Optimal Price Reaction Strategies in the Presence of Active and Passive Competitors

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Abstract: Many markets are characterized by pricing competition. Typically, competitors are involved that adjust their prices in response to other competitors with different frequencies. We analyze stochastic dynamic pricing models under competition for the sale of durable goods. Given a competitor's pricing strategy, we show how to derive optimal response strategies that take the anticipated competitor's price adjustments into account. We study resulting price cycles and the associated expected long-term profits. We show that reaction frequencies have a major impact on a strategy's performance. In order not to act predictable our model also allows to include randomized reaction times. Additionally, we study to which extent optimal response strategies of active competitors are affected by additional passive competitors that use constant prices. It turns out that optimized feedback strategies effectively avoid a decline in price. They help to gain profits, especially, when aggressive competitors are involved.

1 INTRODUCTION

In many markets, firms have to deal with competition and stochastic demand. Sellers are required to choose appropriate pricing decisions to maximize their expected profits. In E-commerce, it has become easy to observe and to change prices. Hence, dynamic pricing strategies that take the competitors' strategies into account will be used increasingly. However, optimal price reactions are not easy to find. While some market participants use mostly constant prices others use automated price adjustment strategies. Applications can be found in a variety of contexts that involve perishable (e.g., fashion goods, seasonal products, event tickets) as well as durable goods (e.g., books, natural resources, gasoline). In many markets, it can be observed, that the application of response strategies typically leads to cyclic price patterns over time, cf. Edgeworth cycles, see, e.g., Maskin, Tirole (1988), Noel (2007). We want to explain such effects from a theoretical perspective.

In this paper, we study oligopoly pricing models in a stochastic dynamic framework. In our model, the sales probabilities are allowed to be an arbitrary function of time and the competitors' prices. Our aim is to take into account (i) various competitors' strategies, (ii) different (randomized) reaction times, and (iii) additional passive competitors that use constant prices.

Selling products is a classical application of revenue management theory. The problem is closely related to the field of dynamic pricing, which is summarized in the books by Talluri, van Ryzin (2004), Phillips (2005), and Yeoman, McMahon-Beattie (2011). The survey by Chen, Chen (2015) provides an excellent overview of recent pricing models under competition.

In the article by Gallego, Wang (2014) the authors consider a continuous time multi-product oligopoly for differentiated perishable goods. They use optimality conditions to reduce the multi-dimensional dynamic pricing problem to a one-dimensional one. Gallego, Hu (2014) analyze structural properties of equilibrium strategies in more general oligopoly models for the sale of perishable products. The solution of their model is based on a deterministic version of the model. Martinez-de-Albeniz, Talluri (2011) consider duopoly and oligopoly pricing models for identical products. They use a general stochastic counting process to model customer's demand.

Further related models are studied by Yang, Xia (2013) and Wu, Wu (2015). Dynamic pricing models under competition that also include strategic customers are analyzed by Levin et al. (2009) and Liu, Zhang (2013). Dynamic pricing competition models with limited demand information are analyzed by Tsai, Hung (2009), Adida, Perakis (2010) and Chung

et al. (2012) using robust optimization and learning approaches. Many models consider continuous time models with finite horizon and limited inventory. In most existing models, discounting is not included and the demand is assumed to be of a special functional form. We consider infinite horizon models with unlimited inventory (i.e., products can be reproduced or reordered). Demand is allowed to depend generally on time as well as the prices of all market participants.

While many publications concentrate on (the existence of) equilibrium strategies, we do not assume that all market participants act rationally. In many markets it can be observed that automated strategies that are used by firms are relatively simple and aggressive. The most common strategy is to slightly undercut the competitor's price, cf. Kephart et al. (2000). In order to be able to respond to various potentially suboptimal pricing strategies we provide applicable solution algorithms that allow to compute optimal response strategies.

The main contribution of this paper is threefold. We (i) derive optimal price response strategies that anticipate competitors' prices, (ii) we quantify the impact of different (randomized) reaction times on expected long-term profits of all market participants, and (iii) we are able to explain different types of price cycles.

This paper is organized as follows. In Section 2, we describe the stochastic dynamic oligopoly model with infinite time horizon (durable goods). We allow sales probabilities to depend on competitor prices as well as on time (seasonal effects). The state space is characterized by time and the actual competitors' prices. The stochastic dynamic control problem is expressed in discrete time. In Section 3, we consider a duopoly competition. The competitor is assumed to frequently adjust its prices using a predetermined strategy. We assume that the price reactions of competitors as well as their reaction times can be anticipated. We set up a firm's Hamilton-Jacobi-Bellman equation and use recursive methods (value iteration) to approximate the value function. We are able to compute optimal feedback prices as well as expected long-term profits of the two competing firms. Evaluating price paths over time, we are able to explain specific price cycles. Furthermore, the results obtained are generalized to scenarios with randomized reaction times and mixed strategies.

In Section 4, we analyze optimal response strategies in the presence of active and passive competitors. We study how the duopoly game of two active competitors is affected by additional passive competitors. We show how to compute optimal pricing strategies and to evaluate expected profits. We also illustrate how the cyclic price paths of the active competitors are affected by different price levels of passive competitors. Finally, we evaluate the expected profits when different strategies are played against each other. Conclusions and managerial recommendations are offered in final Section 5.

2 MODEL DESCRIPTION

We consider the situation where a firm wants to sell goods (e.g., gasoline, groceries, technical devices) on a digital market platform (e.g., Amazon, eBay). We assume that several sellers compete for the same market, i.e., customers are able to compare prices of different competitors.

We assume that the time horizon is infinite. We assume that firms are able to reproduce or reorder products (promise to deliver), and the ordering is decoupled from pricing decisions. If a sale takes place, shipping costs c have to be paid, $c \geq 0$. A sale of one item at price a, $a \geq 0$, leads to a net profit of $a - c$. Discounting is also included in the model. For the length of one period, we will use the discount factor δ, $0 < \delta < 1$.

Since in many practical applications prices cannot be continuously adjusted, we consider a discrete time model. The sales intensity of our product is denoted by λ. Due to customer choice, the sales intensity will particularly depend on our offer price a and the competitors' prices. We also allow the sales intensity to depend on time, e.g., the time of the day or the week. We assume that the time dependence is periodic and has an integer cycle length of J periods. In our model, the sales intensity λ is a general function of time, our offer price a and the competitors' prices $\vec{p}$. Given the prices a and $\vec{p}$ in period t, the jump intensity λ satisfies, $t = 0, 1, 2, ..., a \geq 0, \vec{p} \geq \vec{0}$,

$$\lambda_t(a, \vec{p}) = \lambda_{t \bmod J}(a, \vec{p}). \qquad (1)$$

In our discrete time model, we assume the sales probabilities (for one period) to be Poisson distributed. I.e., the probability to sell exactly i items within one period of time is given by, $t = 0, 1, 2, ...,$ $a \geq 0, \vec{p} \geq \vec{0}, i = 0, 1, 2, ...,$

$$P_t(i, a, \vec{p}) = \frac{\lambda_t(a, \vec{p})^i}{i!} \cdot e^{-\lambda_t(a, \vec{p})}. \qquad (2)$$

For each period t, a price a has to be chosen. We call strategies $(a_t)_t$ admissible if they belong to the class of Markovian feedback policies; i.e., pricing decisions $a_t \geq 0$ may depend on time t and the current

prices of the competitors. By A we denote the set of admissible prices. A list of variables and parameters is given in the Appendix, cf. 3.

By X_t we denote the random number of sales in period t. Depending on the chosen pricing strategy $(a_t)_t$, the random accumulated profit from time/period t on (discounted on time t) amounts to, $t = 0, 1, 2, ...,$

$$G_t := \sum_{s=t}^{\infty} \delta^{s-t} \cdot (a_s - c) \cdot X_s. \quad (3)$$

The objective is to determine a non-anticipating (Markovian) pricing policy that maximizes the expected total profit $E(G_0)$.

In the following sections, we will solve dynamic pricing problems that are related to (1) - (3). In the next section, we study a duopoly situation. We assume that the competitor frequently adjusts his/her prices and show how to derive optimal response strategies. We analyze the impact of different reaction times as well as randomized reaction times. We also consider the case in which the competitor plays mixed strategies. In Section 4, we compute pricing strategies for oligopoly scenarios with active and passive competitors.

3 OPTIMAL REACTION STRATEGIES IN A DUOPOLY

3.1 Fixed Reaction Times

In some applications, sellers are able to anticipate transitions of the market situation. Such information can be used to optimize expected profits. In particular, the price responses of competitors as well as their reaction time can be taken into account. In this case, a change of the market situation $\vec{p}$ can take place within a period. A typical scenario is that a competitor adjusts its price in response to our price with a certain delay. In this section, we assume that the pricing strategy and the reaction time of the competitor is known; i.e., we assume that choosing a price a at time t is followed by a state transition (e.g., a competitor's price reaction) and the current market situation $\vec{p}$ changes to a subsequent state described by a transition function F, which can depend on $\vec{p}$ and a.

In the following, we want to derive optimal price response strategies to a given competitor's strategy. For simplicity, we consider the sale of one type of product in a duopoly situation. We assume that the state of the system (the market situation) is one-dimensional and simply characterized by the competitor's price p, i.e., we let $\vec{p} := p$.

In real-life applications, a firm is not able to adjust its prices immediately after the price reaction of the competing firm. Hence, we assume that in each period the price reaction of the competing firm takes place with a delay of h periods, $h < 1$. I.e., after an interval of size h the competitor adjusts its price from p to $F(a)$, see Figure 1.
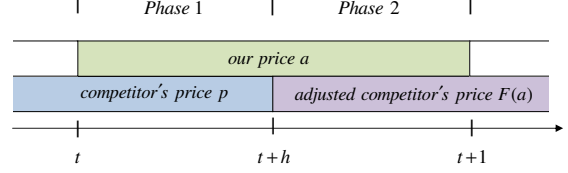

Figure 1: Sequence of price reactions in case of a duopoly.

Thus in period t, the probability to sell exactly i items during the first interval of size h is $P_t^{(h)}(i, a, p) := Pois(h \cdot \lambda_t(a, p))$, while for the rest of the period the sales probability changes to $P_t^{(1-h)}(i, a, F(a)) = Pois((1-h) \cdot \lambda_t(a, F(a)))$.

We will use value iteration to approximate the value function, which represents the present value of future profits. For a given "large" number T, $T \gg J$, we let $V_T(p) = 0$ for all p, and compute, $t = 0, 1, 2, ..., T-1$, $0 < h < 1$, $p \in A$,

$$V_t(p) = \max_{a \in A} \left\{ \sum_{i_1 \geq 0} P_t^{(h)}(i_1, a, p) \right.$$
$$\cdot \sum_{i_2 \geq 0} P_{t+h}^{(1-h)}(i_2, a, F(a))$$
$$\left. \cdot \left((a-c) \cdot (i_1 + i_2) + \delta \cdot V_{t+1}(F(a)) \right) \right\}. \quad (4)$$

The associated pricing strategy $a_t^*(p)$, $t = 0, 1, 2, ..., J-1$, $p \in A$, is determined by the argmax of

$$a_t^*(p) = \arg\max_{a \in A} \left\{ \sum_{i_1 \geq 0} P_t^{(h)}(i_1, a, p) \right.$$
$$\cdot \sum_{i_2 \geq 0} P_{t+h}^{(1-h)}(i_2, a, F(a))$$
$$\left. \cdot \left((a-c) \cdot (i_1 + i_2) + \delta \cdot V_{t+1}(F(a)) \right) \right\}. \quad (5)$$

In case $a_t^*(p)$ is not unique, we choose the largest one.

Remark 3.1. Our recursive solution approach also allows to solve problems with perishable products and finite horizons T. Equations (4)-(5) just have to be evaluated for all $t = 0, 1, 2, ..., T-1$.

To illustrate our approach we will consider a numerical example for durable goods. We assume that the competitor applies one of the most common strategies: the competitor undercuts our current price by ε down to a certain minimum (e.g., the shipping costs c). The sales dynamics of the following example above are based on a large data set from the Amazon market for used books, see Schlosser et al. (2016).

Definition 3.1. We define the sales probabilities $P_t^{(h)}(i,a,p) := Pois\left(h \cdot e^{\vec{x}(a,p)'\vec{\beta}}/(1 + e^{\vec{x}(a,p)'\vec{\beta}})\right)$, using linear combinations of the following five regressors $\vec{x} = \vec{x}(a,p)$ given coefficients $\vec{\beta} = (\beta_1,...,\beta_5)$:

(i) constant / intercept

$$x_1(a,p) = 1$$

(ii) rank of price a compared to price p

$$x_2(a,p) = 1 + \left(1_{\{p<a\}} + 1_{\{p \leq a\}}\right)/2$$

(iii) price gap between price a and price p

$$x_3(a,p) = a - p$$

(iv) total number of competitors

$$x_4(a,p) = 1$$

(v) average price level

$$x_5(a,p) = (a+p)/2$$

Example 3.1. We assume a duopoly. Let $c = 3$, $\delta = 0.99$, $0 \leq h \leq 1$, and let $F(a) := \max(a - \varepsilon, c)$, $\varepsilon = 1$, $a \in A := \{1,2,...,100\}$. For the computation of the value function, we let $T := 1000$. We assume the sales probabilities $P_t^{(h)}(\cdot,a,p)$, see Definition 3.1, where $\vec{\beta} = (-3.89, -0.56, -0.01, 0.07, -0.02)$.

Figure 2a and Figure 3a show optimal response strategies for different reaction times h=0.1 and h=0.9. The case $h = 0.1$ illustrates a fast reaction time of the competitor; $h = 0.9$ represents a slow reaction of the competitor. If $h = 0.5$ both competing firms react equally fast. In all three cases the optimal response strategy are of similar shape. If the competitor's price is either very low or very large, it is optimal to set the price to a certain moderate level. If the competitor's price is somewhere in between (intermediate range), it is best to undercut that price by one price unit ε. If h is larger, the upper price level is increasing and the intermediate range is bigger.

The application of optimal response strategies leads to cyclic price patterns over time, cf. Edgeworth cycles, see, e.g., Maskin, Tirole (1988), Kephart et al.
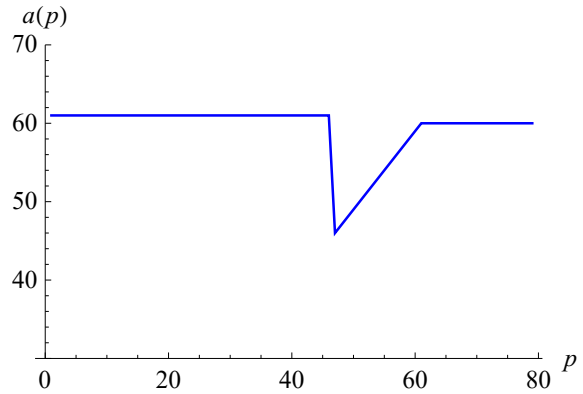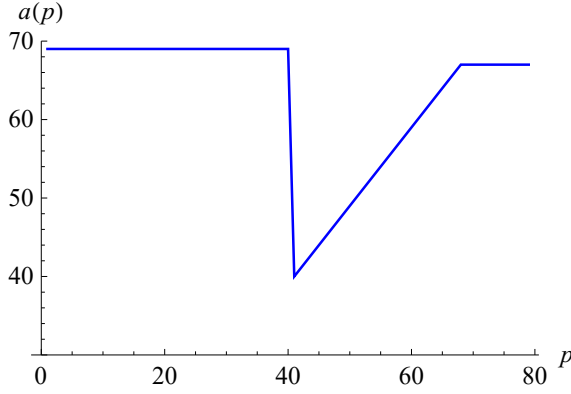

(a) Optimal response policy.
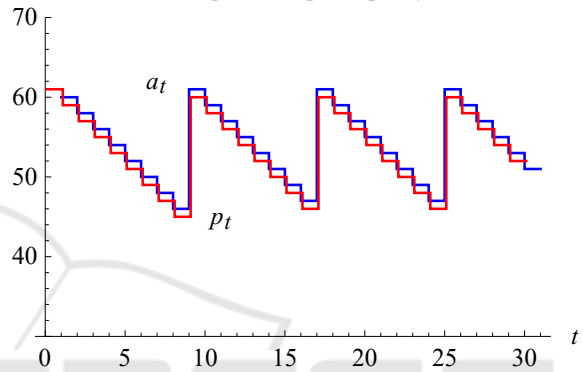
(b) Evaluated price paths over time.

Figure 2: Optimal response policy and price paths for Example 3.1 with $h = 0.1$.

(2000), or Noel (2007). The resulting price paths are shown in Figure 2b and Figure 3b. If the reaction time of the competitor is longer, we observe that the cycle length and the amplitude of the price patterns are increasing. Note, roughly $h \cdot 100\%$ of the time our firm is offering the lowest price; i.e., the parameter h can also be used to model situations in which one firm is able to adjust its prices more often than another firm.

In addition, we are able to analyze the impact of the reaction time on expected long-term profits of our firm as well as the competitor. We assume that the competitor faces the same sales probabilities and shipping costs as we do. The competitor's expected profits can be recursively evaluated by, cf. (4), $t = 0, 1, 2, ..., T-1$, $0 < h < 1$, $a \in A$, $V_{T+h}^{(c)}(a) = 0$,

$$V_{t+h}^{(c)}(a) = \sum_{i_2 \geq 0} P_{t+h}^{(1-h)}(i_2, F(a), a)$$

$$\cdot \sum_{i_1 \geq 0} P_{t+1}^{(h)}\left(i_1, F(a), a_{t+1 \bmod J}^*(F(a))\right)$$

$$\cdot \left((F(a) - c) \cdot (i_1 + i_2) + \delta \cdot V_{t+h+1}^{(c)}\left(a_{t+1 \bmod J}^*(F(a))\right)\right).$$
(6)

(a) Optimal response policy.
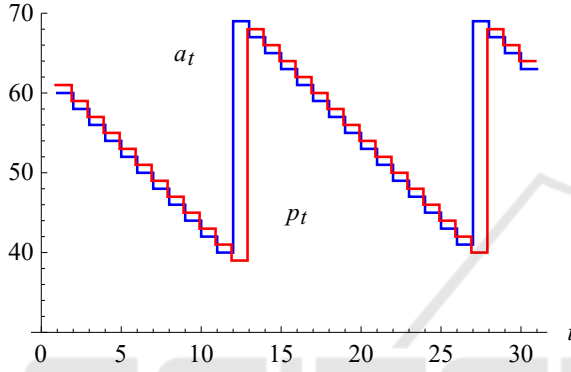
(b) Evaluated price paths over time.

Figure 3: Optimal response policy and price paths for Example 3.1 with $h = 0.9$.

Due to the cyclic price paths, the expected future profits $V_0(p)$ and $V_h^{(c)}(a)$ are (almost) independent of the initial states/prices. Figure 4 depicts V as well as the competitor's expected profits $V^{(c)}$ as a function of h. We observe that the expected profit V is linear increasing in the competitor's reaction time; the competitor's profit $V^{(c)}$ is decreasing in h. Note, the impact of h is substantial. The disadvantage of the player that stops the undercutting phase can already be compensated if our reaction time is smaller than 0.46, i.e., if h exceeds the value 0.54.

3.2 Randomized Reaction Times

Due to the significant impact of reaction times, firms will try to minimize their reaction times by anticipating their competitor's time of adjustment. In order not to act predictable, firms will randomize their reaction times. Moreover, firms will try to gain advantage by updating their prices more frequently.

In case the reaction time is not deterministic, the model can be adjusted. If the distribution of the reaction time of competitors is known, the Hamilton-Jacobi-Bellman (HJB) equation, cf. (4), can be mod-
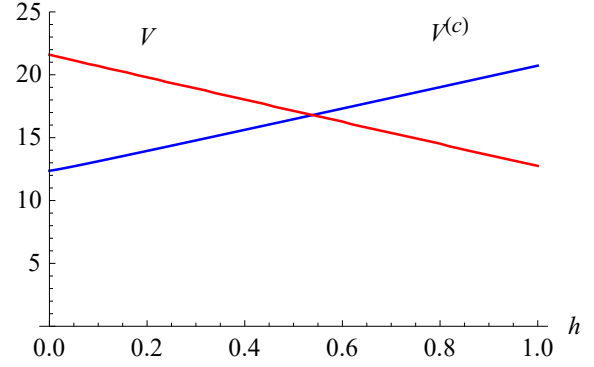

Figure 4: Expected profit for different reaction times of the competitor; Example 3.1.

ified. The different reaction scenarios just have to be considered with the corresponding probability. Note, the reaction times of different competitors can be observed in the long run.

Strategic firms will try to optimally time their price adjustments. In order not to act predictable, firms might use randomized strategies. In the following, we consider such a scenario. We assume that each firm adjusts its price with a certain intensity (e.g., on average once a period of size 1). We model that approach as follows: we assume that at each point in time d, $d = t + \Delta, t + 2\Delta, ..., t + 1$, $0 < \Delta \ll 1$, our firm adjusts its price with probability q, $0 < q \ll 1$; i.e., on average we adjust our price q/Δ times a period of size 1. Similarly, the competitor adjusts its price with probability $q^{(c)}$, $0 < q^{(c)} \ll 1$.

The competitor applies a certain strategy $F(a)$. By a^- we denote our current price at time d, the beginning of the sub-period $(d, d + \Delta)$. With probability $q^{(c)}$, the competitor adjusts its price from p to $F(a^-)$. With probability q, we adjust the price a^- to price a. Since q and $q^{(c)}$ are assumed to be "small" we do not consider the case in which both firms adjust their prices at the same time. The related value function is given by, $a^-, p \in A$, $t = 0, \Delta, 2\Delta, ..., T - \Delta$, $\tilde{V}_T(a^-, p) = 0$,

$$\tilde{V}_t(a^-, p) = (1 - q - q^{(c)})$$
$$\cdot \sum_{i \geq 0} P_t^{(\Delta)}(i, a^-, p) \cdot \left((a^- - c) \cdot i + \delta^\Delta \cdot \tilde{V}_{t+\Delta}(a^-, p) \right)$$
$$+ q^{(c)} \cdot \sum_{i \geq 0} P_t^{(\Delta)}(i, a^-, F(a^-))$$
$$\cdot \left((a^- - c) \cdot i + \delta^\Delta \cdot \tilde{V}_{t+\Delta}(a^-, F(a^-)) \right)$$
$$+ q \cdot \max_{a \in A} \left\{ \sum_{i \geq 0} P_t^{(\Delta)}(i, a, p) \right.$$
$$\left. \cdot \left((a - c) \cdot i + \delta^\Delta \cdot \tilde{V}_{t+\Delta}(a, p) \right) \right\}. \quad (7)$$

The optimal price $\tilde{a}_t^*(a^-, p), t = 0, \Delta, 2\Delta, ..., J - \Delta$, is determined by the arg max of (7). The competitor's expected profit corresponds to, $t = 0, \Delta, 2\Delta, ..., T - \Delta$, $\tilde{V}_T^{(c)}(a^-, p) = 0$,

$$\tilde{V}_t^{(c)}(a^-, p) = (1 - q - q^{(c)})$$
$$\cdot \sum_{i \geq 0} P_t^{(\Delta)}(i, p, a^-) \cdot \left((p - c) \cdot i + \delta^\Delta \cdot \tilde{V}_{t+\Delta}^{(c)}(a^-, p) \right)$$
$$+ q^{(c)} \cdot \sum_{i \geq 0} P_t^{(\Delta)}(i, F(a^-), a^-)$$
$$\cdot \left((F(a^-) - c) \cdot i + \delta^\Delta \cdot \tilde{V}_{t+\Delta}^{(c)}(a^-, F(a^-)) \right)$$
$$+ q \cdot \sum_{i \geq 0} P_t^{(\Delta)} \left(i, p, \tilde{a}_{t \bmod J}^*(a^-, p) \right)$$
$$\cdot \left((p - c) \cdot i + \delta^\Delta \cdot \tilde{V}_{t+\Delta}^{(c)} \left(\tilde{a}_{t \bmod J}^*(a^-, p), p \right) \right). \quad (8)$$

Example 3.2. We assume the duopoly setting of Example 3.1. We let $c = 3$, $F(a) := \max(a - \varepsilon, c)$, $\varepsilon = 1$, $a \in A := \{1, 2, ..., 100\}$, $\delta = 0.99$, $\Delta = 0.1$. We use $T := 1000$. We consider different reaction probabilities q and $q^{(c)}$.

1 contains the expected profits $(\tilde{V}, \tilde{V}^{(c)})$ of the two competing firms for different reaction probabilities. We observe that $\tilde{V}$ is increasing in q and decreasing in $q^{(c)}$. For $\tilde{V}^{(c)}$ it is the other way around. It turns out, that the ratio $q/q^{(c)}$ of the adjustment frequencies is a critical quantity.

Table 1: Expected profits $\tilde{V}$ and $\tilde{V}^{(c)}$ for different reaction probabilities q, $q^{(c)} = 0.05, 0.1, 0.2$, $\delta = 0.99$, $\Delta = 0.1$; Example 3.2.

$q^{(c)} \backslash q$	0.05	0.1	0.2
0.05	(16.53, 17.07)	(16.80, 16.81)	(17.01, 16.62)
0.1	(16.26, 17.36)	(16.48, 17.09)	(16.75, 16.84)
0.2	(16.03, 17.59)	(16.22, 17.37)	(16.48, 17.12)

The overall adjustment frequency plays a minor role as long as the ratio $q/q^{(c)}$ is the same. Hence, the expected profits of both firms can be approximated by the profits from the model with deterministic reaction time, cf. Section 3.1, where $h = q/q^{(c)}$, i.e., the percentage of time our firm has the most recent price.

Figure 5b illustrates the (simulated) price paths for the parameter setting of Example 3.2. Figure 5a shows the deterministic case of Example 3.1 for $h = 0.5$. We observe that overall the price patterns have similar characteristics. However, in the randomized case, the timing of the price reactions is not predictable. While in the deterministic $h = 0.5$ case (cf.
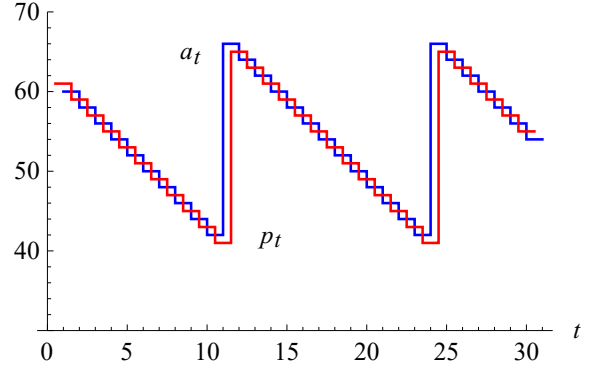
(a) Deterministic reaction times $h = 0.5$, Example 3.1.
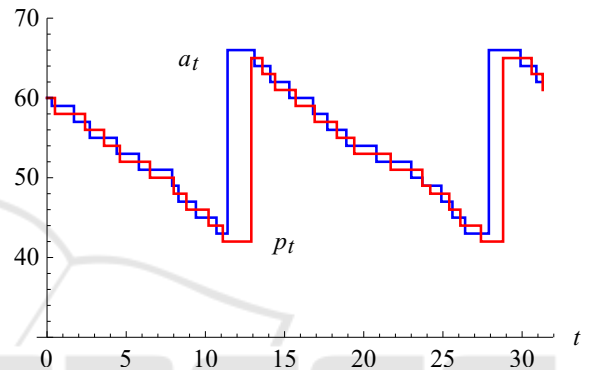
(b) Randomized reaction times for $\Delta = 0.1$, $q = q^{(c)} = 0.1$; Example 3.2.

Figure 5: Evaluated price paths over time.

Section 3.1) we have $\tilde{V} = 16.44$ and $\tilde{V}^{(c)} = 17.13$, in the randomized case ($\Delta = 0.1$, $q = q^{(c)} = 0.1$) the expected profits are $\tilde{V} = 16.48$ and $\tilde{V}^{(c)} = 17.09$. I.e., in both models the advantage of the aggressive player is basically the same; in the model with randomized reaction times the advantage is slightly smaller.

3.3 Mixed Competitors' Strategies

Our results show that if the competitor's strategy is known, suitable response strategies can be computed. Hence, firms might try to randomize their strategies. In this section, we will analyze scenarios in which competitors play a mixed pricing strategy.

We assume that the competitor plays strategy $F_k(a)$, $a \in A$, with probability π_k, $1 \leq k \leq K < \infty$, $\sum_k \pi_k = 1$.

We assume deterministic reaction times. We adjust our model, cf. Section 3.1, by using a weighted sum of the potential price reactions. The Hamilton-Jacobi-Bellman (HJB) equation can be written as, $t = 0, 1, 2, ..., T - 1$, $0 < h < 1$, $p \in A$,

$$V_t(p) = \max_{a \in A} \left\{ \sum_{i_1 \geq 0} P_t^{(h)}(i_1, a, p) \right.$$

$$\cdot \sum_k \pi_k \cdot \sum_{i_2 \geq 0} P_{t+h}^{(1-h)}(i_2, a, F_k(a))$$

$$\left. \cdot \left((a-c) \cdot (i_1 + i_2) + \delta \cdot V_{t+1}(F_k(a)) \right) \right\}, \quad (9)$$

where $V_T(p) = 0$ for all p. The associated pricing strategy $a_t^*(p)$, $t = 0, 1, 2, ..., J-1$, $0 < h < 1$, $p \in A$, is determined by the argmax of (9). The resulting competitor's expected profits can be computed by (starting from, e.g., $V_{T+h}^{(c)}(a) = 0$), $t = 0, 1, 2, ..., T-1$, $0 < h < 1$, $a \in A$,

$$V_{t+h}^{(c)}(a) = \sum_j \pi_k \cdot \sum_{i_2 \geq 0} P_{t+h}^{(1-h)}(i_2, F_k(a), a)$$

$$\cdot \sum_{i_1 \geq 0} P_{t+1}^{(h)}\left(i_1, F_k(a), a_{t+1 \bmod J}^*(F_k(a)) \right)$$

$$\cdot \left((F_k(a) - c) \cdot (i_1 + i_2) + \delta \cdot \tilde{V}_{t+h+1}^{(c)}\left(a_{t+1 \bmod J}^*(F_k(a)) \right) \right).$$

$$(10)$$

The models described above allow computing suitable pricing strategies in various competitive markets. As long as the number of competing firms is small, the value function and the optimal prices can be computed. Note, due to the coupled state transitions in general the value function has to be computed for all states in advance. When the number of competitors is large this can cause serious problems since the state space can grow exponentially (curse of dimensionality). Hence, the approach is suitable, if the number of competitors is small and their strategies are known. If the number of competitors is large and the firm's strategies are unknown, we recommend using simple but robust strategies, see Schlosser et al. (2016).

4 COMPETITION WITH ACTIVE AND PASSIVE SELLERS

If the pricing strategies and the reaction times of different competitors are known the model can be extended to an oligopoly setting. For each additional competitor the state space of the model has to be extended by one dimension. Note, only active competitors that frequently adjust their prices should be taken into account. Inactive customers will be treated as external fixed effects.

In the following, we assume one active competitor and Z passive competitors. The prices of the passive competitors are denoted by $\vec{z} = (z_1, ..., z_Z)$, $z_j \geq 0$, $j = 1, ..., Z$, and assumed to be constant over time. The active competitor plays a (non-randomized) strategy $F(a)$ that refers to our price a (not the passive one). The Hamilton-Jacobi-Bellman (HJB) equation can be written as, $t = 0, 1, 2, ..., T-1$, $0 < h < 1$, $p \geq 0$, $V_T(p, \vec{z}) = 0$ for all $p, \vec{z}$,

$$V_t(p, \vec{z}) = \max_{a \in A} \left\{ \sum_{i_1 \geq 0} P_t^{(h)}(i_1, a, p, \vec{z}) \right.$$

$$\cdot \sum_{i_2 \geq 0} P_{t+h}^{(1-h)}(i_2, a, F(a), \vec{z})$$

$$\left. \cdot \left((a-c) \cdot (i_1 + i_2) + \delta \cdot V_{t+1}(F(a), \vec{z}) \right) \right\}. \quad (11)$$

The associated pricing strategy amounts to, $t = 0, 1, 2, ..., J-1$, $0 < h < 1$, $p \in A$,

$$a_t^*(p, \vec{z}) = \arg\max_{a \in A} \left\{ \sum_{i_1 \geq 0} P_t^{(h)}(i_1, a, p, \vec{z}) \right.$$

$$\cdot \sum_{i_2 \geq 0} P_{t+h}^{(1-h)}(i_2, a, F(a), \vec{z})$$

$$\left. \cdot \left((a-c) \cdot (i_1 + i_2) + \delta \cdot V_{t+1}(F(a), \vec{z}) \right) \right\}. \quad (12)$$

The competitor's profits can be computed by (starting from, e.g., $V_{T+h}(a, \vec{z}) = 0$ for all $a, \vec{z}$), $t = 0, 1, 2, ..., T-1$, $0 < h < 1$, $a \geq 0$,

$$V_{t+h}^{(c)}(a, \vec{z}) = \sum_{i_2 \geq 0} P_{t+h}^{(1-h)}(i_2, F(a), a, \vec{z})$$

$$\cdot \sum_{i_1 \geq 0} P_{t+1}^{(h)}\left(i_1, F(a), a_{t+1 \bmod J}^*(F(a), \vec{z}), \vec{z} \right)$$

$$\cdot \left((F(a) - c) \cdot (i_1 + i_2) + \delta \cdot V_{t+h+1}^{(c)}\left(a_{t+1 \bmod J}^*(F(a), \vec{z}), \vec{z} \right) \right).$$

$$(13)$$

Note, the value function does not need to be computed for all price combinations of passive competitors in advance. The value function and the associated pricing policy can be computed separately for specific market situations (e.g., just when they occur).

In the following, we consider an example with active and passive competitors.

Example 4.1. We assume the duopoly setting of Example 3.1. We let $F(a) := \max(a - \varepsilon, c)$, $\varepsilon = 1$, $c = 3$, $h = 0.5$, $a \in A := \{1, 2, ..., 100\}$, $\delta = 0.99$, and $T = 1000$. Furthermore, we consider an additional passive competitor with the constant price z, $z = 15, 20, 25$.
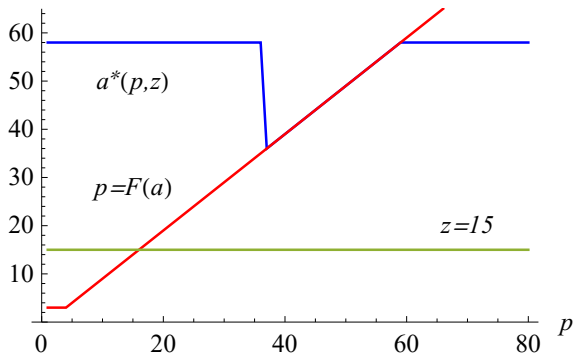
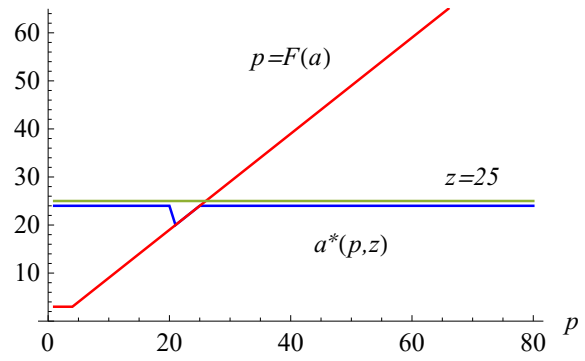(a) Optimal response strategy.
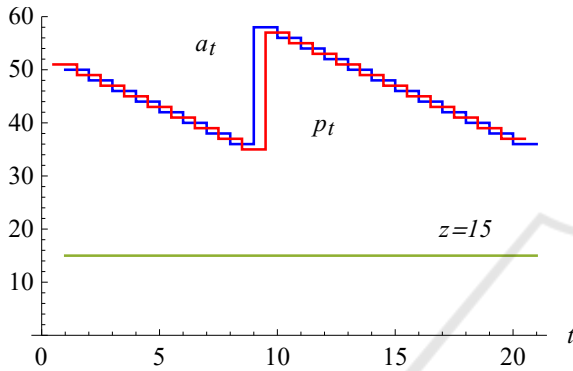
(b) Evaluated price paths over time.

Figure 6: Optimal response strategy and evaluated price paths for Example 4.1; $h = 0.5$, $z = 15$.

(a) Optimal response strategy.

(b) Evaluated price paths over time.

Figure 7: Optimal response strategy and evaluated price paths for Example 4.1; $h = 0.5$, $z = 25$.

The results of the three cases $z = 15$, $z = 20$, and $z = 25$ are illustrated in Figure 6, 7 and 8. We observe three different characteristics. If the passive competitor's price is low ($z = 15$) the cyclic price battle between our firm and the aggressive firm takes place at a higher price level, see Figure 6b. The response strategies of the three firms are displayed in Figure 6a.
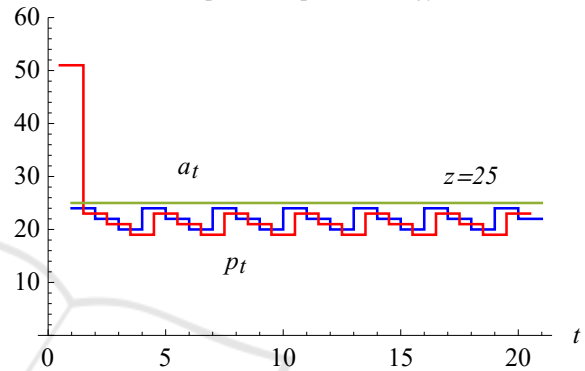
If the price of passive firm is sufficiently high ($z = 20$), then the cyclic price paths of the two active firms take place below that level. If the constant price is "moderate" ($z = 20$), then a mixture of the characteristics shown in Figure 6 and 7 is optimal. We also observe that it is not advisable to place price offers that slightly exceed competitors' prices, cf. Figure 8.

At the end of this section, we want to generally evaluate the outcome when different (time homogeneous) strategies are played against each other. We assume time homogeneous demand and $h = 0.5$. If firm 1 plays a pure strategy S_1 and firm 2 plays the pure strategy S_2 then the associated expected profits can be computed by, $t = 0, 1, 2, ..., T - 1$, $V_T^{(1)}(a) = V_T^{(2)}(a) = 0$, for all $a \geq 0$,

$$V_t^{(1)}(a) = \sum_{i_1 \geq 0} P^{(0.5)}(i_1, S_1(a), a)$$

$$\cdot \sum_{i_2 \geq 0} P^{(0.5)}(i_2, S_1(a), S_2(S_1(a)))$$

$$\cdot \left((S_1(a) - c) \cdot (i_1 + i_2) + \delta \cdot V_{t+1}^{(1)}(S_2(S_1(a))) \right), \tag{14}$$

$$V_t^{(2)}(a) = \sum_{i_1 \geq 0} P^{(0.5)}(i_1, S_2(a), a)$$

$$\cdot \sum_{i_2 \geq 0} P^{(0.5)}(i_2, S_2(a), S_1(S_2(a)))$$

$$\cdot \left((S_2(a) - c) \cdot (i_1 + i_2) + \delta \cdot V_{t+1}^{(2)}(S_1(S_2(a))) \right). \tag{15}$$

By S_U we denote the response strategy $F(a) := \max(a - \varepsilon, c)$, which slightly undercuts the competitor's price. By S_{RU} we denote the optimal response strategy to S_U. By S_{RRU} we denote the optimal response strategy to S_{RU}, cf. (11)-(12). Considering Example 4.1 with $z = 20$, the expected profits of the different strategy combinations are summarized in 2.

We observe that the aggressive strategy S_U yields very good results with the exception when the competitor also plays S_U. The strategy S_{RU} yields good results in all three constellations. Strategy S_{RRU} is excellent when played against S_{RU} but yields only mod-
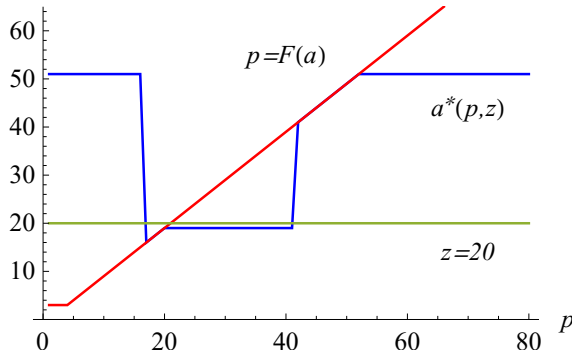
(a) Optimal response strategy.
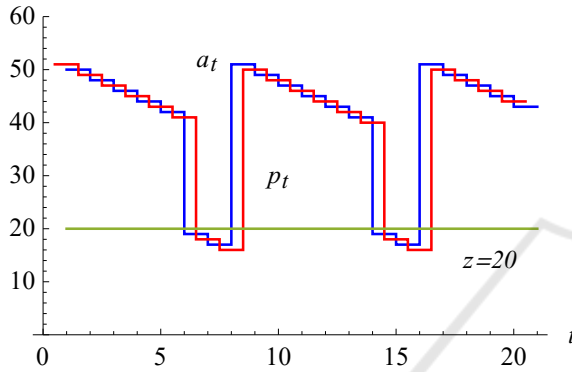
(b) Evaluated price paths over time.

Figure 8: Optimal response strategy and evaluated price paths for Example 4.1; $h = 0.5$, $z = 20$.

Table 2: Expected profits $V_0^{(1)}(50)$ of firm 1 when its strategy $S_1 = \{S_U, S_{RU}, S_{RRU}\}$ is played against a strategy $S_2 = \{S_U, S_{RU}, S_{RRU}\}$, $z = 20$, Example 4.1.

$S_1 \backslash S_2$	S_U	S_{RU}	S_{RRU}
S_U	1.69	9.85	9.91
S_{RU}	9.42	9.66	9.58
S_{RRU}	8.76	10.62	8.76

erate results in the other two cases. For other z values, cf. Example 4.1, the results are similar. Iterating mutual strategy responses further is a way to identify equilibrium strategies.

Finally, our example shows that optimal response strategies have a significant impact on expected profits. They help to gain profits, especially, when aggressive competitors are involved. On the other hand, we learn that it is also important to know the competitors' strategies. In practical applications, the competitors' price reactions can be inferred from market data over time.

5 CONCLUSION

With a rise in E-commerce it has become easier to observe and to adjust prices automatically. As a result, dynamic pricing strategies are applied by an increasing number of firms. This paper analyzes stochastic dynamic infinite horizon oligopoly models characterized by active and passive competitors. We set up a dynamic pricing model including discounting and shipping costs. The sales probabilities are allowed to depend on time and can arbitrarily depend on our price as well as the competitors' prices. Hence, our model is suitable for practical applications. Data-driven estimations of sales intensities under pricing competition can be used to calibrate the model.

Given a competitor's response strategy, we are able to compute optimal reaction strategies that take the anticipated competitors' price adjustments into account. In general, it is optimal to slightly undercut competitor's prices. However, when the price falls below a certain lower bound it is advisable to raise the price to an optimally chosen upper level. Our examples show that the model can be used to explain and to study Edgeworth price cycles.

We also verify that reaction times have a significant impact on long-term profits. Hence, firms will try to strategically time their price adjustments. In order not to act predictable firms might use randomized strategies. Using a generalized version of our model, we show how to derive optimal response strategies when reaction times are randomized. We observe that the ratio of frequencies of the competitors' prices adjustments is crucial for the firm's expected profits, i.e., to be able to adjust prices more often than the competitors do is an important competitive advantage.

In an extension of the model, we have considered additional players with fixed price strategies. We have presented a solution approach that allows deriving optimal response strategies. We have analyzed how the presence of additional passive competitors affects the price battle of active players that frequently adjust their prices. The solution approach is even applicable when the number of passive competitors is large. Our technique to compute prices remains simple and is easy to implement.

Moreover, we have evaluated the outcome when different reaction strategies are played against each other. It turned out that our optimized feedback strategies effectively avoid a decline in price. Especially, when competitors play aggressive strategies it is important to react in a reasonable way in order not to loose potential profits. Our approach allows to derive and to study price response strategies for various real-life applications especially in E-commerce.

Iterating mutual strategy responses, cf. Table 2, may also be the key to identify equilibrium strategies. Note, mutual strategy responses do not necessarily have to converge as pure strategy equilibria might not exist, see Kephart et al. (2000). In such cases, the approach used in Section 3.3 might help to identify equilibria in mixed strategies.

In future research we will use market data to estimate competitors' response strategies. We will also extend the model to study the sale of perishable products with finite initial inventory levels.

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APPENDIX

Table 3: List of variables and parameters.

t	time / period
X	random number sold items
G	random future profits
c	shipping costs
δ	discount factor
F	competitor's reaction strategy
Z	number of passive competitors
A	set of admissible prices
$V, V^{(c)}$	value functions
a	offer price
$\vec{p}, \vec{z}$	competitors' prices
λ	sales intensity
P	sales probability
J	cycle length
h	reaction time
$q, q^{(c)}$	reaction probabilities