Paraconsistent Logic with Multiple Fuzzy Linguistic Truth-values

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Abstract: This paper extends the two-valued paraconsistent logic into an one in which a proposition takes a truth-value from a set of multiple fuzzy linguistic terms. More specifically, we propose the corresponding inference rule and semantics, and finally prove the soundness of our new fuzzy logical system and its completeness. Moreover, we use an example to illustrate the applicability of our logic system in real life.

1 INTRODUCTION

Paraconsistent logic is a branch of non-classical logic in which the inconsistency can be accepted but the contradiction cannot imply any proposition (Da Costa, 1958; Da Costa et al., 1995; Akama and Da Costa, 2016) (while in a classic logic system, the contradiction can imply any proposition). Paraconsistent logic is very useful (Priest et al., 1989; Tanaka et al., 2012; Abe, 2016). In fact, its basic idea can be applied to other kinds of logic system like paraconsistent relevant logic (Kamide, 2013, 2016) and paraconsistent deontic logic (Costa and Carnielli, 1986). Also it is very useful in artificial intelligence. For example, when an expert system cannot deal with contrary options of different experts, the way in which paraconsistent logic cope with inconsistence will be so helpful to cope with the issue. Because of the trait of holding contradictions, paraconsistent logic will push forward the development of artificial intelligence to a new stage and inject energy constantly.

As paraconsistent logic, fuzzy logic (Zadeh, 1965, 1983, 1996) is widely applied as well (Yager and Zadeh, 1992; Zhan et al., 2014). The main idea behind fuzzy logic is using fuzzy sets and fuzzy inference rules to simulate the synthetic reasoning of human mind. It is so accordant with human mind's customary vague thinking that it has been applied to many aspects of our life, such as the control systems of air conditioning, washing machine, robot, and so on.

Lots of studies about fuzzy temporal logic (Mukherjee and Dasgupta, 2013; Poli, 2015) and fuzzy modal logic (Vidal et al., 2015; Jing et al., 2014) have already done, but not many are on fuzzy paraconsistent logic. Turunen et al. (2010) firstly link paraconsistent logic and fuzzy logic together by introducing the paraconsistent semantics for Pavelka style fuzzy sentential logic. They emphasise that they do not introduce a new non-classical logic but introduce paraconsistent semantics of Pavelka style fuzzy sentential logic based on Balnap's four valued paraconsistent logic and Lukasiewicz Pavelka's logic system. Rodrguez et al. (2014) went further to introduce another paraconsistent algebraic semantics for Lukasiewicz-Pavelka logic and remove some limitations of their work in 2010. Although their work is very significant, they only dealt with the theoretical aspect, but did not show the practical value of their theory. In addition, they just unidirectionally construct a paraconsistent semantic for fuzzy logic. In the opposite direction of their research (*i.e.*, constucting a fuzzy semantic for paraconsistent logic), they did not get involved. Arnon (2014) introduced proof systems and semantics for two paraconsistent extensions of the system T of Anderson et al. (1978), and prove strong soundness, completeness, and decidability for both in his article. The semantics of both systems is based on excluding just one element from the set of designated values.

The basic idea behind fuzzy logic can be applied in other kinds of logic, so a new fuzzy semantic can be born. In fact, based on the classical intuitionistic logic, Turunen (1992) developed a kind of fuzzy in-

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tuitionistic logic. Though he did not use the term of Zadeh's fuzzy set, he used exactly the same thought as Zadeh. He also believed that the truth-value of a proposition should be in a finite set rather than a binary set with 0 and 1. Thiele and Kalenka (1993) introduced fuzzy temporal logic based on the classical two-valued temporal logic. Moon et al. (2004) introduced fuzzy branching temporal logic. In recent years, many applications of fuzzy temporal logic have been proposed, such as the control of the robot behaviour (Ijsselmuiden et al., 2014), the prediction of railway custom flow (Dou et al., 2014), and so on. Although the idea of fuzzy logic has been used to extend intuitionistic logic, modal logic, and temporal logic early, few researchers have proposed the complete fuzzy paraconsistent logic and concerned its practicability.

Fuzzy logic and paraconsistent logic both have extinct characteristics. Paraconsistent logics are specially tailored to deal with inconsistency, while fuzzy logics are primarily used to deal with graded truth and vagueness (Ertola et al., 2013). Both of them are developing rapidly and independently in their own area. If we can apply the idea of fuzzy logic to paraconsistent logic, it will help both to develop together. In fact, this is possible and necessary. In paraconsistent logics, the truth value of a proposition is only 0 or 1, which is not always the case in real life. People often cannot decide absolutely *true* or *false*, *right* or *wrong*, *good* or *bad*, but people are accustomed to some inexact fuzzy concepts, such as *little true*, *very right*, *very good*, *relatively large*, and so on. In this case, two-valued logic fails to meet the needs, so it is necessary to take some fuzzy elements into account and turn the original two-valued one into multi-valued one, so that it can be applied to wider spread areas in real life and artificial intelligence.

The rest of this paper is organised as follows. Section 2 recaps some basic concepts and notations in fuzzy set theory. Section 3 constructs a new semantic with fuzzy linguistic truth-value for paraconsistent logic. Section 4 presents the axiom system of our logic. Section 5 proves its soundness and completeness. Section 6 gives an example to show how the fuzzy paraconsistent logic can be used to solve a real problem. Finally, Section 7 concludes the paper with future work.

2 PRELIMINARIES

This section will recap basic concepts and notations of fuzzy set (Zadeh, 1965), which we will use to build up our fuzzy paraconsistent logic.

Definition 1 (Fuzzy Set)**.** *Let U be a crisp set, a fuzzy set F on U is defined by a membership function:*

$$\mu_F : U \to [0,1].$$

Specifically, $\mu_F(u) \in [0,1]$ represents the membership degree of u in F.

Definition 2 (Linguistic Truth-value)**.** *The linguistic truth-value set is defined as follows:*

$$\begin{aligned} LTS = \{&absolute\text{-}true, very\text{-}true,\\ &moderate\text{-}true, slightly\text{-}true,\\ &slightly\text{-}false, moderate\text{-}false,\\ &very\text{-}false, absolute\text{-}false\}. \end{aligned} \quad (1)$$

For convenience, we denote

$$LTTS = \{\textit{absolute-true, very-true, moderate-true, slightly-true}\}, \quad (2)$$

$$LTFS = \{\textit{absolute-false, very-false, moderate-false, slightly-false}\}. \quad (3)$$

In this paper, we let $\overline{\tau} \in LTS$ represent a complement to τ. Pairs of the linguistic truth-values that are complement to each other include: *absolute-true* and *absolute-false*, *very-true* and *very-false*, *moderate-true* and *moderate-false*, *slightly-true* and *slightly-false*.

Definition 3 (Membership Function of Linguistic Truth-value)**.** *For any $x \in [0,1]$,*

$$\mu_{absolute\text{-}false}(x) = \begin{cases} 1 & \textit{if } x = 0,\\ 0 & \textit{otherwise;} \end{cases} \quad (4)$$

$$\mu_{very\text{-}false}(x) = e^{-80x^2}; \quad (5)$$

$$\mu_{moderate\text{-}false}(x) = e^{-140(x-0.25)^2}; \quad (6)$$

$$\mu_{slightly\text{-}false}(x) = e^{-200(x-0.45)^2}; \quad (7)$$

$$\mu_{slightly\text{-}true}(x) = \mu_{slightly\text{-}false}(1-x); \quad (8)$$

$$\mu_{moderate\text{-}true}(x) = \mu_{moderate\text{-}false}(1-x); \quad (9)$$

$$\mu_{very\text{-}true}(x) = \mu_{very\text{-}false}(1-x); \quad (10)$$

$$\mu_{absolute\text{-}true}(x) = \begin{cases} 1 & \textit{if } x = 1,\\ 0 & \textit{otherwise.} \end{cases} \quad (11)$$

The curves of membership functions of the above linguistic truth-value is shown in Fig. 1.

Definition 4 (Operators on Linguistic Truth-value)**.**

$$\mu_{A\wedge B}(x,y) = \min\{\mu_A(x), \mu_B(y)\}, \quad (12)$$

$$\mu_{A\vee B}(x,y) = \max\{\mu_A(x), \mu_B(y)\}, \quad (13)$$

$$\mu_{A\to B}(x,y) = \max\{1-\mu_A(x), \mu_B(y)\}, \quad (14)$$

$$\mu_{\neg A}(x) = 1-\mu_A(x). \quad (15)$$

Definition 5 (Fuzzy Modus Ponens Rule)**.** *Suppose A and A′ are fuzzy sets on domain X, and B and B′ are*

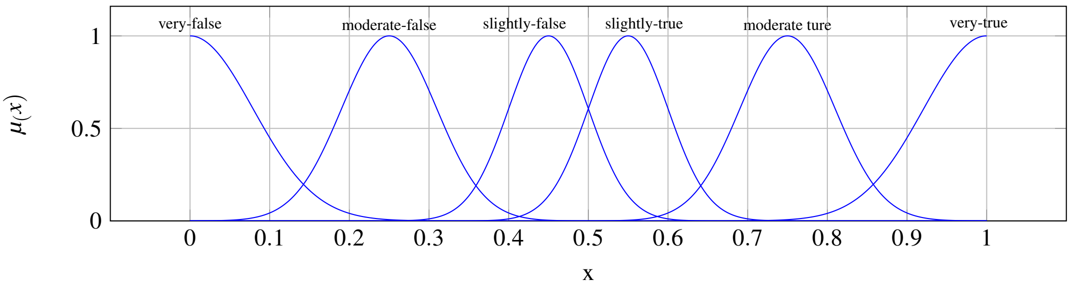

Figure 1: Membership function of linguistic truth-value.

the fuzzy sets on domain Y. If we know $A \to B$ and A', then we can get B', which is defined as follows:

$$\mu_{B'}(y) = \sup\{\min\{\mu_{A'}(x), \mu_{A \to B}(x,y)\} \mid x \in X\}. \quad (16)$$

Definition 6 (Linguistic Approximation). *$\tau \in LTTS$ is called the linguistic approximation of τ^* (denoted as $\tau = \odot\tau^*$) when $\forall \tau_1 \in LTTS$,*

$$ED(\tau, \tau^*) \leq ED(\tau_1, \tau^*),$$

where ED is the Euclidean Distance, which is defined as follows: for two fuzzy sets A and B,

$$ED(A,B) = \sqrt{\Sigma\{(\mu_A(x) - \mu_B(x))^2 \mid x \in [0,1]\}}.$$

3 LOGIC SYSTEM

This section will present the syntax and semantics of our logic system. Basically, the syntax of our fuzzy paraconsistent logic (denoted as F_n $(1 \leq n \leq \omega)$) is the same as that of paraconsistent logic systems C_n $(1 \leq n \leq \omega)$ (Costa et al., 2005), but our semantics are different. However, for the sake of completely understanding our whole logic system, we still present its syntax here.

3.1 Syntax

Just like C_n, F_n is a series of logic system F_1, F_2, ..., F_n, ..., F_ω. Each logic system is strictly stronger than those which follow it. In particular, F_ω is the weakest logic system.

Definition 7 (Language). *The language of F_n is denoted as L_0, which consists of the following three kinds of initial symbol:*

1. *proposition symbol: $p_0, p_1, \ldots, p_k, \ldots$;*
2. *connection symbol: $\neg$, $\wedge$, $\vee$, $\to$; and*
3. *punctuation: left parenthesis (and right parenthesis).*

Definition 8 (Formula). *The initial symbols in L_0 can be combined arbitrarily. A finite sequence of a combination of initial symbols in L_0 is called a formula in L_0 iff it can be generated by limited applications of the following rules:*

1. *an atomic proposition is a formula;*
2. *if A is a formula, then $\neg A$ is also a formula;*
3. *if A and B are formulas, then $(A \wedge B)$, $(A \vee B)$ and $(A \to B)$ are also formulas.*

In this paper, we use capital letters $A, B, C\ldots$ to represent a formula. The set consisting of all the formulas in L_0 is denoted as $Form(L_0)$.

And there are some special notations in F_n:

1. $A^0 =_{df} \neg(A \wedge \neg A)$, which means that proposition A should comply with the law of contradiction.
2. $A^{n+1} = (A^n)^0$=$\neg(A^n \wedge \neg A^n)$ and $A^1 = A^0$.
3. $A^{(n)}$=$A^1 \wedge A^2 \wedge \ldots \wedge A^n$, which intuitively means that A acts in full accordance with the way in which it acts in classical logic.
4. $\neg^{(n)}A$=$\neg A \wedge A^{(n)}$

3.2 Semantics

Definition 9 (Valuation). *A value V is a mapping $V : Form(L_0) \to LTS$ such that:*

1. *if $V(A) = \tau \in LTFS$, then $V(\neg A) = \overline{\tau}$;*
2. *if $V(\neg\neg A) = \tau \in LTS$, then $V(A) = \tau$;*
3. *if $V(B^{(n)}) \in LTTS$, $V(A \to B) \in LTTS$, $V(A \to \neg B) \in LTTS$, then*

$$\mu_{V(A)}(x) = \max\{\sup_{y \in Y}\{\min\{\mu_{V(B^{(n)})}(x), \mu_{\neg B \to \neg A}(y,x)\}\}, \sup_{y \in Y}\{\min\{\mu_{V(B^{(n)})}(x), \mu_{B \to \neg A}(y,x)\}\};$$

4. *if $V(A) = \tau$, then*

$$V(A \to B) = \odot(\min(\overline{\tau}, V(B)));$$

5. $V(A \wedge B) = \odot(\min(V(A), V(B))$;
6. $V(A \vee B) = \odot(\max(V(A), V(B))$; *and*
7. *if* $V(A^{(n)}) \in LTTS$, $V(B^{(n)}) \in LTTS$, *then* $V((A \wedge B)^{(n)}) \in LTTS$, $V((A \vee B)^{(n)}) \in LTTS$, $V((A \rightarrow B)^{(n)}) \in LTTS$.

In the above definition, $V(A) = \tau \in LTS$ means that the truth-value of A is τ, $V(A) \in LTTS$ means the credibility of A is high, and $V(A) \in LTFS$ means the credibility of A is low. So, in the above definition:

- The first property means when the credibility of A is low, the truth-value of $\neg A$ is the complement of A. Instead, when the credibility of A is high, the truth-value of $\neg A$ cannot simply be the complement of A. It intuitively means A and $\neg A$ cannot own low credibility at the same time, instead they can have high credibility simultaneously.
- The second property means that the truth-value of $\neg\neg A$ implies that of A, but not vice versa.
- The third property means that if the credibility of the proposition "B satisfies the contradictory law" is high, then the law of reduction to absurdity is established.
- The fourth, fifth, sixth properties redefine the semantic of implication, conjunction and disjunction.
- The last property means that if the credibility of the proposition "A and B satisfy the contradictory law" is high, then the credibility of the compound proposition of A and B is high.

By the above definition, given $V(A)$ and $V(B)$, we can obtain the valuation of $A \wedge B$ is showed in Table 1, the valuation of $A \vee B$ is showed in Table 2, and the valuation of $A \rightarrow B$ is showed in Table 3.

The following definition extends the concept of model in paraconsistent logic into our fuzzy paraconsistent logic.

Definition 10 (Model). *A value V is called a model of formula set Γ iff for any formula $A \in \Gamma$, $V(A) \in LTTS$.*

Definition 11 (Semantic Consequence). *A formula A is called the semantic consequence of Γ, denoted as $\Gamma \models A$, iff for any model V of Γ, $V(A) \in LTTS$. When Γ is empty, we denote $\models A$ and say A is of commonly high credibility.*

4 AXIOM SYSTEM

This section will present the axiom system of our logic.

4.1 Axioms

The axioms of F_n ($1 \leq n \leq \omega$) is the same as the axioms of C_n ($1 \leq n \leq \omega$). That is, they are formulas that have one of the following forms:

1. $A \rightarrow (B \rightarrow A)$
2. $(A \rightarrow B) \rightarrow ((A \rightarrow (B \rightarrow C)) \rightarrow (A \rightarrow B))$
3. $A \rightarrow (B \rightarrow (A \wedge B))$
4. $(A \wedge B) \rightarrow A$
5. $(A \wedge B) \rightarrow B$
6. $(A \rightarrow C) \rightarrow ((B \rightarrow C) \rightarrow (A \vee B \rightarrow C))$
7. $A \rightarrow (A \vee B)$
8. $B \rightarrow (A \vee B)$
9. $A \vee \neg A$
10. $\neg\neg A \rightarrow A$
11. $B^{(n)} \rightarrow ((A \rightarrow B) \rightarrow ((A \rightarrow \neg B) \rightarrow \neg A))$
12. $(A^{(n)} \wedge B^{(n)}) \rightarrow ((A \wedge B)^{(n)} \wedge (A \vee B)^{(n)} \wedge (A \rightarrow B)^{(n)})$

$A^{(n)}$ intuitively means that A acts in full accordance with the way in which it acts in classical logic. So it can be seen what axioms 11 and 12 intuitively mean. Also, we can see that connection symbol $\wedge, \vee, \rightarrow$ and $\neg^{(n)}$ in our logic respectively have the properties of the conjunctive, disjunction, implication and negation in the classical logic.

Furthermore, it can be proved that all the axioms are commonly of high credibility by the three operators as showed in Tables 1, 2 and 3.

4.2 Inference Rules

The inference rule of F_n has only one and it is Modus Ponens.

Definition 12 (Modus Ponens rule). *If we know that $(V(A) = \tau_1)$ implies $V(B) = \tau_2$ and know $V(A) = \tau_1'$, then we can get $V(B) = \tau_2'$, which is defined as follows:*

$$\tau_2' = \odot(\tau'), \tag{17}$$

where τ' is defined as:

$$\mu_{\tau'}(y) = sup_{x \in X}\{\mu_{\tau_2'}(x) \wedge max\{1 - \mu_{\tau_1}(x), \mu_{\tau_2}(y)\}\}. \tag{18}$$

Table 1: The linguistic truth true table of the conjunctive operator in our fuzzy paraconsisitent logic.

V(A) \ V(B)	absolute-true	very-true	moderate-true	slightly-true	slightly-false	moderate-false	very-false	absolute-false
absolute-true	absolute-true	very-true	moderate-true	slightly-true	slightly-false	moderate-false	very-false	absolute-false
very-true	very-true	very-true	moderate-true	slightly-true	slightly-false	false	very-false	absolute-false
moderate-true	moderate-true	moderate-true	moderate-true	slightly-true	slightly-false	false	very-false	absolute-false
slightly-true	slightly-true	slightly-true	slightly-true	slightly-true	slightly-false	moderate-false	very-false	absolute-false
slightly-false	slightly-false	slightly-false	slightly-false	slightly-false	slightly-false	moderate-false	slightly-false	absolute-false
moderate-false	moderate-false	moderate-false	moderate-false	moderate-false	moderate-false	moderate-false	very-false	absolute-false
very-false	very-false	very-false	very-false	very-false	very-false	very-false	very-false	absolute-false
absolute-false	absolute-false	absolute-false	absolute-false	absolute-false	absolute-false	absolute-false	absolute-false	absolute-false

Table 2: The linguistic true truth table of the disjunctive operator in our fuzzy paraconsisitent logic.

V(A) \ V(B)	absolute-true	very-true	moderate-true	slightly-true	slightly-false	moderate-false	very-false	absolute-false
absolute-true	absolute-true	absolute-true	absolute-true	absolute-true	absolute-true	absolute-true	absolute-true	absolute-true
very-true	absolute-true	very-true	very-true	very-true	very-true	very-true	absolute-true	absolute-true
moderate-true	absolute-true	very-true	moderate-true	moderate-true	moderate-true	moderate-true	moderate-true	moderate-true
slightly-true	absolute-true	very-true	moderate-true	slightly-true	slightly-true	slightly-true	moderate-true	moderate-true
slightly-false	absolute-true	very-true	moderate-true	slightly-true	slightly-false	moderate-false	very-false	absolute-false
moderate-false	absolute-true	very-true	moderate-true	slightly-true	moderate-false	moderate-false	very-false	absolute-false
very-false	absolute-true	very-true	moderate-true	slightly-true	very-false	very-false	very-false	absolute-false
absolute-false	absolute-true	very-true	moderate-true	slightly-true	absolute-false	absolute-false	absolute-false	absolute-false

Table 3: The linguistic truth value table of the complement operator in our fuzzy paraconsisitent logic.

$V(A)$	absolute-true	very-true	moderate-true	slightly-true	slightly-false	moderate-false	very-false	absolute-false
$V(\neg A)$	absolute-false	very-false	moderate-false	slightly-false	slightly-true	moderate-true	very-true	absolute-true

4.3 Proof

Definition 13 (Proof). *We say there is a proof from formula set Γ to formula A, if there is a finite sequence of formulas $A_1, A_2, \ldots, A_m$, such that A_m is A and for every j $(1 \le j \le m)$, A_j satisfies one of the following conditions:*

1. *A_j is an axiom of F_n;*
2. *A_j is a formula in Γ; and*
3. *there are i and k $(i, k < j)$ such that A_j is obtained by A_i and A_k with Modus Ponens.*

Definition 14. *If we have a proof from formula set Γ to formula A, we call A is Γ deductible in F_n, denoted as $\Gamma \vdash A$. When Γ is empty, we denote it as $\vdash A$ and say A is a theorem of F_n.*

Theorem 1. *All the axioms and rules in classical proposition logic are set up in F_n $(1 \le n \le \omega)$. In particular, Deduction Theorem is set up in $F_n (1 \le n \le \omega)$.*

Theorem 2. *In $F_n (1 \le n \le \omega)$, we have:*

$$\vdash (A \to \neg A) \to A,$$
$$\vdash A^{(n)} \to (\neg A)^{(n)},$$
$$\vdash A^{(n)},$$
$$B^{(n)}, A \to B \vdash \neg B \to \neg A.$$

Theorem 3. *In $F_n (1 \le n \le \omega)$, the following formulas which hold in classic propositional logic do not hold:*

$$A \wedge \neg A \to B,$$
$$A \to \neg\neg A,$$
$$(\neg A \wedge (A \vee B)) \to B,$$
$$(A \to B) \to (\neg B \to \neg A),$$
$$\neg(A \wedge \neg A).$$

For the sake of page limit, we cannot give out the detailed proof of the above theorem here, but we will do in the extended version of this paper.

5 SOUNDNESS AND COMPLETENESS

This section will prove the soundness and completeness of our logic.

5.1 Soundness

Intuitively, we say a logic system is sound, meaning that for a formula of a logic, if it is correct in the sense of syntax, then it is correct in the sense of semantics. Formally, we have:

Theorem 4. *F_n is sound,* i.e., $\Gamma \vdash A \Rightarrow \Gamma \models A$.

Proof. Since $\Gamma \vdash A$, we have a sequence A_1, A_2, $\ldots$, A_m, such that A_m is A, and for every j $(1 \le j \le m)$, A_j satisfies one of the following conditions:

1. A_j is an axiom of F_n;
2. A_j is a formula in Γ; and
3. there are i and $k (i, k < j)$ such that A_j is obtained by A_i and A_k with Modus Ponens.

When $j = 1$, A_1 is an axiom of F_n or a formula in Γ, and then obviously we have $\Gamma \models A_1$. When $j > 1$, suppose $\Gamma \models A_j$ is suitable for every positive integer that is less than j. If A_j is an axiom of F_n or a formula in Γ, and then obviously we have $\Gamma \models A_j$. If A_j is obtained by using Modus Ponens rule, then $\exists$ i and k $(i, k < j)$, such that A_k is $A_i \to A_j$. Accordingly, we have $\Gamma \models A_i$ and $\Gamma \models A_i \to A_j$. So we have $V(A_i) \in LTTS$, $V(A_i \to A_j) \in LTTS$ for any model V. Let $V(A_i) = \tau$, then we have $\bar{\tau} \in LTFS$ and further we have $max\{\bar{\tau}, V(A_j)\} \in LTTS$. Therefore, we can get $max\{\bar{\tau}, V(A_j)\} = V(A_j) \in LTTS$. Thus, we have $\Gamma \models A_j$.

By the method of induction, we know that for all j, $\Gamma \models A_j$, so $\Gamma \models A_m$, *i.e.*, $\Gamma \models A$. □

5.2 Completeness

Intuitively, we say a logic system is completeness, meaning that for a logic formula, if it is correct in the sense of semantics, then it is correct in the sense of syntax.

Definition 15. *Γ is a set of formulas, $\Gamma \subseteq Form(L_0)$. Let $\overline{\Gamma}$ denote the set of all formulas A such that $\Gamma \vdash A$.*

1. *We say that a set Γ of formulas is trivial iff $\overline{\Gamma} = Form(L_0)$; otherwise, it is non-trival.*
2. *Γ is inconsistent iff there is at least one formula A such that both A and $\neg A$ belong to $\overline{\Gamma}$; otherwise, Γ is consistent.*

Non-trival is an important concept in paraconsistent logic. If a formula set Γ can deduce all the formulas, then it does not need to be studied. That is why we say that it is trivial. Classical logic allows contradictories to imply everything, so inconsistent logic is trivial. Nonetheless, paraconsistent logic admits the existence of inconsistent but is not a trivial theory. So, a nontrivial and inconsistent theory is just what paraconsistent logic is studying, but any trivial and inconsistent theory needs not to be studied.

Definition 16. *Γ is maximal non-trivial iff it is non-trivial and, for any formula A, if $A \notin \Gamma$, then $\Gamma \cup \{A\}$ is trivial.*

Theorem 5. *Every non-trivial set of formulas is contained in a maximal non-trivial set.*

Proof. The proof is the same as that in classical logic, so for the sake of space it is omitted . □

Theorem 6. *Every maximal non-trivial set of formulas has a model.*

Proof. Define a mapping

$$V : Form(L_0) \to LTS$$

satisfying that for a formula A, if $A \in \Gamma$ then $V(A) \in LTTS$; otherwise, $V(A) \in LTFS$. It is then easy to see that V satisfies all the conditions in the definition of a valuation (*i.e.*, Definition 9). □

Intuitively, the following theorem of completeness means that all the formulas with high credibility in Γ can be deduced from F_n.

Theorem 7. $F_n(1 \leq n \leq \omega)$ *is complete,* i.e., $\Gamma \models A \Rightarrow \Gamma \vdash A$.

Proof.

$$\begin{aligned}
\Gamma \models A \Leftrightarrow & \text{for all the model of } \Gamma, V(A) \in LTTS \\
\Leftrightarrow\ & \nexists V \text{ such that } V \text{ is the model of } \Gamma \\
& \qquad \text{and } V(A) \in LTFS \\
\Leftrightarrow\ & \nexists V \text{ such that } V \text{ is the model of } \Gamma \\
& \qquad \text{and } V(\neg A) \in LTTS \\
\Rightarrow & \Gamma \cup \{\neg A\} \text{ has no model} \\
\Rightarrow & \Gamma \cup \{\neg A\} \text{ is trivial} \\
\Rightarrow & \overline{\Gamma \cup \{\neg A\}} = \text{the set of all the formulas} \\
\Rightarrow & \Gamma \cup \{\neg A\} \text{ can deduce all the formulas} \\
\Rightarrow & \Gamma \cup \{\neg A\} \vdash \neg\neg A \\
\Rightarrow & \Gamma \cup \{\neg A \vee A\} \vdash \neg\neg A \\
\Rightarrow & \Gamma \vdash \neg\neg A \\
\Rightarrow & \Gamma \vdash A.
\end{aligned}$$

□

6 ILLUSTRATION

The technology of expert system is one of the most successful applications of artificial intelligence. An expert system is to collect as more expert knowledge as possible and typically translate them into a series of rules in the form of "if … then …". According to these rules, the computer will be able to solve a problem like an expert. Many traditional expert systems are built upon the basis of classical logic, which has some significant limitations. For example, different experts in the same field may have different opinions for some deep problems, and thus it may lead to some inconsistency in the knowledge base. However, even if there are some inconsistent knowledge, we should not give up the whole knowledge base because there are some useful and consistent knowledge. So, required is a certain degree tolerance of contradictions. This is exactly the practical value of fuzzy paraconsistent logic.

Let us examine an example of a medical expert system (Yang, 2005). Suppose that for the disease d_1 and d_2, doctors 1 and 2 have their own diagnostic rules as follows:

- The rules of doctor 1:
  1. if a patient gets symptom s_1 and s_2, then the patient suffers from disease d_1;
  2. if a patient gets symptom s_1 and s_3, then the patient suffers from disease d_2;
  3. if a patient has disease d_1, then the patient does not suffer from disease d_2; and
  4. if a patient has disease d_2, then the patient does not suffer from disease d_1.
- The rules of doctor 2:
  1. if a patient gets symptom s_1 and s_4, then the patient suffers from disease d_1; and
  2. if a patient gets symptom s_3 and does not get symptom s_1, then the patient suffers from disease d_2.

Now suppose we get two patients. Patient a gets symptom s_1, s_3, and s_4 but does not get symptom s_2; and patient b gets symptom s_2, s_3, and s_4 but does not get symptom s_1. According to doctor 1, patient a has disease d_2 but does not have disease d_1. Rather, according to doctor 2, patient a has disease d_1. That is, there is a contradiction about the diagnosis of patient a. Nonetheless, this contradiction does not influence upon the diagnosis of patient b. Patient b has disease d_2 according to doctor 2 and does not have disease d_1 according to doctor 1. Although the knowledge base contains contrary knowledge about patient a, it can still be used to diagnose b. So, the knowledge base with contradictions is still useful and should not be abandoned. Fuzzy paraconsistent logic has greater practical value just because it can solve such problems that paraconsistent logic cannot solve.

Sometimes it is insufficient that a symptom is merely confirmed the presence or absence, we need to determine how serious the symptom is. If we design an objective indicator to measure the severity of certain symptoms, such as body temperature can be an indicator of the severity of fever, the concentration of a substance can be an indicator of the severely of virus infection, the diagnose will be more accurate.

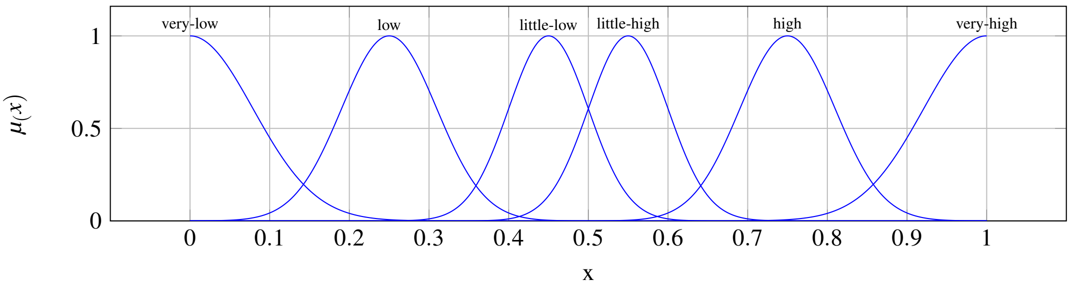

Figure 2: Membership functions of d_1 and d_2.

Suppose symptom s_1 is fever. Normally, the oral temperature of an adult is 37 degree centigrade. Thus, it is reasonable to use the difference between body temperature and 37 as the indicator of the severity of fever. Specifically, we assume there are six grades for the severity: *very-slight, moderate-slight, a little-slight, a little-severe, moderate-severe*, and *very-serious*. For the proposition of "someone being a high fever", we can say *very-false, moderate-false, slightly-false, slightly-true, moderate-true,* and *very-true*. The six kinds of linguistic truth-value can correspond to the six grades of severity of fever. Hence, we can draw the membership function of six linguistic truth-values of s_1 as shown in Figure 1.

In our new medical expert system, we still lack the figures of membership functions of other three symptoms.Without losing generality, we can set them as shown in Figure 1.

After we get the exact data of four symptoms, the next step is to define the fuzzy rules. The diagnosed rules of doctors 1 and 2 can be simply written as follows:

1. $s_1 \wedge s_2 \to d_1$,
2. $s_1 \wedge s_4 \to d_1$,
3. $s_1 \wedge s_3 \to d_2$,
4. $\neg s_1 \wedge s_3 \to d_2$.

According to the above four rules, we can set the corresponding fuzzy rules as shown in Figure 3.

We divide the possibility of having a disease into six grades: *very-low, low, little-low, little-high, high,* and *very-high*. Then we can define the membership functions of d_1 and d_2 as follows:

$$\mu_{very\text{-}low}(x) = e^{-80x^2}, \quad (19)$$

$$\mu_{low}(x) = e^{-140(x-0.25)^2}, \quad (20)$$

$$\mu_{little\text{-}low}(x) = e^{-200(x-0.45)^2}, \quad (21)$$

$$\mu_{little\text{-}high}(x) = \mu_{little\text{-}low}(1-x), \quad (22)$$

$$\mu_{high}(x) = \mu_{low}(1-x), \quad (23)$$

$$\mu_{very\text{-}high}(x) = \mu_{very\text{-}low}(1-x), \quad (24)$$

1. If (s_1 is very-false) and (s_2 is very-true) then (d_1 is very-low) (1)
2. If (s_1 is moderate-false) and (s_2 is very-true) then (d_1 is low) (1)
3. If (s_1 is slightly-false) and (s_2 is very-true) then (d_1 is little-low) (1)
4. If (s_1 is slightly-true) and (s_2 is slightly-true) then (d_1 is little-high) (1)
5. If (s_1 is miderate-true) and (s_2 is moderate-true) then (d_1 is high) (1)
6. If (s_1 is very-true) and (s_2 is very-true) then (d_1 is very-high) (1)
7. If (s_1 is very-true) and (s_4 is very-true) then (d_1 is very-high) (1)
8. If (s_1 is miderate-true) and (s_4 is moderate-true) then (d_1 is high) (1)
9. If (s_1 is slightly-true) and (s_4 is slightly-true) then (d_1 is little-high) (1)
10. If (s_1 is slightly-false) and (s_4 is very-true) then (d_1 is little-low) (1)
11. If (s_1 is moderate-false) and (s_4 is very-true) then (d_1 is low) (1)
12. If (s_1 is very-false) and (s_4 is very-true) then (d_1 is very-low) (1)
13. If (s_1 is very-true) and (s_3 is very-true) then (d_2 is very-high) (1)
14. If (s_1 is miderate-true) and (s_3 is moderate-true) then (d_2 is high) (1)
15. If (s_1 is very-false) and (s_3 is very-true) then (d_2 is very-high) (1)
16. If (s_1 is moderate-false) and (s_3 is moderate-true) then (d_2 is high) (1)
17. If (s_1 is slightly-false) and (s_3 is slightly-true) then (d_2 is little-high) (1)
18. If (s_1 is slightly-true) and (s_3 is slightly-true) then (d_2 is little-high) (1)

Figure 3: Fuzzy rules.

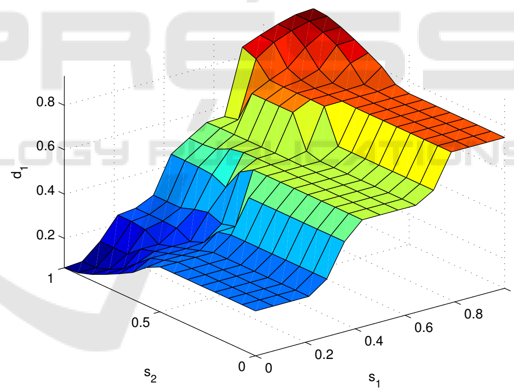

Figure 4: The possibility of d_1 changes with those of s_1 and s_2.

where $x \in [0,1]$. The curves of the above membership functions is shown in Figure 2.

According to the situation of symptoms, this inference fuzzy system can output the possibility of having a disease. With the help of the rule viewer of Matlab, we can see the whole output situation of the fuzzy reasoning system, as shown in Figure 4. There each coordinate has a corresponding meaning. For example, point (0.8, 0.5, 0.581) represents that when the input value of s_1 is 0.8 and that of s_2 is 0.5, the possibility of having disease d_2 is of 0.581.

From the above example, we can see our fuzzy paraconsistent logic has greater practical value than

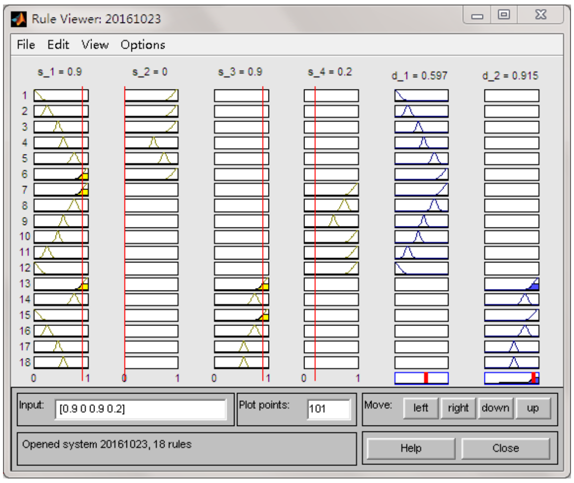

Figure 5: When s_1 and s_3 are serious and s_4 is slight, more likely the patient suffers from d_2.

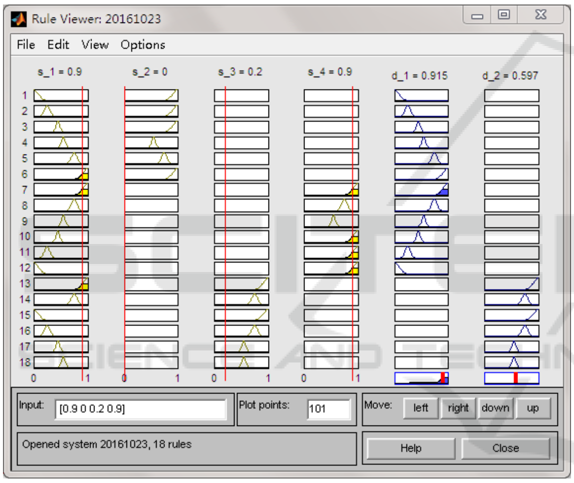

Figure 6: When s_1 and s_4 are serious and s_3 is slight, more likely the patient suffers from d_1.

paraconsistent logic. In fact, in the medical expert system above, there is a contradiction about the diagnosis of patients a with symptom s_1, s_3 and s_4, so that according to the two-valued paraconsistent logic we cannot decide whether a is suffering from disease d_1 or d_2. However, in the medical expert system with ours of multiple linguistic truth-values, we can calculate the possibilities of suffering from diseases d_1 and d_2. So, we can base on the more accurate data to decide what kind of treatment should be taken.

Figures 5 and 6 show how the possibility of a disease varies with the severity degree of symptoms. The data of Figure 5 is $\{0.9, 0, 0.9, 0.2, 0.597, 0.915\}$, meaning that if s_1 is severe, s_3 is severe and s_4 is slight, then the possibility of d_2 is very large, and so it is better to use drugs that can properly treat disease d_2. The data of Figure 6 is $\{0.9, 0, 0.2, 0.9, 0.915, 0.597\}$, meaning that if s_1 is severe, s_4 is severe and s_3 is slight, then the possibility of d_1 is very large, and so it is better to use drugs that can properly treat disease d_1.

Paraconsistent logic advocates that contradictions should be tolerated, but have to be limited in a certain range, which is correct. However, from another viewpoint, it actually equals to leave the contradictions to fend for themselves, which seems a little irresponsible. So the way in which paraconsistent logic deals with contradictions somehow is improper. Rather, we can see from the above example that our fuzzy paraconsistent logic can make up for this shortcoming. Actually, it can provide the weights of the both sides of a contradiction, so that people can make better decisions when facing a contradiction.

7 CONCLUSION

Paraconsistent logic is the unique logic that can deal with an inconsistent theory, so it has a wide application in many areas, especially in artificial intelligence. However, it is still a semantically two-valued logic. Obviously, absolutely true or false is not enough in real life because people tend to use the vague phrases like very-true, slightly-true, slightly-false, very-false, and so on. Therefore, this paper enables a proposition in paraconsistent logic to take its truth value from a set of multiple linguistic terms, so that it can be applied to a wider scope in real life. Moreover, we also prove the soundness and completeness of this kind of paraconsistent logic with multiple linguistic truth-values. In addition, this paper illustrates the practical value of our fuzzy paraconsistent logic by a real life example, but its significance is far more than that. Actually, the potential applications of our fuzzy paraconsistent logic are not limited to expert systems, other more areas such as legal, political and economic areas are applicable, too. Inconsistence and fuzzification are ubiquitous, so our fuzzy paraconsistent logic has great potential waiting to be explored.

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REFERENCES

Abe, J. M. (2016). Paraconsistent logics and applications. In *New Approaches in Intelligent Control*, volume 107 of *Intelligent Systems Reference Library*, pages 273–300. Springer.

Akama, S. and Da Costa, N. C. (2016). Why paraconsistent logics? In *Towards Paraconsistent Engineering*, volume 110 of *Intelligent Systems Reference Library*, pages 7–24. Springer.

Anderson, A. R., Belnap, N. D., and Dunn, J. M. (1978). Entailment: The logic of relevance and necessity. *Philosophical Books*, 19(2):75–77.

Arnon, A. (2014). Paraconsistent fuzzy logic preserving non-falsity. *Fuzzy Sets and Systems*, 292:75–84.

Costa, N. D. and Carnielli, W. (1986). On paraconsistent deontic logic. *Philosophia*, 16(3-4):293–305.

Costa, N. D., D.Krause, and O.Bueno (2005). Paraconsistent logics and paraconsistency. *Philosophy of Logic*, 4:791–911.

Da Costa, N. C., Béziau, J.-Y., Bueno, O. A., et al. (1995). Aspects of paraconsistent logic. *Logic Journal of the IGPL*, 3(4):597–614.

Da Costa, N. C. A. (1958). On the theory of inconsistent formal systems. *Bulletin De La Socit De Chimie Biologique*, 40(7-8):1179–1187.

Dou, F., Jia, L., Wang, L., Xu, J., and Huang, Y. (2014). Fuzzy temporal logic based railway passenger flow forecast model. *Computational Intelligence and Neuroscience*, 2014:Article No. 42.

Ertola, R., Esteva, F., Flaminio, T., Godo, L., and Noguera, C. (2013). Exploring paraconsistency in degree-preserving fuzzy logics. In *Proceedings of the 8th Conference of the European Society for Fuzzy Logic and Technology*, pages 117–124.

Ijsselmuiden, J., Nch, D., Grosselfinger, A. K., Arens, M., and Stiefelhagen, R. (2014). Automatic understanding of group behavior using fuzzy temporal logic. *Journal of Ambient Intelligence and Smart Environments*, 6(6):623–649.

Jing, X., Luo, X., and Zhang, Y. (2014). A fuzzy dynamic belief logic system. *International Journal of Intelligent Systems*, 29(7):687–711.

Kamide, N. (2013). Formalizing inconsistency-tolerant relevant human reasoning: A decidable paraconsistent relevant logic with constructible falsity. In *Proceedings of 2013 IEEE International Conference on Systems, Man, and Cybernetics*, pages 1865–1870.

Kamide, N. (2016). A decidable paraconsistent relevant logic: Gentzen system and routley-meyer semantics. *Mathematical Logic Quarterly*, 62(3):177–189.

Moon, S., Lee, K., and Lee, D. (2004). Fuzzy branching temporal logic. *IEEE Transactions on Systems Man and Cybernetics: Part B Cybernetics*, 34(2):1045–1055.

Mukherjee, S. and Dasgupta, P. (2013). A fuzzy real-time temporal logic. *International Journal of Approximate Reasoning*, 54(9):1452–1470.

Poli, V. S. R. (2015). Fuzzy temporal predicate logic for incomplete information. In *Proceedings of 2015 International Conference on Fuzzy Theory and Its Applications*, pages 86–90.

Priest, G., Tanaka, K., and Weber, Z. (1989). *Paraconsistent Logic*. München.

Rodrguez, J., Turunen, E., Da, R., and Montero, J. (2014). Another paraconsistent algebraic semantics for lukasiewiczpavelka logic. *Fuzzy Sets and Systems*, 242(242):132–147.

Tanaka, K., Berto, F., Mares, E., and Paoli, F. (2012). *Paraconsistency: Logic and applications*, volume 26. Springer.

Thiele, H. and Kalenka, S. (1993). On fuzzy temporal logic. In *Proceeding of the 2nd IEEE International Conference on Fuzzy Systems*, pages 1027–1032.

Turunen, E. (1992). On fuzzy intuitionistic logic. *Kybernetika*, 28(7):72–77.

Turunen, E., Öztürk, M., and Tsoukis, A. (2010). Paraconsistent semantics for pavelka style fuzzy sentential logic. *Fuzzy Sets and Systems*, 161(14):1926–1940.

Vidal, A., Esteva, F., and Godo, L. (2015). On modal extensions of product fuzzy logic. *Journal of Logic and Computation*, page exv046.

Yager, R. R. and Zadeh, L. A. (1992). *An Introduction to Fuzzy Logic Applications in Intelligent systems*, volume 165 of *The International Series in Engineering and Computer Science*. Springer.

Yang, W. (2005). Theoretical significance and practical value of paraconsistent logic. *Journal of Renmin University of China*, (2):63–69.

Zadeh, L. A. (1965). Fuzzy sets. *Information and Control*, 8(65):338–353.

Zadeh, L. A. (1983). The role of fuzzy logic in the management of uncertainty in expert systems. *Fuzzy sets and systems*, 11(1):199–227.

Zadeh, L. A. (1996). Fuzzy logic=computing with words. *IEEE transactions on fuzzy systems*, 4(2):103–111.

Zhan, J., Luo, X., Feng, C., and Ma, W. (2014). A fuzzy logic based bargaining model in discrete domains: Axiom, elicitation and property. In *Proceedings of 2014 IEEE International Conference on Fuzzy Systems*, pages 424–431.