} \widetilde{\mathcal{B}}_c, \quad \mathcal{L} = \widetilde{\mathcal{B}}_c - \mathcal{B}_2 \mathcal{H}. \tag{13}$$

Using the matrix $\mathcal{L}$, we transform the state in the H_∞ problem (1)-(3),(9)-(10) as follows:

$$Z(t) = (\mathcal{L}, \mathcal{B}_2)\, z(t), \tag{14}$$

where $z(t) \in E^n$ is a new state.

Due to the results of (Glizer et al., 2007), the transformation (14) is invertible.

Let us partition the matrix $\mathcal{H}$ into blocks as:

$$\mathcal{H} = \big(\mathcal{H}_1, \mathcal{H}_2\big), \tag{15}$$

where the blocks $\mathcal{H}_1$ and $\mathcal{H}_2$ are of the dimensions $(r-q) \times (n-r)$ and $(r-q) \times q$, respectively.

Based on the results of (Glizer and Kelis, 2015b) (Lemma 1), we directly obtain the following two assertions.

Proposition 2. *Let the assumptions A1-A2 be valid. Then, the transformation (14) converts the H_∞ control problem (1)-(3),(9)-(10) to the new H_∞ control problem for the system*

$$\frac{dz(t)}{dt} = Az(t) + Bu(t) + Fw(t), \quad z(0) = 0, \quad t \geq 0, \tag{16}$$

$$v(t) = \mathrm{col}\{Cz(t),\ Mu(t)\}, \tag{17}$$

and the cost functional

$$J(u,w) = \Big(\|v(t)\|_{L^2}\Big)^2 - \gamma^2 \Big(\|w(t)\|_{L^2}\Big)^2, \tag{18}$$

where

$$A = (\mathcal{L}, \mathcal{B}_2)^{-1} \mathcal{A} (\mathcal{L}, \mathcal{B}_2), \tag{19}$$

$$B = (\mathcal{L}, \mathcal{B}_2)^{-1} \mathcal{B} = \begin{pmatrix} O_{(n-r)\times q} & O_{(n-r)\times(r-q)} \\ I_q & O_{q\times(r-q)} \\ \mathcal{H}_2 & I_{r-q} \end{pmatrix}, \tag{20}$$

$$F = (\mathcal{L}, \mathcal{B}_2)^{-1} \mathcal{F}, \tag{21}$$

$$C = \mathcal{C} (\mathcal{L}, \mathcal{B}_2), \tag{22}$$

$$M = \mathcal{M}. \tag{23}$$

Let us partition the matrix C, given by (22), into blocks as:

$$C = \big(C_1, C_2\big), \tag{24}$$

where the blocks C_1 and C_2 are of the dimensions $p_1 \times (n-r+q)$ and $p_1 \times (r-q)$, respectively.

Corollary 1. *Let the assumptions A1-A2 be valid. Then, the matrix $D \overset{\triangle}{=} C^T C$ has the block form*

$$D = \begin{pmatrix} C_1^T C_1 & C_1^T C_2 \\ C_2^T C_1 & C_2^T C_2 \end{pmatrix}$$
$$= \begin{pmatrix} D_1 & O_{(n-r+q)\times(r-q)} \\ O_{(r-q)\times(n-r+q)} & D_2 \end{pmatrix}, \tag{25}$$

where

$$D_1 \overset{\triangle}{=} C_1^T C_1 = \mathcal{L}^T \mathcal{D} \mathcal{L}, \ \ D_2 \overset{\triangle}{=} C_2^T C_2 = \mathcal{B}_2^T \mathcal{D} \mathcal{B}_2. \tag{26}$$

The matrix D_1 is symmetric positive semi-definite, while the matrix D_2 is symmetric positive definite.

Due to (9)-(10),(23), the new H_∞ control problem (16)-(18) is singular. Similarly to the problem (1)-(3), we say that the controller $u^*[z(t)]$ solves the problem (16)-(18) if it guarantees the fulfilment of the inequality

$$J\big(u^*, w\big) \leq 0 \tag{27}$$

along trajectories of (16) for all $w(t) \in L^2[0,+\infty; E^m]$.

Lemma 1. *Let the assumptions A1-A2 be valid. If the controller $u^0[Z(t)]$ solves the H_∞ control problem (1)-(3), then the controller $u^0\big[(\mathcal{L}, \mathcal{B}_2) z(t)\big]$ solves the H_∞ control problem (16)-(18). Vice versa, if the controller $u^*[z(t)]$ solves the H_∞ control problem (16)-(18), then the controller $u^*\Big[\big(\mathcal{L}, \mathcal{B}_2\big)^{-1} Z(t)\Big]$ solves the H_∞ control problem (1)-(3).*

Proof. Let us start with the first statement of the lemma. Since the controller $u^0[Z(t)]$ solves the problem (1)-(3), then the inequality (4) is satisfied along trajectories of (1) for all $w(t) \in L^2[0,+\infty; E^m]$. Now, let us make the state transformation (14) in the problem (1)-(3). Due to this transformation and Proposition 2, the problem (1)-(3) becomes the problem (16)-(18). The inequality (4) becomes the inequality $J\Big(u^0\big[(\mathcal{L}, \mathcal{B}_2) z(t)\big], w(t)\Big) \leq 0$ along trajectories of (16) for all $w(t) \in L^2[0,+\infty; E^m]$, meaning that the controller $u^0\big[(\mathcal{L}, \mathcal{B}_2) z(t)\big]$ solves the problem (16)-(18). This completes the proof of the first statement. The second statement is proven similarly. □

Remark 2. *Due to Lemma 1, the initially formulated problem (1)-(3) is equivalent to the new problem (16)-(18). From the other hand, due to Proposition 2 (see (20)) and Corollary 1, the latter is simpler than the former. Therefore, in the sequel of this paper, we deal with the H_∞ control problem (16)-(18). We consider this problem as an original one and call it the Singular H_∞ Control Problem (SHICP).*

4 REGULARIZATION OF THE SHICP

4.1 Partial Cheap Control H_∞ Problem

To study the SHICP, we replace it with a regular H_∞ control problem, which is close in a proper sense to the SHICP. This new H_∞ control problem has the same equation of dynamics (16). However, the output equation in the new problem differs from the one in the SHICP. This output equation has the "regular" form, i.e., the rank of the matrix of coefficients for the control in this equation equals r (the dimension of the control vector), and it is close to the one in the SHICP.

Since $r \le p_2$, then $q < p_2$ and $r - q \le p_2$. Therefore, there exists a $\big(p_2 \times (r-q)\big)$-matrix $\mathcal{M}_2$ such that

$$\mathcal{M}_2^T \mathcal{M}_1 = O_{(r-q)\times q}, \quad \mathcal{M}_2^T \mathcal{M}_2 = I_{r-q}. \tag{28}$$

Based on this observation, we choose the regular output equation as:

$$v_\varepsilon(t) = \mathrm{col}\{Cz(t),\ M_\varepsilon u(t)\}, \tag{29}$$

where

$$M_\varepsilon = \big(\mathcal{M}_1, \varepsilon\mathcal{M}_2\big), \tag{30}$$

$\varepsilon > 0$ is a small parameter.

Using (10), (28) and (30), we obtain

$$N_\varepsilon \stackrel{\triangle}{=} M_\varepsilon^T M_\varepsilon = \begin{pmatrix} \Lambda & O_{q\times(r-q)} \\ O_{(r-q)\times q} & \varepsilon^2 I_{r-q} \end{pmatrix}, \tag{31}$$

$\mathrm{rank} N_\varepsilon = r$, and N_ε is positive definite for all $\varepsilon > 0$.

The cost functional, corresponding to the output equation (29), is

$$\begin{aligned} J_\varepsilon(u,w) &= \Big(\|v_\varepsilon(t)\|_{L^2}\Big)^2 - \gamma^2\Big(\|w(t)\|_{L^2}\Big)^2 \\ &= \int_0^{+\infty} \Big(z^T(t)Dz(t) + u^T(t)N_\varepsilon u(t) \\ &\qquad - \gamma^2 w^T(t)w(t)\Big)dt, \end{aligned} \tag{32}$$

where the matrix D is given by (25)-(26).

Remark 3. *Due to the smallness of the parameter ε and the form of the matrix N_ε, the H_∞ control problem (16),(29),(32) is a partial cheap control problem, i.e., the problem where the cost only of some (but not all) control coordinates is small. A total cheap control problem, i.e., the problem where the cost of all control coordinatesis small, has been extensively investigated in the literature in different settings. Thus, in ((Bikdash et al., 1993); (Glizer, 1999); (Glizer, 2005); (Glizer, 2009a); (Glizer, 2012c); (Glizer, 2012b); (Glizer, 2014); (Kurina and Hoai, 2014); (Mahadevan and Muthukumar, 2011); (O'Malley and Jameson, 1977); (Popescu and Gajic, 1999); (Saberi and Sannuti, 1987); (Seron et al., 1999)) (see also references therein) various optimal control problems for both, finite and infinite horizon total cheap control cost functionals, were studied. In (Turetsky et al., 2014), a robust trajectory tracking problem with a total cheap control was considered. In ((Glizer, 2000); (Glizer, 2016); (Glizer and Kelis, 2015a); (Glizer and Kelis, 2017); (Petersen, 1986); (Shinar et al., 2014); (Starr and Ho, 1969); (Turetsky and Glizer, 2007)) different differential games with total cheap control of at least one player were analyzed. In ((Glizer, 2009b); (Glizer, 2012a); (Glizer, 2013)), some H_∞ total cheap control problems were solved. However, partial cheap control problems were considered only in few works in the literature. Thus, in (O'Reilly, 1983) and (Glizer and Kelis, 2016), an infinite horizon linear-quadratic optimal control problem with partial cheap control for homogeneous and nonhomogeneous systems, respectively, was studied. In (Glizer and Kelis, 2015b), a zero-sum linear-quadratic differential game with partial cheap control for the minimizing player was analyzed. To the best of our knowledge, an H_∞ control problem with partial cheap control has not yet been considered in the literature. In what follows, we call the problem (16), (29), (32) the H_∞ Partial Cheap Control Problem (HIPCCP).*

Remark 4. *We say that the controller $u_\varepsilon^*[z(t)]$ solves the HIPCCP if it guarantees the fulfilment of the inequality*

$$J_\varepsilon\big(u_\varepsilon^*, w\big) \le 0 \tag{33}$$

along trajectories of (16) for all $w(t) \in L^2[0,+\infty;E^m]$.

4.2 Solvability Conditions of the HIPCCP

Since the matrix N_ε is positive definite for all $\varepsilon > 0$, then we can apply Proposition 1 to solve the HIPCCP. For this purpose, we write down the Riccati matrix algebraic equation

$$PA + A^T P + P\big(S_w - S_u(\varepsilon)\big)P + D = 0, \tag{34}$$

and the linear system

$$\frac{dz(t)}{dt} = \Big(A - S_u(\varepsilon)P\Big)z(t), \quad t \ge 0 \tag{35}$$

where

$$S_w = \gamma^{-2}FF^T, \quad S_u(\varepsilon) = BN_\varepsilon^{-1}B^T, \tag{36}$$

the matrix D is defined in Corollary 1.

By virtue of Proposition 1, we have the following assertion.

Proposition 3. *Let, for a given $\varepsilon > 0$, the equation*

(34) have a symmetric solution $P = P(\varepsilon)$ such that the trivial solution of the system (35) for $P = P(\varepsilon)$ is asymptotically stable. Then, the controller

$$u_\varepsilon^*[z(t)] = -N_\varepsilon^{-1} B^T P(\varepsilon) z(t) \tag{37}$$

solves the HIPCCP.

5 ASYMPTOTIC ANALYSIS OF THE EQUATION (34)

5.1 Equivalent Transformation of (34)

Substitution of the block representations of the matrices B and N_ε (see the equations (20) and (31)) into the expression for $S_u(\varepsilon)$ (see (36)), yields after a routine algebra the following block representation of this matrix:

$$S_u(\varepsilon) = \begin{pmatrix} S_{u_1} & S_{u_2} \\ S_{u_2}^T & (1/\varepsilon^2) S_{u_3}(\varepsilon) \end{pmatrix}, \tag{38}$$

where

$$S_{u_1} = \begin{pmatrix} O_{(n-r)\times(n-r)} & O_{(n-r)\times q} \\ O_{q\times(n-r)} & \Lambda^{-1} \end{pmatrix},$$
$$S_{u_2} = \begin{pmatrix} O_{(n-r)\times(r-q)} \\ \Lambda^{-1}\mathcal{H}_2^T \end{pmatrix},$$
$$S_{u_3}(\varepsilon) = \varepsilon^2 \mathcal{H}_2 \Lambda^{-1} \mathcal{H}_2^T + I_{r-q}, \tag{39}$$

Λ is given by (10).

Due to (38)-(39), the left-hand side of the equation (34) has a singularity at $\varepsilon = 0$. In order to remove this singularity, we seek the solution $P(\varepsilon)$ of this equation in the block-form

$$P(\varepsilon) = \begin{pmatrix} P_1(\varepsilon) & \varepsilon P_2(\varepsilon) \\ \varepsilon P_2^T(\varepsilon) & \varepsilon P_3(\varepsilon) \end{pmatrix}, \tag{40}$$

where the matrices $P_1(\varepsilon)$, $P_2(\varepsilon)$ and $P_3(\varepsilon)$ have the dimensions $(n-r+q)\times(n-r+q)$, $(n-r+q)\times(r-q)$ and $(r-q)\times(r-q)$, respectively, and

$$P_1^T(\varepsilon) = P_1(\varepsilon), \quad P_3^T(\varepsilon) = P_3(\varepsilon). \tag{41}$$

We also partition the matrices A and S_w into blocks as:

$$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}, \quad S_w = \begin{pmatrix} S_{w_1} & S_{w_2} \\ S_{w_2}^T & S_{w_3} \end{pmatrix}, \tag{42}$$

where the blocks A_1, A_2, A_3 and A_4 have the dimensions $(n-r+q)\times(n-r+q)$, $(n-r+q)\times(r-q)$, $(r-q)\times(n-r+q)$ and $(r-q)\times(r-q)$, respectively; the blocks S_{w_1}, S_{w_2} and S_{w_3} have the form

$$S_{w_1} = \gamma^{-2} F_1 F_1^T, \quad S_{w_2} = \gamma^{-2} F_1 F_2^T, \quad S_{w_3} = \gamma^{-2} F_2 F_2^T, \tag{43}$$

F_1 and F_2 are the upper and lower blocks of the matrix F of the dimensions $(n-r+q)\times s$ and $(r-q)\times s$, respectively, i.e.,

$$F = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}. \tag{44}$$

Substitution of the equations (25), (38), (40) and (42) into the equation (34) converts this equation after a routine algebra into the following equivalent set:

$$\begin{aligned}
&P_1(\varepsilon)A_1 + \varepsilon P_2(\varepsilon)A_3 + A_1^T P_1(\varepsilon) + \varepsilon A_3^T P_2^T(\varepsilon) \\
&\quad + P_1(\varepsilon)\left(S_{w_1} - S_{u_1}\right)P_1(\varepsilon) \\
&\quad + \varepsilon P_2(\varepsilon)\left(S_{w_2}^T - S_{u_2}^T\right)P_1(\varepsilon) \\
&\quad + \varepsilon P_1(\varepsilon)\left(S_{w_2} - S_{u_2}\right)P_2^T(\varepsilon) \\
&\quad + P_2(\varepsilon)\left(\varepsilon^2 S_{w_3} - S_{u_3}(\varepsilon)\right)P_2^T(\varepsilon) + D_1 = 0, \\
&P_1(\varepsilon)A_2 + \varepsilon P_2(\varepsilon)A_4 + \varepsilon A_1^T P_2(\varepsilon) + \varepsilon A_3^T P_3(\varepsilon) \\
&\quad + \varepsilon P_1(\varepsilon)\left(S_{w_1} - S_{u_1}\right)P_2(\varepsilon) \\
&\quad + \varepsilon^2 P_2(\varepsilon)\left(S_{w_2}^T - S_{u_2}^T\right)P_2(\varepsilon) \\
&\quad + \varepsilon P_1(\varepsilon)\left(S_{w_2} - S_{u_2}\right)P_3(\varepsilon) \\
&\quad + P_2(\varepsilon)\left(\varepsilon^2 S_{w_3} - S_{u_3}(\varepsilon)\right)P_3(\varepsilon) = 0, \\
&\varepsilon P_2^T(\varepsilon)A_2 + \varepsilon P_3(\varepsilon)A_4 + \varepsilon A_2^T P_2(\varepsilon) + \varepsilon A_4^T P_3(\varepsilon) \\
&\quad + \varepsilon^2 P_2^T(\varepsilon)\left(S_{w_1} - S_{u_1}\right)P_2(\varepsilon) \\
&\quad + \varepsilon^2 P_3(\varepsilon)\left(S_{w_2}^T - S_{u_2}^T\right)P_2(\varepsilon) \\
&\quad + \varepsilon^2 P_2^T(\varepsilon)\left(S_{w_2} - S_{u_2}\right)P_3(\varepsilon) \\
&\quad + P_3(\varepsilon)\left(\varepsilon^2 S_{w_3} - S_{u_3}(\varepsilon)\right)P_3(\varepsilon) + D_2 = 0.
\end{aligned} \tag{45}$$

5.2 Zero-order Asymptotic Solution of the Set (45)

We look for the zero-order asymptotic solution $\{P_{10}, P_{20}, P_{30}\}$ of the system (45). Equations for the terms of this asymptotic solution are obtained by setting formally $\varepsilon = 0$ in (45), which yields the set of the equations

$$P_{10}A_1 + A_1^T P_{10} + P_{10}\left(S_{w_1} - S_{u_1}\right)P_{10} - P_{20}P_{20}^T + D_1 = 0, \tag{46}$$
$$P_{10}A_2 - P_{20}P_{30} = 0, \tag{47}$$
$$\left(P_{30}\right)^2 - D_2 = 0. \tag{48}$$

The equation (48) has the solution

$$P_{30} = P_{30}^* \stackrel{\triangle}{=} \left(D_2\right)^{1/2}, \tag{49}$$

where the superscript "1/2" denotes the unique symmetric positive definite square root of the corresponding symmetric positive definite matrix.

Due to (49), the equation (47) yields the expression for P_{20}

$$P_{20} = P_{10}A_2\big(D_2\big)^{-1/2}, \tag{50}$$

where the superscript "$-1/2$" denotes the inverse matrix for the unique symmetric positive definite square root of the corresponding symmetric positive definite matrix.

Now, substituting (50) into (46), we obtain after some rearrangement the equation with respect to P_{10}

$$P_{10}A_1 + A_1^T P_{10} + P_{10}S_1P_{10} + D_1 = 0, \tag{51}$$

where

$$S_1 = S_{w_1} - S_{u_1} - A_2D_2^{-1}A_2^T. \tag{52}$$

Consider the matrix

$$\bar{M} = \begin{pmatrix} \mathcal{M}_1 & O_{p_2\times(r-q)} \\ O_{p_1\times q} & C_2 \end{pmatrix}. \tag{53}$$

Using the equation (10) and Corollary 1 yields

$$\bar{N} \stackrel{\triangle}{=} \bar{M}^T\bar{M} = \begin{pmatrix} \Lambda & O_{q\times(r-q)} \\ O_{(r-q)\times q} & D_2 \end{pmatrix}. \tag{54}$$

By virtue of the results of (Glizer and Kelis, 2015b) (Lemma 5), the matrix S_1 can be represented in the form

$$S_1 = S_{w_1} - \bar{B}\bar{N}^{-1}\bar{B}^T, \tag{55}$$

where

$$\bar{B} = \big(\ \widetilde{B}\ , A_2\ \big), \quad \widetilde{B} = \begin{pmatrix} O_{(n-r)\times q} \\ I_q \end{pmatrix}. \tag{56}$$

In what follows, we assume:

(A3) Riccati matrix algebraic equation (51) has a symmetric solution $P_{10} = P_{10}^*$ such that the trivial solution of the system

$$\frac{dx(t)}{dt} = \big(A_1 - \bar{B}\bar{N}^{\,-1}\bar{B}^{\,T}P_{10}^*\big)x(t), \quad t \geq 0 \tag{57}$$

is asymptotically stable.

5.3 H_∞-Control Interpretation of the Equation (51)

Consider the H_∞ control problem for the system

$$\frac{d\bar{x}(t)}{dt} = A_1\bar{x}(t) + \bar{B}\bar{u}(t) + F_1\bar{w}(t), \quad t \geq 0, \quad \bar{x}(t) = 0, \tag{58}$$

$$\bar{v}(t) = \mathrm{col}\{C_1\bar{x}(t), \bar{M}\bar{u}\}, \tag{59}$$

where $\bar{x}(t) \in E^{n-r+q}$ is a state vector; $\bar{u}(t) \in E^r$ is a control, $\bar{w}(t) \in E^m$ is a disturbance.

The cost functional of this problem is

$$\begin{aligned} \bar{J}(\bar{u},\bar{w}) &= \Big(\|\bar{v}(t)\|_{L^2}\Big)^2 - \gamma^2\Big(\|\bar{w}(t)\|_{L^2}\Big)^2 \\ &= \int_0^{+\infty}\Big(\bar{x}^{\,T}(t)D_1\bar{x}(t) + \bar{u}^{\,T}(t)\bar{N}\bar{u}(t) \\ &\qquad -\gamma^2\bar{w}^{\,T}(t)\bar{w}(t)\Big)dt. \end{aligned} \tag{60}$$

We call the H_∞ control problem (58)-(60) the Reduced H_∞ Control Problem (RHICP).

We say that the controller $\bar{u}^{\,*}[\bar{x}(t)]$ solves the RHICP if it guarantees the fulfilment of the inequality

$$\bar{J}\big(\bar{u}^{\,*},\bar{w}\big) \leq 0 \tag{61}$$

along trajectories of (58) for all $\bar{w}(t) \in L^2[0,+\infty;E^m]$.

Subject to the assumption (A3) and by virtue of Proposition 1, the RHICP is solvable. The controller

$$\bar{u}^{\,*}[\bar{x}(t)] = -\bar{N}^{\,-1}\bar{B}^{\,T}P_{10}^*\bar{x}(t) \tag{62}$$

solves this H_∞ control problem.

Thus, the equation (51) is connected with the RHICP by the solvability conditions of the latter.

Note, that the RHICP can be derived in another way, directly from the HIPCCP. Namely, let us partition the state vector $z(t)$ and the control vector $u(t)$ of the latter problem into blocks as:

$$z(t) = \mathrm{col}\big(x(t), y(t)\big),\ \ x(t) \in E^{n-r+q},\ \ y(t) \in E^{r-q}, \tag{63}$$

$$u(t) = \mathrm{col}\big(u_1(t), u_2(t)\big),\ \ u_1(t) \in E^q,\ \ u_2(t) \in E^{r-q}. \tag{64}$$

Now, using the block representations of the matrices B, C, A, F, M_ε, N_ε, $\widetilde{B}$ (see (20), (24), (42), (44), (30), (31), (56)) and Corollary 1, we can rewrite the HIPCCP (16), (29), (32) in the following equivalent form:

$$\begin{aligned} \frac{dx(t)}{dt} &= A_1x(t) + A_2y(t) \\ +\widetilde{B}u_1(t) + F_1w(t),\ \ t &\geq 0,\ \ x(0) = 0, \\ \frac{dy(t)}{dt} &= A_3x(t) + A_4y(t) \\ +\mathcal{H}_2u_1(t) + u_2(t) + F_2w(t),\ \ t &\geq 0,\ \ y(0) = 0, \end{aligned} \tag{65}$$

$$v_\varepsilon(t) = \mathrm{col}\{C_1x(t) + C_2y(t), \mathcal{M}_1u_1(t) + \varepsilon\mathcal{M}_2u_2(t)\}, \tag{66}$$

$$\begin{aligned} J_\varepsilon(u,w) &= \Big(\|v_\varepsilon(t)\|_{L^2}\Big)^2 - \gamma^2\Big(\|w(t)\|_{L^2}\Big)^2 \\ &= \int_0^{+\infty}\Big(x^T(t)D_1x(t) + y^T(t)D_2y(t) \\ &\quad + u_1^T(t)\Lambda u_1(t) + \varepsilon^2u_2^T(t)u_2(t) \\ &\quad -\gamma^2w^T(t)w(t)\Big)dt. \end{aligned} \tag{67}$$

The transformation of the control $\widetilde{u}_2(t) = \varepsilon u_2(t)$ converts the H_∞ control problem (65)-(67) to the equivalent problem

$$\frac{dx(t)}{dt} = A_1x(t) + A_2y(t) + \widetilde{B}u_1(t) + F_1w(t), \quad t \geq 0, \ \ x(0) = 0,$$
$$\varepsilon\frac{dy(t)}{dt} = \varepsilon\big(A_3x(t) + A_4y(t) + \mathcal{H}_2u_1(t)\big) + \widetilde{u}_2(t) + \varepsilon F_2w(t), \quad t \geq 0, \ \ y(0) = 0, \tag{68}$$

$$\widetilde{v}(t) = \mathrm{col}\{C_1x(t) + C_2y(t), \mathcal{M}_1u_1(t) + \mathcal{M}_2\widetilde{u}_2(t)\}, \tag{69}$$

$$\widetilde{J}(u_1, \widetilde{u}_2, w) = \Big(\|\widetilde{v}(t)\|_{L^2}\Big)^2 - \gamma^2\Big(\|w(t)\|_{L^2}\Big)^2 = \int_0^{+\infty}\Big(x^T(t)D_1x(t) + y^T(t)D_2y(t) + u_1^T(t)\Lambda u_1(t) + \widetilde{u}_2^T(t)\widetilde{u}_2(t) - \gamma^2 w^T(t)w(t)\Big)dt. \tag{70}$$

The problem (68)-(70) is an H_∞ control problem with a singularly perturbed dynamics. Namely, the system (68) is singularly perturbed.

Remark 5. *H_∞ control problems with singularly perturbed dynamics were studied extensively in the literature in various settings (see e.g. (Glizer and Fridman, 2000) and references therein).*

Now, we are going to show that the slow subproblem, associated with the problem (68)-(70), coincides with the RHICP. This slow subproblem is obtained similarly to ((Glizer, 2009b)) by setting formally $\varepsilon = 0$ in (68)-(70). Note, that the setting $\varepsilon = 0$ in the second equation of (68) yields $\widetilde{u}_2(t) = 0$, $t \geq 0$. Based on this observation and re-denoting $x(t)$, $y(t)$, $u_1(t)$, $w(t)$, $\widetilde{v}(t)$, $\widetilde{J}$ with $x_s(t)$, $y_s(t)$, $u_{1,s}(t)$, $w_s(t)$, $\widetilde{v}_s(t)$, $\widetilde{J}_s$, respectively, we obtain

$$\frac{dx_s(t)}{dt} = A_1x_s(t) + A_2y_s(t) + \widetilde{B}u_{1,s}(t) + F_1w_s(t), \quad t \geq 0, \ \ x_s(0) = 0, \tag{71}$$

$$\widetilde{v}_s(t) = \mathrm{col}\{C_1x_s(t) + C_2y_s(t), \mathcal{M}_1u_{1,s}(t)\}, \tag{72}$$

$$\widetilde{J}_s = \Big(\|\widetilde{v}_s(t)\|_{L^2}\Big)^2 - \gamma^2\Big(\|w_s(t)\|_{L^2}\Big)^2 = \int_0^{+\infty}\Big(x_s^T(t)D_1x_s(t) + y_s^T(t)D_2y_s(t) + u_{1,s}^T(t)\Lambda u_{1,s}(t) - \gamma^2 w_s^T(t)w_s(t)\Big)dt. \tag{73}$$

Remark 6. *Since the variable $y_s(t)$ in the problem (71)-(73) does not satisfy any equation for $t \in [0,+\infty)$, we can choose this variable to satisfy a desirable property of the system (71)-(72). This means that the variable $y_s(t)$ can be considered as an additional control variable in this system. Thus, the functional (73), calculated along trajectories of this system, depends on the control variable $\hat{u}_s(t) \stackrel{\triangle}{=} \mathrm{col}\big(u_{1,s}(t), y_s(t)\big)$ and the disturbance $w_s(t) \in L^2[0,+\infty; E^q]$, i.e., $\widetilde{J}_s = \widetilde{J}_s(\hat{u}_s, w_s)$. Thus, the slow subproblem, associated with the HIPCCP, is to find a controller $\hat{u}_s^*[x_s(t)]$ that ensures the inequality $\widetilde{J}_s(\hat{u}_s^*, w_s) \leq 0$ along trajectories of (71) for all $w_s \in L_2[0,+\infty; E^m]$. This H_∞ control problem is called the Slow H_∞ Control Subproblem (SHICSP) associated with the HIPCCP.*

Due to Remark 6, the RHICP equation of dynamics (58) and the integral form of the cost functional (60) for $\bar{u}(t) = \hat{u}_s(t)$ coincide with the equation of dynamics (71) and the integral form of the cost functional (73), respectively, in the SHICSP associated with the HIPCCP. This means that the RHICP and the SHICSP are identical to each other. Thus, due to (62), the controller

$$\hat{u}_s^*[x_s(t)] \stackrel{\triangle}{=} \mathrm{col}\big(u_{1,s}^*[x_s(t)], y_s^*[x_s(t)]\big) = -\bar{N}^{\,-1}\bar{B}^{\,T}P_{10}^*x_s(t) \tag{74}$$

solves the SHICSP. Using (54) and (56), the blocks $u_{1,s}^*[x_s(t)]$ and $y_s^*[x_s(t)]$ of this controller can be represented in the form $u_{1,s}^*[x_s(t)] = -\Lambda^{-1}\widetilde{B}^{\,T}P_{10}^*x_s(t)$, $y_s^*[x_s(t)] = -D_2^{-1}A_2^TP_{10}^*x_s(t)$.

5.4 Justification of the Asymptotic Solution to the Equation (34)

We assume that:
(A4) The trivial solution of the system

$$\frac{dx(t)}{dt} = \big(A_1 + S_1P_{10}^*\big)x(t), \quad x(t) \in E^{n-r+q}, \quad t \geq 0 \tag{75}$$

is asymptotically stable, where P_{10}^* is the solution of the equation (51) mentioned in the assumption (A3).

Let us denote

$$P_{20}^* \stackrel{\triangle}{=} P_{10}^*A_2\big(D_2\big)^{-1/2}. \tag{76}$$

Lemma 2. *Let the assumptions (A1)-(A2),(A4) be valid. Then, there exists a positive number ε_0, such that for all $\varepsilon \in (0, \varepsilon_0]$ the equation (34) has the symmetric solution $P(\varepsilon)$ of the block-form (40), and the blocks $P_i(\varepsilon)$, ($i = 1,2,3$) of this solution satisfy the inequalities*

$$\|P_i(\varepsilon) - P_{i0}^*\| \leq a\varepsilon, \quad i = 1,2,3, \quad \varepsilon \in (0, \varepsilon_0], \tag{77}$$

where $a > 0$ is some constant independent of ε.

Proof. Based on the Implicit Function Theorem (Schwartz, 1967) (Chapter III, paragraph 8), the lemma is proven similarly to (Kokotovic et al., 1986) (Theorem 4.2). □

Lemma 3. *Let the assumptions (A1)-(A4) be valid. Then, there exists a positive number* $\varepsilon_1 \le \varepsilon_0$*, such that for any* $\varepsilon \in (0, \varepsilon_1]$ *the trivial solution of the system (35) with* $P = P(\varepsilon)$ *is asymptotically stable.*

Proof. Substitution of the block representations of the vector $z(t)$ and the matrices $S(\varepsilon)$, $P(\varepsilon)$, A (see (63) and (38), (40), (42)) into the system (35) yields after a routine algebra the following equivalent system:

$$\begin{aligned}\frac{dx(t)}{dt} &= \big(A_1 - S_{u_1}P_1(\varepsilon) - \varepsilon S_{u_2}P_2^T(\varepsilon)\big)x(t)\\ &+\big(A_2 - \varepsilon S_{u_1}P_2(\varepsilon) - \varepsilon S_{u_2}P_3(\varepsilon)\big)y(t), \quad t \ge 0,\\ \varepsilon\frac{dy(t)}{dt} &= \big(\varepsilon A_3 - \varepsilon S_{u_2}^T P_1(\varepsilon) - S_{u_3}(\varepsilon)P_2^T(\varepsilon)\big)x(t)\\ &+\big(\varepsilon A_4 - \varepsilon^2 S_{u_2}^T P_2(\varepsilon) - S_{u_3}(\varepsilon)P_3(\varepsilon)\big)y(t), \quad t \ge 0.\end{aligned} \tag{78}$$

Remember that the parameter $\varepsilon > 0$ is small. Therefore, the system (78) is singularly perturbed (Kokotovic et al., 1986). To prove the asymptotic stability of the trivial solution to this system, we use the results of (Kokotovic et al., 1986) (Corollary 3.1). By virtue of these results, if the trivial solutions of the slow and fast subsystems associated with the singularly perturbed system (78) are asymptotically stable, then for all sufficiently small $\varepsilon > 0$ the trivial solution of the system (78) itself is asymptotically stable.

The slow subsystems associated with the system (78) is obtained in the following two steps. First, setting formally $\varepsilon = 0$ in (78), using the equation (39), the inequalities (77), and re-denoting $x(t)$ and $y(t)$ with $x_s(t)$ and $y_s(t)$, respectively, we obtain the system

$$\begin{aligned}\frac{dx_s(t)}{dt} &= \big(A_1 - S_{u_1}P_{10}^*\big)x_s(t) + A_2 y_s(t), \quad t \ge 0,\\ 0 &= \big(P_{20}^*\big)^T x_s(t) + P_{30}^* y_s(t), \quad t \ge 0.\end{aligned} \tag{79}$$

Then, eliminating $y_s(t)$ from (79), and using the equations (49) and (50) yield the slow subsystem associated with (78)

$$\frac{dx_s(t)}{dt} = \Big(A_1 - \big(S_{u_1} + A_2(D_2)^{-1}A_2^T\big)P_{10}^*\Big)x_s(t),\ t \ge 0. \tag{80}$$

Comparing the equations (52) and (55), we obtain that $S_{u_1} + A_2(D_2)^{-1}A_2^T = \bar{B}\bar{N}^{-1}\bar{B}^T$. Therefore, the differential equation (80) coincides with the differential equation (57). Hence, due to the assumption (A3), the trivial solution of (80) is asymptotically stable.

The fast subsystem associated with (78) is obtained from the second equation of this system in the following formal way. First, we remove from this equation the term depending on $x(t)$. Second, we make in the obtained equation the transformation of variables $t = \varepsilon\xi$, $y_f(\xi) = y(\varepsilon\xi)$, where ξ and $y_f(\xi)$ are new independent variable and state variable. Finally, setting formally $\varepsilon = 0$ in the transformed equation yields the fast subsystem

$$\frac{dy_f(\xi)}{d\xi} = -P_{30}^* y_f(\xi), \quad \xi \ge 0. \tag{81}$$

Since the matrix $P_{30}^* = (D_2)^{1/2}$ is positive definite, the trivial solution of the differential equation (81) is asymptotically stable. Therefore, by virtue of the above mentioned results of (Kokotovic et al., 1986), there exists a positive number ε_1 such that, for all $\varepsilon \in (0, \varepsilon_1]$, the trivial solution of the system (78) is asymptotically stable. Since the system (35) with $P = P(\varepsilon)$ is equivalent to (78), the trivial solution of the former also is asymptotically stable for all $\varepsilon \in (0, \varepsilon_1]$. Thus, the lemma is proven. □

Corollary 2. *Let the assumptions (A1)-(A4) be valid. Then, for all* $\varepsilon \in (0, \varepsilon_1]$*, the controller (37) solves the HIPCCP.*

Proof. The corollary is a direct consequence of Proposition 3, Lemma 2 and Lemma 3. □

6 SOLUTION OF THE SHICP

6.1 Controller for the SHICP: Formal Design

First of all, let us note the following. Due to the equations (17), (18), (29), (32),

$$J\big(u_\varepsilon^*[z(t)], w(t)\big) \le J_\varepsilon\big(u_\varepsilon^*[z(t)], w(t)\big) \tag{82}$$

along trajectories of the equation (16) for all $\varepsilon \in (0, \varepsilon_1]$ and all $w(t) \in L^2[0, +\infty; E^m]$. Therefore, the controller $u_\varepsilon^*[z(t)]$, solving the HIPCCP (see Corollary 2), also solves the SHICP. However, the design of $u_\varepsilon^*[z(t)]$ is a complicated task, because it requires the solution of a high dimension system of nonlinear algebraic equations depending on a parameter. To overcome this difficulty, we propose in this subsection another (simplified) controller for the SHICP.

Consider the matrix

$$P_0^*(\varepsilon) = \begin{pmatrix} P_{10}^* & \varepsilon P_{20}^* \\ \varepsilon\big(P_{20}^*\big)^T & \varepsilon P_{30}^* \end{pmatrix}, \quad \varepsilon \in (0, \varepsilon_1]. \tag{83}$$

Based on the matrix $P_0^*(\varepsilon)$, we consider the following auxiliary controller, obtained from the controller $u_\varepsilon^*[z(t)]$ (see (37)) by replacing $P(\varepsilon)$ with $P_0^*(\varepsilon)$:

$$u_{\rm aux}[z(t)] = -N_\varepsilon^{-1}B^T P_0^*(\varepsilon)z(t). \tag{84}$$

Substitution of the block representations for the matrices B, N_ε, $P_0^*(\varepsilon)$ and the vector $z(t)$ (see (20), (31), (83) and (63)) into (84), and use of the block form of the matrix $\widetilde{B}$ (see (56)) yield after a routine algebra the block representation for the vector $u_{\rm aux}[z(t)]$

$$u_{\rm aux}[z(t)] = -\begin{pmatrix} \Lambda^{-1}\Big[K_1(\varepsilon)x(t)+\varepsilon K_2(\varepsilon)y(t)\Big] \\ \\ \frac{1}{\varepsilon}\Big[\big(P_{20}^*\big)^T x(t) + P_{30}^* y(t)\Big] \end{pmatrix}, \tag{85}$$

where Λ and $\mathcal{H}_2$ are defined in (10) and (15), respectively, $K_1(\varepsilon) \stackrel{\triangle}{=} \widetilde{B}^T P_{10}^* + \varepsilon\mathcal{H}_2^T\big(P_{20}^*\big)^T$, $K_2(\varepsilon) \stackrel{\triangle}{=} \widetilde{B}^T P_{20}^* + \varepsilon\mathcal{H}_2^T P_{30}^*$.

Now, calculating the point-wise (with respect to $z(t)\in E^n$) limit of the upper block in (85) for $\varepsilon\to 0^+$, we obtain the simplified controller for the SHICP

$$u_{\varepsilon,0}^*[z(t)] = \begin{pmatrix} -\Lambda^{-1}\widetilde{B}^T P_{10}^* x(t) \\ \\ -\frac{1}{\varepsilon}\Big[\big(P_{20}^*\big)^T x(t)+P_{30}^* y(t)\Big] \end{pmatrix}. \tag{86}$$

Remark 7. *Comparing the controller $u_{\varepsilon,0}^*[z(t)]$ with the controller $\hat{u}_s^*[x_s(t)]$, solving the SHICSP (see (74)), one can conclude that the upper blocks of these controllers coincide with each other.*

6.2 Properties of the Controller (86)

Let for given $\varepsilon > 0$ and $w(t)\in L^2[0,+\infty;E^m]$, $z_0^*\big(t,\varepsilon;w(\cdot)\big)$, $t\geq 0$ be the solution of the initial-value problem (16) with $u(t)=u_{\varepsilon,0}^*[z(t)]$. Let

$$K_0^* \stackrel{\triangle}{=} \Big(\big(P_{20}^*\big)^T, P_{30}^*\Big). \tag{87}$$

Theorem 1. *Let the assumptions (A1)-(A4) be valid. Then, there exists a positive number ε_0^* such that for all $\varepsilon\in(0,\varepsilon_0^*]$ and $w(t)\in L_2[0,+\infty;E^m]$ the following inequality is satisfied*

$$J\big(u_{\varepsilon,0},w\big) \leq -\int_0^{+\infty}\big[z_0^*\big(t,\varepsilon;w(\cdot)\big)\big]^T\big(K_0^*\big)^T K_0^* z_0^*\big(t,\varepsilon;w(\cdot)\big)dt \tag{88}$$

along trajectories of the system (16).

Proof. To save the space, we present here a sketch of the proof.

First, we are going to show that the controller $u_{\varepsilon,0}^*[z(t)]$ solves the HIPCCP for all sufficiently small $\varepsilon>0$, i.e., for all such ε and all $w(t)\in L^2[0,+\infty;E^m]$, the following inequality is satisfied:

$$J_\varepsilon\big(u_{\varepsilon,0}^*,w\big)\leq 0. \tag{89}$$

Substitution of $u_{\varepsilon,0}^*[z(t)]$ into this problem and use of the block representations for the matrices A, B, F, D, N_ε, and for the vectors $z(t)$, $u_{\varepsilon,0}^*[z(t)]$ (see (42), (20), (44), (25), (31) and (63), (86)) transform the equation of dynamics (16) and the functional (32) of the HIPCCP as follows:

$$\frac{dz(t)}{dt} = \hat{A}(\varepsilon)z(t)+Fw(t), \quad z(0)=0, \tag{90}$$

$$\hat{J}_\varepsilon(w) \stackrel{\triangle}{=} J_\varepsilon(u_{\varepsilon,0}^*,w) = \int_0^{+\infty}\Big(z^T(t)\hat{D}z(t)-\gamma^2 w^T(t)w(t)\Big)dt, \tag{91}$$

where

$$\hat{A}(\varepsilon) = \begin{pmatrix} \hat{A}_1 & \hat{A}_2 \\ \hat{A}_3(\varepsilon) & \hat{A}_4(\varepsilon)\end{pmatrix}, \ \hat{D} = \begin{pmatrix} \hat{D}_1 & \hat{D}_2 \\ \hat{D}_2^T & \hat{D}_3\end{pmatrix},$$

$\hat{A}_1 = A_1 - \widetilde{B}\Lambda^{-1}\widetilde{B}^T$, $\hat{A}_2 = A_2$, $\hat{A}_3(\varepsilon) = A_3 - \mathcal{H}_2\Lambda^{-1}\widetilde{B}^T P_{10}^* - (1/\varepsilon)\big(P_{20}^*\big)^T$, $\hat{A}_4(\varepsilon) = A_4-(1/\varepsilon)P_{30}^*$, $\hat{D}_1 = D_1 + P_{10}^*\widetilde{B}\Lambda^{-1}\widetilde{B}^T P_{10}^*$, $\hat{D}_2 = P_{20}^*P_{30}^*$, $\hat{D}_3 = D_2 + \big(P_{30}^*\big)^2$.

Due to (91), the inequality (89) is equivalent to the following inequality for all sufficiently small $\varepsilon>0$ and all $w(t)\in L^2[0,+\infty;E^m]$:

$$\hat{J}_\varepsilon(w)\leq 0. \tag{92}$$

Similarly to Corollary 3, it is shown that the trivial solution to the differential equation in (90) is asymptotically stable for all sufficiently small $\varepsilon>0$. This observation yields the following limit equality for any $w(t)\in L^2[0,+\infty;E^m]$ and any sufficiently small $\varepsilon>0$:

$$\lim_{t\to+\infty} z_0\big(t,\varepsilon;w(\cdot)\big)=0. \tag{93}$$

Now, let us consider the Riccati matrix algebraic equation with respect to the matrix $\hat{P}$

$$\hat{P}\hat{A}(\varepsilon)+\hat{A}^T(\varepsilon)\hat{P}+\hat{P}S_w\hat{P}+\hat{D}=0. \tag{94}$$

Similarly to Lemma 2, it is shown that for all sufficiently small $\varepsilon>0$ the equation (94) has a symmetric solution $\hat{P}=\hat{P}^*(\varepsilon)$. Using this observation, we consider the Lyapunov-like function $V(z,\varepsilon)=z^T\hat{P}^*(\varepsilon)z$, $z\in E^n$. Analyzing the behavior of $V(z,\varepsilon)$ along trajectories of the equation in (90) and using the equation (93), we prove the validity of the inequality (92) and, therefore, the inequality (89). The latter, along with the equations (18), (32), (87) and (91), yields the statement of the theorem. □

Theorem 2. *Let the assumptions (A1)-(A4) be valid. Then, there exists a positive number* ε_1^* *such that for all* $\varepsilon \in (0,\varepsilon_1^*]$ *and* $w(t) \in L^2[0,+\infty;E^m]$ *the integral in the right-hand part of (88), being nonnegative, satisfies the inequality*

$$\int_0^{+\infty} \left[z_0^*\big(t,\varepsilon;w(\cdot)\big)\right]^T \left(K_0^*\right)^T K_0^* z_0^*\big(t,\varepsilon;w(\cdot)\big)dt \le a\varepsilon\Big(\|w(t)\|_{L^2}\Big)^2, \quad (95)$$

where $a > 0$ *is some constant independent of* ε *and* $w(\cdot)$.

Proof. Here, we also present a sketch of the proof. Let the $n\times n$-matrix $\Phi(t,\varepsilon)$ be the fundamental matrix solution of the equation $dz(t)/dt = \hat{A}(\varepsilon)z(t)$, i.e., this matrix satisfies the following initial-value problem:

$$\frac{d\Phi(t,\varepsilon)}{dt} = \hat{A}(\varepsilon)\Phi(t,\varepsilon),\ \ t\ge 0,\ \ \Phi(0,\varepsilon) = I_n. \quad (96)$$

Then,

$$z_0^*\big(t,\varepsilon;w(\cdot)\big) = \int_0^t \Phi(t-\sigma,\varepsilon)Fw(\sigma)d\sigma,\ \ t\ge 0. \quad (97)$$

Let us partition the vector-valued function $z_0^*\big(t,\varepsilon;w(\cdot)\big)$ into blocks as $z_0^*\big(t,\varepsilon;w(\cdot)\big) = \mathrm{col}\Big(x_0^*\big(t,\varepsilon;w(\cdot)\big), y_0^*\big(t,\varepsilon;w(\cdot)\big)\Big)$, $x_0^*\big(t,\varepsilon;w(\cdot)\big) \in E^{n-r+q}$, $y_0^*\big(t,\varepsilon;w(\cdot)\big) \in E^{r-q}$. Now, a proper asymptotic analysis of the problem (96), the use of the equation (97) and the Cauchy-Bunyakovsky-Schwarz integral inequality yield the existence of a positive number ε_1^* such that for all $\varepsilon \in (0,\varepsilon_1^*]$ and all $w(t) \in L^2[0,+\infty;E^m]$ the following inequalities are satisfied:

$$\left\|x_0^*\big(t,\varepsilon;w(\cdot)\big) - \hat{x}\big(t;w(\cdot)\big)\right\| \le a_1\varepsilon^{1/2}\|w(t)\|_{L^2},\ \ t\ge 0, \quad (98)$$

$$\left\|y_0^*\big(t,\varepsilon;w(\cdot)\big) - \hat{y}\big(t;w(\cdot)\big)\right\| \le a_1\varepsilon^{1/2}\|w(t)\|_{L^2},\ \ t\ge 0, \quad (99)$$

where $a_1 > 0$ is some constant independent of ε and $w(\cdot)$,

$$\hat{x}\big(t;w(\cdot)\big) = \int_0^t \hat{\Phi}_x(t-\sigma)F_1w(\sigma)d\sigma,\ \ t\ge 0, \quad (100)$$

$$\hat{y}\big(t;w(\cdot)\big) = \int_0^t \hat{\Phi}_y(t-\sigma)F_1w(\sigma)d\sigma,\ \ t\ge 0, \quad (101)$$

the $(n-r+q)\times(n-r+q)$-matrix-valued function $\hat{\Phi}_x(t)$ is the solution of the initial-value problem

$$\frac{d\hat{\Phi}_x(t)}{dt} = \big(\hat{A}_1 - A_2(P_{30}^*)^{-1}(P_{20}^*)^T\big)\hat{\Phi}_x(t),\ \ t\ge 0,$$

$$\hat{\Phi}_x(0) = I_{n-r+q},$$

and the $(r-q)\times(n-r+q)$-matrix-valued function $\hat{\Phi}_y(t)$ has the form $\hat{\Phi}_y(t) = -(P_{30}^*)^{-1}(P_{20}^*)^T\hat{\Phi}_x(t)$. The latter expression, along with the equations (100)-(101), yields for all $t\ge 0$, $w(t)\in L^2[0,+\infty;E^m]$:

$$(P_{20}^*)^T\hat{x}\big(t;w(\cdot)\big) + P_{30}^*\hat{y}\big(t;w(\cdot)\big) = 0. \quad (102)$$

Now, using the equations (87), (102) and the inequalities (98)-(99), we obtain the following inequality for all $\varepsilon\in(0,\varepsilon_1^*]$ and all $w(t)\in L^2[0,+\infty;E^m]$:

$$\left\|K_0^* z_0\big(t,\varepsilon;w(\cdot)\big)\right\| \le a_2\varepsilon^{1/2}\|w(t)\|_{L^2},$$

where $a_2 > 0$ is some constant independent of ε and $w(\cdot)$. This inequality directly yields the inequality (95). □

The following corollary is a direct consequence of Theorem 2.

Corollary 3. *Let the assumptions (A1)-(A4) be valid. Then, there exists a positive number* ε_2^* *and a function* $g(\varepsilon)$, $(0\le g(\varepsilon)\le a\varepsilon$, $\varepsilon\in(0,\varepsilon_2^*]$, *the constant* $a > 0$ *is defined in Theorem 2), such that for all* $\varepsilon\in(0,\varepsilon_2^*]$ *the controller* $u_{\varepsilon,0}^*[z(t)]$ *solves the singular* H_∞ *control problem for the system (16) with the following functional:*

$$J_g(u,w) = \int_0^{+\infty}\Big[z^T(t)Dz(t) - (\gamma_g(\varepsilon))^2 w^T(t)w(t)\Big]dt,$$

where the performance level $\gamma_g(\varepsilon)$ *has the form* $\gamma_g(\varepsilon) = \sqrt{\gamma^2 - g(\varepsilon)} > 0$.

Remark 8. *Due to Theorems 1, 2 and Corollary 3, the controller* $u_{\varepsilon,0}^*[z(t)]$ *solves not only the original SHICP (16), (18), but also the singular* H_∞ *control problem with the same dynamics (16) and the new cost functional* $J_g(u,w)$. *The latter has a smaller performance level* $\gamma_g(\varepsilon)$ *than SHICP. This performance level satisfies the limit equality* $\lim_{\varepsilon\to 0^+}\gamma_g(\varepsilon) = \gamma$.

7 EXAMPLE

To illustrate the theoretical results of the paper, we consider the following example of the problem (16)-(18) with the data: $n = r = m = 2$, $q = 1$ and

$$A = \begin{pmatrix} 0 & 2 \\ 8 & 1 \end{pmatrix},\ B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix},\ D = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix},$$

$$F = \begin{pmatrix} -1 & 2 \\ -4 & 3 \end{pmatrix},\ \ M = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ \ \gamma = 1. \quad (103)$$

Using these data, we obtain:

$$u_{\varepsilon,0}^*[z(t)] = \begin{pmatrix} -\sqrt{2}x(t) \\ -\frac{1}{\varepsilon}\Big(4\sqrt{2}x(t) + y(t)\Big) \end{pmatrix}, \quad (104)$$

where $z(t) = \mathrm{col}\big(x(t), y(t)\big)$.

In Table 1, the minimum H_∞ performance level γ_ε (H_∞-norm) of the system (16)-(17), subject to the data (103) and the control $u(t) = u^*_{\varepsilon,0}[z(t)]$, is presented for various values of ε.

Table 1: Minimum H_∞ performance level.

ε	0.5	0.25	0.1	0.05	0.025
γ_ε	2.455	1.736	1.193	0.994	0.988

It is seen that γ_ε decreases for the decreasing ε. Moreover, for sufficiently small ε, the value of γ_ε becomes smaller than the performance level $\gamma = 1$ in the H_∞ control problem of this example.

8 CONCLUSIONS

An H_∞ control problem for a linear system was considered. The feature of the problem is that the matrix of coefficients for the control in the quadratic cost functional is singular but, in general, non-zero. The control coordinates presenting in the cost functional are regular, while the other ones are singular. Under proper assumptions, the linear system was transformed equivalently to the system consisting of three modes. The first mode is not controlled directly, the second mode is controlled by the regular control coordinates, while the third mode is controlled by the entire control. Due to this transformation, the initially formulated H_∞ control problem was converted to a new singular H_∞ control problem. This new problem was solved by a regularization approach, i.e., by its approximate transformation to an auxiliary regular H_∞ control problem. The latter has the same equation of dynamics and a similar cost functional augmented by an integral of the squares of the singular control coordinates with a small positive weight ε^2, ($\varepsilon > 0$). Hence, the auxiliary problem is an H_∞ partial cheap control problem. An asymptotic solution of the ε-dependent Riccati matrix algebraic equation, associated with this partial cheap control problem by the solvability conditions, was constructed and justified. Based on this asymptotic solution, a simplified controller for the H_∞ partial cheap control problem was designed. It was shown that this controller also solves the singular H_∞ control problem. Moreover, it was shown that this controller also solves a singular H_∞ control problem with a smaller performance level, depending on ε. This smaller performance level tends to the original one for $\varepsilon \to 0^+$.

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