# Process Optimization for Cutting Steel-Plates

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Abstract: In this paper, we consider the two-stage three-dimensional guillotine cutting stock problem with usable leftover. There are two properties that distinguish our problem formulation from others. First, we allow the items to be rotated. Second, we consider the case in which leftover material is to be reused in subsequent production cycles. Third, we solve the problem in three dimensions. The optimization problem is formulated as a mixed integer linear program. To verify the approach, several examples show that our formulation performs well.

## 1 INTRODUCTION

In many areas of industrial production large or long pieces of material need to be cut into smaller ones. E.g., it needs to be decided from which reel of raw material a cable of a specific length will be produced. This is an example for a one-dimensional problem. An example for a two-dimensional problem is to cut patterns from pieces of large leather. In our case, we want to cut items out of slabs of steel. This is a three-dimensional problem.

The cutting stock problem is similar to the bin packing problem and the knapsack problem. In the bin packing problem objects of different sizes must be packed into a finite number of bins. The classical bin packing problem minimizes the number of bins that are used. For the knapsack problem, there is only one bin of a certain volume and the objects to pack have a value and a volume. Then, the objective is to maximize the value of the packed objects. Both problems are, in general, NP-hard problems.

In this paper, we formulate a problem that is similar to the above mentioned problems and solves the cutting stock problem. The problem is formulated such that the optimization program becomes linear and we can find an optimal solution according to our objective function.

The general problem of cutting stock is explored widely in the literature. The earliest work we found, that minimizes scrap material, is (Kantorovich, 1960). We cite the English translation, which is, to the best of our knowledge, a translation from the original Russian publication from 1939. (Kurt Eisemann, 1957), and (Gilmore and Gomory, 1961), utilize linear programming to solve cutting stock problems. P. Gilmore and Gomory investigate the one dimensional problem of cutting items from stock of several standard lengths. They devise the method of column generation in their work. P. Gilmore and Gomory continued their work in (Gilmore and Gomory, 1965). Here, they extend their formulation to multistage cutting stock problems of two and more dimensions.

(Farley, 1988), adapt the approach of (Gilmore and Gomory, 1961), with some practical adaptations. In particular, they introduce guillotine cuts, which will be explained in detail in the following section.

Solutions not utilizing linear programming have also been published, e.g., tree-search algorithms, (Christofides and Whitlock, 1977). The survey (Hinxman, 1980), summarizes many more publications. (Dyckhoff et al., 1985), give a detailed catalog of criteria for characterization of real-world cutting stock problems (Dyckhoff et al., 1985). (Yanasse et al., 1991), describe heuristics for the cutting stock problem. (Carnieri et al., 1994), formulate the cutting stock problem as a Knapsack algorithm and also give heuristics to enhance computational efficiency.

(Kalagnanam et al., 2000), formulate the problem for the steel industry, without considering the depth of the slabs. (Martello et al., 2000), present a three-dimensional bin packing problem, while allowing arbitrary, i.e., non-guillotine, cuts. (Morabito and Arenales, 2000), focus on a simplified cutting pattern. In their book *Operations Research: Applications and Algorithms*, (Wayne L Winston and Jeffrey B Goldberg, 2004), summarize and extend some of the aforemen-

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tioned as well as many more methods.

(Silva et al., 2010), consider waste, but lack the third dimension in their problem formulation. They also mention the value of the surplus material on a conceptual level, but do not show it in their problem formulation. (Burke et al., 2011), present an iterative packing methodology based on squeaky wheel optimization. (Furini and Malaguti, 2013), make similar formulations as previously found in the literature, but focus on the run-time of the solution.

The recent results from (Andrade et al., 2013), lay the foundation for our problem formulation. (Andrade et al., 2013), investigate two-stage two-dimensional guillotine cutting stock problems with usable leftover. We will extend their formulation for two-stage three-dimensional guillotine cutting stock problems with usable leftover. The term “stage” will be explained when we describe the constrains that are imposed by our production machinery.

There are three properties that distinguish our problem formulation from others. First, we allow the items to be rotated.

Second, we consider the case in which leftover material is to be reused in subsequent production cycles. I.e., after one production cycle, leftover material is added to the set of slabs.

Third, we solve the problem in three dimensions. On the one hand, this is needed to calculate the weight, which is required in our objective function. On the other hand, this opens up for the possibility to cut an item from a slab that is thicker than necessary, if the value of the leftover material permits or even dictates this decision.

The remainder of the present paper is organized as follows. In Section 2, we present the problem we want to solve and the model we use. In Section 3, we propose an optimization program to solve the problem. In Section 4, we discuss some examples and their solutions. Finally, we summarize our results in Section 5.

## 2 PROBLEM FORMULATION AND MODEL DESCRIPTION

The overall goal is to fulfill customer orders of blocks of steel. We call the ordered blocks of steel *items*. These items are cut from larger *slabs* of steel. As the size of an item is defined by the customer, the size is, in general, different and arbitrary from item to item.

The problem we solve in this work is the decision which item is cut from which slab and how items are geometrically placed on each slab. Due to the production process, or technical and economic restrictions, these problems can have an abundance of constraints. It is not important that the material we are working with is steel. However, the machines that are used for cutting the slabs create certain constraints in our problem formulation.

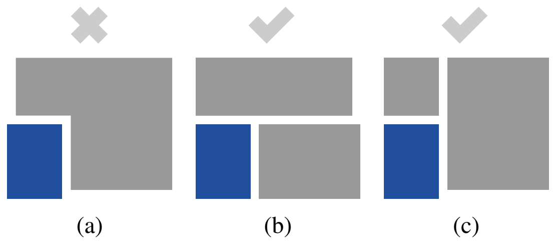

Figure 1: Guillotine cutting in (b) and (c).

In general, when items are cut from slabs, the original slab is cut up into items and surplus material. The surplus material can either be useful in the future or it is so small that it is thrown away. If it is kept we call it *leftover material* and it will be placed in the set of slabs for future usage. If it is thrown away we call it *scrap material*.

One of our goals is to use surplus material more frequently than new slabs. Otherwise, the slabs in stock could increase over time. Our solution for this problem is to attach a value per kilogram to all the slabs. The less a slab weights in total, the smaller its value per kilogram.

The machines cutting the slabs can only do full straight cuts, parallel to the edges of the slab and not stop halfway through the material. This is called guillotine cutting.

Figure 1a shows how the blue item can *not* be cut from the gray slab if it is placed in the bottom left corner. The item has either to be cut as seen in Fig. 1b or Fig. 1c. Leftover material will have different shapes, not only depending on the exact geometrical placement of the item on the slab, but also depending on the exact cuts being made. Currently, in order to calculate the dimensions of the surplus material, the problem formulation assumes a strict cutting order, which will be explained shortly.

Our model follows (Lodi and Monaci, 2003), in using the notion of shelves, as does (Andrade et al., 2013). Shelves are a way to connect the restriction on guillotine cutting and geometric placement of items. Figure 2 shows the general layout of items and shelves on a slab.

In this figure, item 1 and item 2 are on the same shelf, while item 3 is on a separate shelf. To make this clear, a shelf is just a notion to group and place items on a slab. Further, shelf $\nu$ is opened by item $\nu$. The height of all items, that are placed in this shelf after item $\nu$ must be less than or equal to the height of item $\nu$. I.e., the height of shelf $\nu$ is equivalent to the height

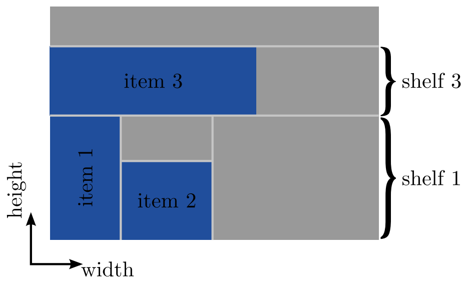

Figure 2: Shelves.

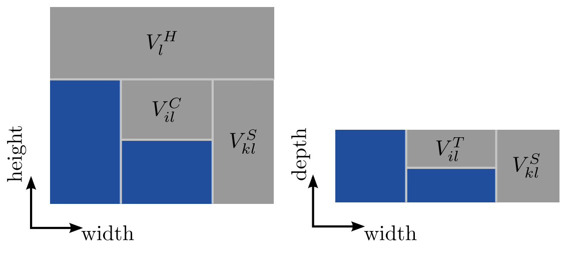

Figure 3: Positions of leftover material.

of item ν. An item is either placed in an existing shelf or opens a new shelf. As a result, if item $\hat{\nu}$ was placed in an existing shelf, then shelf $\hat{\nu}$ does not exist.

As was mentioned earlier, the size and form of the surplus material depends on the order in which cuts are made. Therefore, our problem formulation assumes a specific order of the cuts. This is necessary to calculate the precise dimensions, and hence the value, of the surplus material. The result of the cuts is depicted in Fig. 3.

The figure is ambiguous about the depth of $V_{il}^C$, which is the leftover material filling the height above item $i$ on slab $l$. It has the same depth as the slab, which is clear from the following cutting order.

1. Slabs are cut in the direction of "width" such that shelves are cut out.
2. Each shelf is cut in the direction of "height" such that the widths of items are correct.
3. $V_{il}^C$ are trimmed from the items.
4. $V_{il}^T$ are trimmed from the items.

Trimming in this context means to cut unwanted pieces from the items, e.g., separating $V_{il}^T$ from the item.

(Andrade et al., 2013), describe their model as having "two stages".

> In the first stage, parallel longitudinal (horizontal) guillotine cuts are produced on a plate, without moving it, to produce a set of strips. In the second stage, these strips are pushed, one by one, and the remaining parallel transversal (vertical) guillotine cuts are made on each strip. ((Andrade et al., 2013, p. 2))

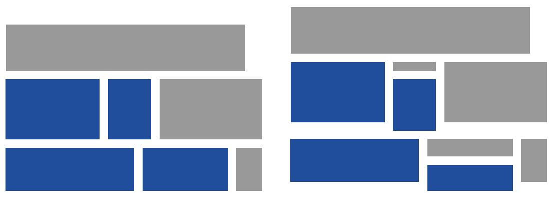

Figure 4: Two-stage cutting patterns.

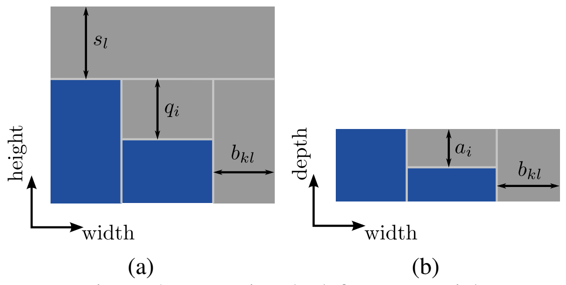

Figure 5: Measuring the leftover material.

Further, (Andrade et al., 2013), distinguish between the exact and non-exact cases of matching items to shelves. Figure 4a shows the exact case. Here, all items in the same shelf have the same height. On the other hand, Fig. 4b shows the non-exact case. Here, trimming is necessary, because items in the same shelf might have a different height. Recall that (Andrade et al., 2013), only account for two dimensions. They do not treat trimming as a separate stage.

Our formulation matches items to slabs in three dimensions considering the non-exact case. Items are trimmed at most two times, but items are not stacked in the third dimension. So we call our problem a two-stage three-dimensional non-exact guillotine cutting stock problem.

The objective function of our optimization problem will maximize the value of the surplus material, accounting for the lost value of cut slabs. A value needs to be attached to the volumes of the pieces of surplus material. In order to calculate the volumes, the measures depicted in Fig. 5 are used.

As we mentioned earlier, the value of each piece of surplus material is a function of its weight. The weight of each piece is calculated by simply multiplying its volume by the weight per volume unit. The constants $g_l^e$ define the boundaries of the weight classes. $\alpha_l^e$ is the value relative to the value of the slab the piece originates from. $l$ denoting the slab, $e$ denoting the weight class. Explicit example values for $g_l^e$ and $\alpha_l^e$ are given in Table 1. E.g., the value per kilogram of a piece of surplus material weighting between 2.1 kg and 5 kg is 0.5 times the value per kilogram of the slab it originates from.

Surplus material is considered waste if one of its sides is too small. See $d_l^{W_{\min}}$, $d_l^{H_{\min}}$, $d_l^{T_{\min}}$ in Table 1.

We allow items to be rotated in the width-height-axis. This is done by copying each item and rotating the copy. We then restrict placement such that only one of these items is produced. More on this in the optimization program itself.

Further, we require items to be sorted by decreasing height, i.e., $h_i \geq h_{i+1}$. This is a result of how the height of shelves is determined in our optimization problem. The first item to be placed in a shelf defines its height. Only items with a larger index are allowed to be put into this shelf. So all items following the first item must have the same or a smaller height, such that they can be placed on any of the existing shelves. Therefore, items are sorted by decreasing height before given an index.

## 3 THE OPTIMIZATION PROGRAM

Below is the complete optimization program, which will be explained afterwards.

Define $\mathbb{F}_N = \{1,\dots,N\}$ for any $N \in \mathbb{N}$. Unless otherwise mentioned, the constraints apply to $\forall i \in \mathbb{F}_n$, $\forall k \in \mathbb{F}_n$, $\forall l \in \mathbb{F}_p$. $n$ and $p$ are number of items and number of slabs, respectively, see Table 1.

The main variables are $x_{ikl}$, i.e., it is one if item $i$ is placed in shelf $k$ on slab $l$.

$$\text{maximize} \quad \sum_{i=1}^{n}\sum_{k=1}^{i}\sum_{l=1}^{p} w_i h_i t_i G_l p_i x_{ikl} \tag{1}$$

$$- \sum_{l=1}^{p} M_l P_l u_l \tag{2}$$

$$- \sum_{l=1}^{p} (P_l^0 - P_l) R_l \tag{3}$$

$$+ \sum_{j=1}^{|V|} F_j \tag{4}$$

subject to all conditions in Table 2,

$$\sum_{l=1}^{p}\sum_{k=1}^{i}\sum_{i\in c_i} x_{ikl} = 1\,,$$

$$\forall c_i = \{j \mid j = i \text{ or } j \text{ is rotated version of } i\}\,, \tag{5}$$

$$\sum_{k=i+1}^{n}\sum_{l=1}^{p} x_{ikl} = 0\,, \tag{6}$$

$$u_l \geq \frac{1}{n}\sum_{i=1}^{n}\sum_{k=1}^{n} x_{ikl} \tag{7}$$

$$b_{kl} = W_l x_{kkl} - \sum_{i=k}^{n} w_i x_{ikl}\,,$$

$$\forall k \in \mathbb{F}_n\,,\ \forall l \in \mathbb{F}_p\,, \tag{8}$$

$$b_k = \sum_{l=1}^{p} b_{kl}\,, \tag{9}$$

$$s_l = H_l u_l - \sum_{k=1}^{n} h_k x_{kkl}\,, \tag{10}$$

$$q_i = \sum_{l=1}^{p}\sum_{k=1}^{i} (h_k - h_i) x_{ikl}\,, \tag{11}$$

$$a_i = \sum_{l=1}^{p}\sum_{k=1}^{i} (T_l - t_i) x_{ikl}\,, \tag{12}$$

$$s_l \geq d_l^{H_{\min}} - \hat{H}(1 - z_l^H)\,, \tag{13}$$

$$s_l \leq \hat{H} z_l^H + d_l^{H_{\min}}\,, \tag{14}$$

$$V_l^H \leq s_l W_l T_l G_l\,, \tag{15}$$

$$V_l^H \leq H_l W_l T_l G_l z_l^H\,, \tag{16}$$

$$b_k \geq \sum_{l=1}^{p} d_l^{W_{\min}} x_{kkl} - (1 - z_k^S)\hat{W}\,, \tag{17}$$

$$b_k \leq \hat{W} z_k^S + \sum_{l=1}^{p} d_l^{W_{\min}} x_{kkl}\,, \tag{18}$$

$$G_l V_{kl}^S \leq M_l z_k^S\,, \tag{19}$$

$$V_{kl}^S \leq b_k h_k T_l G_l\,, \tag{20}$$

$$V_{kl}^S \leq W_l h_k T_l G_l x_{kkl}\,, \tag{21}$$

$$q_i \geq \sum_{l=1}^{p}\sum_{k=1}^{i} d_l^{H_{\min}} x_{ikl} - (1 - z_i^C)\hat{H}\,, \tag{22}$$

$$q_i \leq \hat{H} z_i^C + \sum_{l=1}^{p}\sum_{k=1}^{i} d_l^{H_{\min}} x_{ikl}\,, \tag{23}$$

$$G_l V_{il}^C \leq M_l z_i^C\,, \tag{24}$$

$$V_{il}^C \leq w_i q_i T_l G_l\,, \tag{25}$$

$$V_{il}^C \leq w_i H_l T_l G_l \sum_{k=1}^{i} x_{ikl}\,, \tag{26}$$

$$a_i \geq \sum_{l=1}^{p}\sum_{k=1}^{i} d_l^{T_{\min}} x_{ikl} - \hat{T}(1 - z_i^T)\,, \tag{27}$$

$$a_i \leq \hat{T} z_i^T + \sum_{l=1}^{p}\sum_{k=1}^{i} d_l^{T_{\min}} x_{ikl}\,, \tag{28}$$

$$G_l V_{il}^T \leq M_l z_i^T\,, \tag{29}$$

$$V_{il}^T \leq w_i h_i a_i G_l\,, \tag{30}$$

$$V_{il}^T \leq w_i h_i T_l G_l \sum_{k=1}^{i} x_{ikl}\,, \tag{31}$$

$$R_l = M_l - \underbrace{\sum_{i=1}^{n}\sum_{k=1}^{i} w_i h_i t_i G_l x_{ikl}}_{\text{assigned items}}$$

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Table 1: Inputs and Identifiers.

| **Input or Identifier** | **Description** |
|---|---|
| $w_i, h_i, t_i$ | Width, height and depth of item $i$, with $i = 1, \dots, n$. |
| $p_i$ | Value of item $i$ per kilogram. |
| $W_l, H_l, T_l$ | Width, height and depth of slab $l$, with $l = 1, \dots, p$. |
| $G_l$ | Weight per volume unit of slab $l$, e.g., $7.85 \cdot 10^{-6} \frac{kg}{mm^3}$. |
| $M_l = W_l H_l T_l G_l$ | Weight of slab $l$. |
| $\hat{W} = \max_{l=1,\dots,p} \{W_l\}$ | Greatest width of slabs. |
| $\hat{T} = \max_{l=1,\dots,p} \{T_l\}$ | Greatest height of slabs. |
| $\hat{H} = \max_{l=1,\dots,p} \{H_l\}$ | Greatest depth of slabs. |
| $P_l^0$ | Purchase value of slab $l$ per kilogram. |
| $P_l$ | Current value of slab $l$ per kilogram. |
| $g_l^e$ | Boundaries of weight classes, specific to slab $l$, e.g., $\begin{pmatrix} g_1^1 & g_1^2 & g_1^3 & g_1^4 = g_1^m \end{pmatrix} = \begin{pmatrix} 0 & 2.1 & 5.1 & 10.1 \end{pmatrix}$. |
| $\alpha_l^e$ | Relative value for weight classes, specific to slab $l$, e.g., $\begin{pmatrix} \alpha_1^1 & \alpha_1^2 & \alpha_1^3 & \alpha_1^4 = \alpha_1^m \end{pmatrix} = \begin{pmatrix} 0.2 & 0.5 & 0.6 & 1 \end{pmatrix}$. |
| $d_l^{W_{\min}}, d_l^{H_{\min}}, d_l^{T_{\min}}$ | Minimum width, height, and depth before leftover material is considered waste, e.g., $d_l^{W_{\min}} = d_l^{H_{\min}} = d_l^{T_{\min}} = 10\,cm$ |

$$- \sum_{k=1}^{n} V_{kl}^S - \sum_{i=1}^{n} V_{il}^C$$

$$- \sum_{i=1}^{n} V_{il}^T - V_l^H \,, \qquad (32)$$

$$\begin{aligned} V := \quad & \{V_l^H > 0 \mid l = 1, \dots, p\} \\ & \cup \{V_k^S > 0 \mid k = 1, \dots, n\} \\ & \cup \{V_i^C > 0 \mid i = 1, \dots, n\} \\ & \cup \{V_i^T > 0 \mid i = 1, \dots, n\} \,, \qquad (33) \end{aligned}$$

$$z_{je}^R g_{l(j)}^e \leq V_j \,, \quad \forall j \in \mathbb{F}_{|V|} \,, \ \forall e \in \mathbb{F}_m \,, \qquad (34)$$

$$V_j < g_{l(j)}^{e+1} + M_{l(j)}(1 - z_{je}^R) \,, \quad \forall j \in \mathbb{F}_{|V|} \,, \ \forall e \in \mathbb{F}_{m-1} \,, \qquad (35)$$

$$\sum_{e=1}^{m} z_{je}^R = 1 \,, \ \forall j \in \mathbb{F}_{|V|} \,, \qquad (36)$$

$$F_{je} \leq V_j P_{l(j)}^0 \alpha_{l(j)}^e \,, \quad \forall j \in \mathbb{F}_{|V|} \,, \ \forall e \in \mathbb{F}_m \,, \qquad (37)$$

$$F_{je} \leq M_{l(j)} P_{l(j)}^0 \alpha_{l(j)}^e z_{je}^R \,, \quad \forall j \in \mathbb{F}_{|V|} \,, \ \forall e \in \mathbb{F}_m \qquad (38)$$

In the following, the objective function and the constraints will be discussed in detail.

The objective is to maximize the profit from the production of one batch of items. In general, the profit is calculated by subtracting the costs from the revenue. The costs, on the one hand, consist of the slabs that are used for production, see Eq. (2), and the scrap material, see Eq. (3). The revenue, on the other hand, consists of the produced items, see Eq. (1), and the reusable leftover material, see Eq. (4). Note that the objective function is linear in the variables $x_{ikl}$, $u_l$, $R_l$, and $F_j$, c.f., Table 2.

From Eq. (5) and Eq. (6) we see that all items have to be produced. This means Eq. (1) of the objective is constant and does not influence the optimal solution. However, we will simplify that for the solver.

If we use a slab, it decreases the revenue. I.e., its value is subtracted in Eq. (2). The production will result not only in items that we want to produce. There will also be surplus material. As discussed earlier this can either be leftover material or scrap material.

The scrap material originates from a slab with a purchase value $P_l^0$ and this is the value we lose if we throw it away. As we have already accounted for the current value in Eq. (2), we need to compensate for that in Eq. (3).

The leftover material is a kind of positive revenue. It is not being paid for by a customer at this moment, but it might be in future production cycles. So this value is added to the objective, and thereby to the revenue, in Eq. (4).

The constraints are described in the following. Equation (5) describes that each item should only be produced once. Usage of the set $c_i$ prevents that both, the original and the rotated version of an item, are produced.

Equation (6) means that an item $i$ can only be placed in shelves 1 to $i$. In other words, an item can open a shelf or be placed in an existing one. As items are sorted by height, this ensures equivalence to the case where the height of a shelf is not determined by

Table 2: (Auxiliary) Variables.

| Variable | Description |
|---|---|
| $x_{ikl} \in \{0,1\}$ | Set to 1 if item *i* is placed in shelf *k* on slab *l*. |
| $u_l \in \{0,1\}$ | Set to 1 if slab *l* is used. |
| $b_{kl} \geq 0$ | Remaining width in shelf *k* on slab *l*, see Fig. 5a. |
| $b_k \geq 0$ | Remaining width in shelf *k*, see Eq. (9). |
| $s_l \geq 0$ | Remaining height above top shelf on slab *l*, see Fig. 5a, Eq. (10). |
| $q_i \geq 0$ | Remaining height above item *i* in its assigned shelf, see Fig. 5a, Eq. (11). |
| $a_i \geq 0$ | Remaining depth above item *i* in its assigned shelf, see Fig. 5b, Eq. (12). |
| $V^S_{kl} \geq 0$ | Weight of leftover material filling the *width* of shelf *k* on slab *l*, see Fig. 3. A shelf is located on exactly one slab, so it is $\neq 0$ for exactly one combination of *k* and *l* or all are 0 if the leftover material is waste. |
| $V^S_k = \sum_{l=1}^{p} V^S_{kl} \geq 0$ | Weight $V^S$ of shelf *k*. |
| $V^C_{il} \geq 0$ | Weight of leftover material filling the *height* above item *i* on slab *l*, see Fig. 3. It is $\neq 0$ for only one combination of *i* and *l* or all equal 0 if waste. |
| $V^C_i = \sum_{l=1}^{p} V^C_{il} \geq 0$ | Weight $V^C$ above item *i*. |
| $V^T_{il} \geq 0$ | Weight of leftover material filling the *depth* above item *i* on slab *l*, see Fig. 3. It is $\neq 0$ for only one combination of *i* and *l* or all equal 0 if waste. |
| $V^T_i = \sum_{l=1}^{p} V^T_{il} \geq 0$ | Weight $V^T$ above item *i*. |
| $V^H_l \geq 0$ | Weight of leftover material on the top of slab *l*, see Fig. 3. |
| $z^S_k, z^C_i, z^T_i, z^H_l \in \{0,1\}$ | Set to 0 if associated weights are waste. |
| $V_j \geq 0\,,\ \forall j \in \mathbb{F}_{\|V\|}$ | An element from the set *V*, see Eq. (33). |
| $z^R_{je} \in \{0,1\}\,,\ \forall j \in \mathbb{F}_{\|V\|}$ | Set to 1 if weight *j* is in class *e*. |
| $F_{je}$ | Value of leftover material *j* in weight class *e*. Not equal 0 for one weight class. |
| $F_j = \sum_{e=0}^{m} F_{je} \geq 0\,,\ \forall j \in \mathbb{F}_{\|V\|}$ | The value of weight *j*. |
| $R_l$ | Weight of leftover material that is waste. |

the first, but highest item.

Equation (7) sets $u_l$ to one if slab *l* is used.

Equation (8) and Eq. (9) set the variable $b_k$ which describes the remaining width in shelf *k*, see also Fig. 5. Equation (10), Eq. (11), and Eq. (12) set $s_l$, $q_i$, and $a_i$, respectively. See also Fig. 5.

Equation (13) and Eq. (14) set the variable $z^H_l$ to zero if the associated weight $V^H_l$ is waste. $V^H_l$ is considered waste if $s_l < d^{H_{\min}}_l$. In that case, $z^H_l$ in Eq. (14) can either be 0 or 1, but with Eq. (13) it has to be 0. Then again, if $s_l \geq d^{H_{\min}}_l$, $z^H_l$ in Eq. (13) can either be 0 or 1, while Eq. (14) sets it to 1.

The weights, which are calculated to get the value of the leftover material, are best understood by looking at Fig. 3. The weight $V^H_l$ is calculated in Eq. (15) and Eq. (16). The latter one is set to zero if it is considered waste. This is done so that no value is added for it in the objective function.

In Eq. (17) and Eq. (18) $z^S_k$ is set to 0 if $V^S_{kl}$ is scrap material. Otherwise, $z^S_k$ is set to 1. In both equations, the summation over *l* “picks” the correct slab as $x_{kkl}$ can only be one for one *l*. Beside of that, it works analogously to Eq. (13) and Eq. (14).

The next three equations, Eq. (19), Eq. (20), and Eq. (21), set the weight $V^S_{kl}$. It is set to its actual value if it is usable, otherwise it is set to 0. To be more precise, the equations set the upper limit of the weight and the objective indirectly maximizes the weight such that the upper limit will be reached. Equation (19) limits $V^S_{kl}$ to the weight of the slab or sets it to 0 if it is scrap material. Equation (20) and Eq. (21) need to be interpreted in conjunction. Equation (20) limits $V^S_{kl}$ as if shelf $k$ is on slab $l$, which is not necessarily true. Equation (21) then limits $V^S_{kl}$ to 0 if $x_{kkl}$ equals 0, i.e., only the correct combination of $k$ and $l$ lead to a value unequal 0.

Equation (22) and Eq. (23) set $z^C_i$ in a similar way as Eq. (17) and Eq. (18) set $z^S_k$. The sum to find the correct $d^{H_{\min}}_l$ needs to be a sum over $x_{ikl}$. This is because an item, not being the first item in a shelf, can be located on any shelf.

The next three equations, Eq. (24), Eq. (25), and Eq. (26), set the weight $V^C_{il}$ in the same way as Eq. (19), Eq. (20), and Eq. (21) set $V^S_{kl}$. Equation (26) has an additional sum, compared to the calculation of $V^S_{kl}$. The argument here is similar to the one above. Item $i$ can be located on any shelf, so we sum over the shelves $k$ in order to pick the correct $x_{ikl}$.

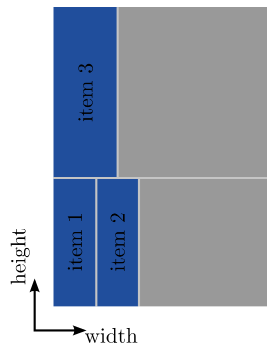
Equation (27) and Eq. (28) set $z^T_i$ in the same way as $z^C_i$ is set. Equation (29), Eq. (30), and Eq. (31) define the weight $V^T_{il}$ in the same way as $V^C_{il}$ is defined.

$R_l$ is the sum of the weight of scrap material from slab $l$. It is calculated in Eq. (32) as the remainder from substracting produced and reusable items from the slab weight. I.e., everything that is not an item or reusable leftover is scrap material.

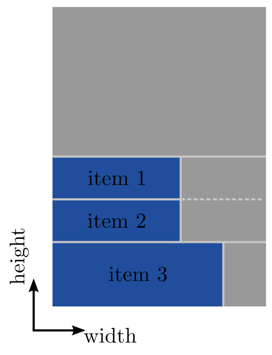
Equation (4) uses the set $V$, which is defined in Eq. (33). $V$ is the set of all weights not of size zero. Recall that these weights are only those that are associated to reusable material.

The rest of the constrains are used to finally calculate the value of the reusable material. $z^R_{je}$ denotes if weight $j$ is in the weight class $e$. Let $l(j)$ denote the slab on which weight $j$ is located. $l(j)$ is linear, because $V_j = V^S_k : l(j) = l \Leftrightarrow V^S_{kl} > 0$, analogously for C, T. Then, $z^R_{je}$ shall equal 1 if $g^e_{l(j)} \leq V_j < g^{e+1}_{l(j)}$. This is accomplished by Eq. (34), Eq. (35), and Eq. (36). Equation (35) will set $z^R_{je}$ to zero if $V_j$ is lighter than the boundary $g^e$. Equation (34) will set $z^R_{je}$ to zero if $V_j$ is heavier than $g^e$. Both these equations allow $z^R_{je}$ to be either 0 or 1 if the above inequality holds. In that case Eq. (36) forces $z^R_{je}$ to be 1 and we have our desired behavior.

With this information we can finally calculate $F_{je}$, which is the value of $V_j$. As the name suggests, it is non-zero only for one specific value of $e$. Unfortunately, we cannot multiply $V_j$ by $z^R_{je}$ as we want our problem to stay linear. Therefore, Eq. (37) sets the upper limit of its value regardless of $z^R_{je}$. Then, Eq. (38) sets an even larger upper limit, but only if $z^R_{je}$ equals 1. Otherwise, it sets it to 0, which is what we want if $V_j$ does not lie in this specific weight class. In our objective function, the sum is above $F_j$, not $F_{je}$. This is explained in Table 2.



(a) Our solution.



(b) Solution from (Andrade et al., 2013).

Figure 6: Example 1, input data No. 1, Table 3.

Table 3: Input data No. 1.

(a) Slabs in stock

| ID | W | H | T |
|---|---|---|---|
| 1 | 500 | 700 | 45 |

(b) Items in order

| ID | w | h | t |
|---|---|---|---|
| −3 | 150 | 400 | 45 |
| −1 | 100 | 300 | 45 |
| −2 | 100 | 300 | 45 |
| 3 | 400 | 150 | 45 |
| 1 | 300 | 100 | 45 |
| 2 | 300 | 100 | 45 |

## 4 EVALUATION

The optimization program was implemented using CVX, which is an add-on for MATLAB. CVX allows a high level description of the program, which is close to the description in Section 3. CVX can use several solvers. The solver we use is Gurobi ((Gurobi Optimization, Inc., 2016)) as it can solve binary integer linear programs.

In all examples, the boundaries and relative values of the weight classes are the ones from Table 1. Also, the weight per volume is given there.

Please note that some of the slabs had to be drawn incomplete in order to keep the page size down. This was done only in areas where no items are placed and is denoted by a dotted line at the top of the slab.

### Example 1

We compare our solution to the one named $\mathcal{M}^L_1$ in (Andrade et al., 2013), which is also implemented in CVX/Gurobi. As mentioned earlier, the solution in (Andrade et al., 2013), solves the problem in two dimensions. Therefore, in our example, the thickness of the ordered items matches the thickness of the slabs in stock.

Later, we will also show an example where the thickness between slab and items is different.

Table 4: Input data No. 2.

(a) Slabs in stock

| ID | W | H | T |
|---|---|---|---|
| 1 | 350 | 650 | 40 |
| 2 | 300 | 300 | 40 |

(b) Items in order

| ID | w | h | t |
|---|---|---|---|
| 1 | 200 | 330 | 40 |
| 2 | 150 | 300 | 40 |
| 3 | 150 | 300 | 40 |

The data from Table 3 produces the solutions shown in Fig. 6. The table shows the rotated items, where the item with identifier $-i$ denotes the rotated version of the item with itentifier $i$. We will omit these in later tables. The items in this specific table are also ordered by height as needed for in our problem formulation.

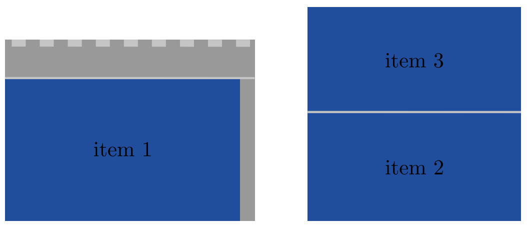
We denote a cut which should not be necessary in the final cutting procedure with a dashed line. Our solution results in two quite large pieces of surplus material. The solution from (Andrade et al., 2013), results in a solution with three smaller pieces of surplus material. We also rotated all items before feeding the input to the algorithm from (Andrade et al., 2013). Still, its solution was to put all three items side by side, i.e., also creating three pieces of surplus material. Subjectively, two larger pieces of surplus material are better than three smaller ones.

Putting the solution from (Andrade et al., 2013) into our objective function, we get a smaller, i.e., worse, objective value. That means, according to our valuation, we have a better overall value of items and slabs, compared to the algorithm from (Andrade et al., 2013). It is also shown that the rotation of items is of importance.

## Example 2

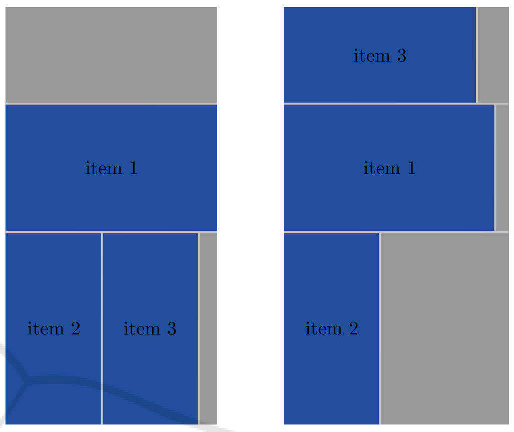
This example, see Fig. 7, underlines how our optimization problem finds really good solutions if slabs can be filled. There is no leftover material on slab 2, so slab 1 produces a nice large piece of leftover material in addition to a smaller one. The smaller one is just slightly above the size limitations of scrap material, so it is put back to the set of slabs for the next run, too.

Assume, we rotate item 1 on slab 1. The values of the used slabs, as well as the value of the produced items stays the same. Still, the value of the objective function decreases, because the sum of the values of the surplus material decreases. Note that, the sum of the weights of the surplus material stays also the same. The value of the surplus material decreases, because the function, described by $g_l^e$ and $\alpha_l^e$, is a concave function. This function is essential for the performance of the optimization program.



(a) Slab 1 (b) Slab 2

Figure 7: Example 2, input data No. 2, Table 4.



(a) Input data No. 3 (b) Input data No. 4

Figure 8: Ex. 3, data No. 3 and 4, Tables 5 and 6.

## Example 3

Two slightly different variations of input data is examined in this example. Table 5 shows input data no. 3, while Table 6 shows input data no. 4. The only difference is the size in the direction of $x$ of the slab.

Figure 8 shows both solutions. Again, it is all about the value of the leftover material. It is clear the solution with the highest value of the leftover material is optimal, because all of the items have to be produced according to the objective function.

With input data No. 4, if the items would be placed as seen in Fig. 8a, then the value of the leftover material would be less than is shown. This directly depends on the weight groups. As a result the weight groups have to be empirically adjusted, according to the slab sizes and items ordered.

Table 5: Input data No. 3.

(a) Slabs in stock

| ID | W | H | T |
|---|---|---|---|
| 1 | 330 | 650 | 40 |

(b) Items in order

| ID | w | h | t |
|---|---|---|---|
| 1 | 200 | 330 | 40 |
| 2 | 150 | 300 | 40 |
| 3 | 150 | 300 | 40 |

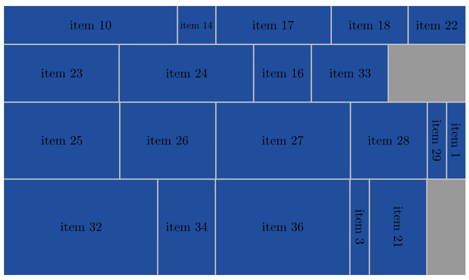
Table 6: Input data No. 4.

(a) Stock

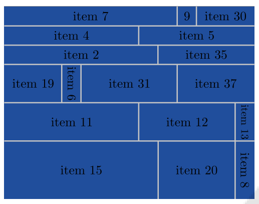
| ID | W | H | T |
|---|---|---|---|
| 1 | 350 | 650 | 40 |

(b) Order

| ID | w | h | t |
|---|---|---|---|
| 1 | 200 | 330 | 40 |
| 2 | 150 | 300 | 40 |
| 3 | 150 | 300 | 40 |



(a) Slab 1



(b) Slab 2

Figure 9: Example 4, input data No. 5, Table 7.

Table 7: Input data No. 5.

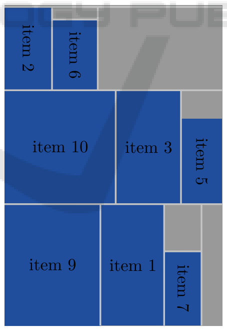
(a) Stock

| ID | W | H | T |
|---|---|---|---|
| 1 | 240 | 140 | 10 |
| 2 | 130 | 100 | 10 |

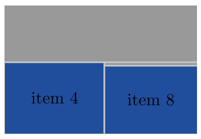
(b) Order

| ID | w | h | t |
|---|---|---|---|
| 1 | 40 | 10 | 10 |
| 2 | 80 | 10 | 10 |
| 3 | 50 | 10 | 10 |
| 4 | 70 | 10 | 10 |
| 5 | 60 | 10 | 10 |
| 6 | 20 | 10 | 10 |
| 7 | 90 | 10 | 10 |
| 8 | 30 | 10 | 10 |
| 9 | 10 | 10 | 10 |
| 10 | 90 | 20 | 10 |
| 11 | 70 | 20 | 10 |
| 12 | 50 | 20 | 10 |
| 13 | 10 | 20 | 10 |
| 14 | 20 | 20 | 10 |
| 15 | 80 | 30 | 10 |
| 16 | 30 | 30 | 10 |
| 17 | 60 | 20 | 10 |
| 18 | 40 | 20 | 10 |
| 19 | 30 | 20 | 10 |
| 20 | 40 | 30 | 10 |
| 21 | 50 | 30 | 10 |
| 22 | 20 | 30 | 10 |
| 23 | 60 | 30 | 10 |
| 24 | 70 | 30 | 10 |
| 25 | 60 | 40 | 10 |
| 26 | 50 | 40 | 10 |
| 27 | 70 | 40 | 10 |
| 28 | 40 | 40 | 10 |
| 29 | 10 | 40 | 10 |
| 30 | 10 | 30 | 10 |
| 31 | 20 | 50 | 10 |
| 32 | 80 | 50 | 10 |
| 33 | 30 | 40 | 10 |
| 34 | 30 | 50 | 10 |
| 35 | 10 | 50 | 10 |
| 36 | 70 | 50 | 10 |
| 37 | 20 | 40 | 10 |

(a) Slab 1



(b) Slab 2



(c) Slab 3

Figure 10: Example 5, input data No. 6, Table 8.

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## Example 4

In Fig. 9 we show the result of an optimization which barely fits on the slabs and where items have many similar dimensions. This is where the optimization problem really shines and a manual solution would be really time consuming if you would even find one.

## Example 5

Compared to the last example, Fig. 10 shows the result of items, which don't have many similar sizes. The result does not look as nice and dense as the previous one, but it still produces many larger pieces of surplus material. Note that slab 1 is not used, i.e., smaller blocks of steel are used first, because they have a smaller value. This is exactly what we want.

## Example 6

Figure 11 is a slight variation of the previous example. The sizes of slab 2 and slab 3 are decreased. As a result not all items fit on these two slabs.

As can be seen from the figure, item 7 is put on slab 1, but it would neatly fit into the upper right corner of slab 3. The optimization problem can't find this solution, because it makes strict assumptions about the layout of the items. Further investigation about cutting patterns and their application to the formulation of the optimization problem might be a direction of future research.

## Example 7

In previous examples, the thickness of items matched

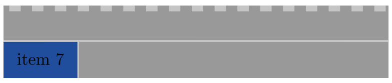
Table 8: Input data No. 6.

(a) Stock

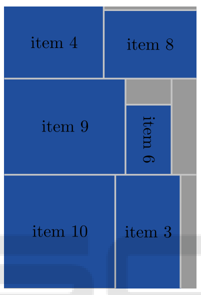
| ID | W | H | T |
|---|---|---|---|
| 1 | 300 | 300 | 45 |
| 2 | 170 | 250 | 45 |
| 3 | 150 | 100 | 45 |

(b) Order

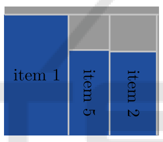
| ID | w | h | t |
|---|---|---|---|
| 1 | 94 | 50 | 45 |
| 2 | 65 | 37 | 45 |
| 3 | 51 | 88 | 45 |
| 4 | 78 | 56 | 45 |
| 5 | 32 | 66 | 45 |
| 6 | 54 | 36 | 45 |
| 7 | 58 | 29 | 45 |
| 8 | 53 | 72 | 45 |
| 9 | 95 | 75 | 45 |
| 10 | 89 | 87 | 45 |



(a) Slab 1



(b) Slab 2



(c) Slab 3

Figure 11: Example 6, input data No. 7, Table 9.

the thickness of the slabs. Now, we show that our formulation can give unexpected results if there are several slabs with different thicknesses to pick from.

In Fig. 12, we show the results of our formulation with input data no. 8, Table 10. One might assume that producing the item on slab 2, which matches its thickness, is the best option. As you can see, our formulation places the item on slab 1, which is considerably thicker than it needs to be. Simply put, the value

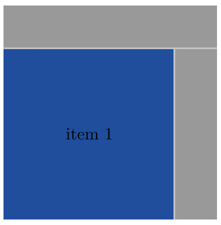
Table 9: Input data No. 7.

(a) Stock

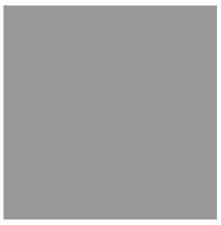
| ID | W | H | T |
|---|---|---|---|
| 1 | 300 | 300 | 45 |
| 2 | 150 | 220 | 45 |
| 3 | 120 | 100 | 45 |

(b) Order

| ID | w | h | t |
|---|---|---|---|
| 1 | 94 | 50 | 45 |
| 2 | 65 | 37 | 45 |
| 3 | 51 | 88 | 45 |
| 4 | 78 | 56 | 45 |
| 5 | 32 | 66 | 45 |
| 6 | 54 | 36 | 45 |
| 7 | 58 | 29 | 45 |
| 8 | 53 | 72 | 45 |
| 9 | 95 | 75 | 45 |
| 10 | 89 | 87 | 45 |



(a) Slab 1



(b) Slab 2

Figure 12: Example 7, input data No. 8, Table 10.

Table 10: Input data No. 8.

(a) Stock

| ID | W | H | T |
|---|---|---|---|
| 1 | 500 | 700 | 25 |
| 2 | 500 | 700 | 15 |

(b) Order

| ID | w | h | t |
|---|---|---|---|
| 1 | 400 | 400 | 15 |

of the leftover material dictates this placement.

Cuts that trim the thickness can take a long time, because the area to cut can be quite large compared to the other cuts. If these trims were not desired, the optimization problem could either be extended by a penalty for these cuts or items and orders could simply be matched in thickness before given to the optimization program.

### Runtime and Quality

Besides example 4, the runtime of our examples was always below one minute, which is a reasonable runtime to collect and place orders. Example 4 took about 5000 seconds, which is considerable more than the other solutions. Limiting the time allowed by the solver for Example 4 to 60 seconds, we still got a feasible solution. I.e., while not as dense as the solution shown in Fig. 9, all items were placed on the slabs.

## 5 SUMMARY

In this paper, we developed an optimization program that solves the problem of cutting steel-plates, where only guillotine cuts are allowed. We consider surplus material as a form of value. Our solution is modeled in three dimensions in order to assign a value to the surplus material. We use the notion of shelves to model the placement of items on slabs and allow items to be rotated.

As has been shown in the examples, our problem formulation gives subjectively good results, i.e., the problem is apparently well solved. We have shown that preferable few, big leftovers are created and small slabs are used first, see Examples 1 and 6 respectively.

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Example 4 showed that our formulation allows the tradeoff between runtime and quality. Example 5 showed good performance, even for items of random size.

In some cases, the placement using shelves might not be optimal and a more advanced placement might yield better results in the case of sparser placements. Consider Example 6, item 7 could fit on slab 3 with a proper cutting order. Although, modeling and finding optimal solutions of placements not following the notion of shelves is much more difficult.

Including the third dimension is necessary to achieve optimal solutions, see Example 7. Our results are strongly dependent on the choice of weight classes and their values, see Examples 2 and 3. A proper optimization of these could be the topic of future research. Although, the values from Table 1 produce good results for the sizes we used in our examples.

In the future, the number of cuts or a penalty for using many different slabs could be incorporated easily into the objective function. But these considerations should be justified by concrete requirements of a factory.

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