What Should We Add to Syntax-oriented Generation of Random Expressions to Meet Pedagogical Needs?

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Abstract: The paper looks for an exercise environment where the teacher program would enable to fix parameters of random generation so that the generated tasks have desired pedagogical properties. In particular, it should be possible to prescribe qualitative content and size of the solution and its parts, opportunities to make certain errors, probabilities of different decisions to be made by the student. We analyse two existing environments for algebraic exercises in Propositional Logic and identify what additional options are necessary and what computational engines or precomputed resources could produce the desired properties.

1 INTRODUCTION

Random generation of expressions has become a widely used technique in Computer-Aided Learning of Mathematics. There are many websites and programs, which enable students to solve randomly generated drill tasks on virtually all technical topics of Basic School. Teachers use random generation of individualized tasks to ensure that students complete their independent work and tests without plagiarism. They often prefer random homework and test tasks even if they require different amount of work from different students. The importance of random generation will increase with the development of MOOC (Massive Open Online Courses) by the universities.

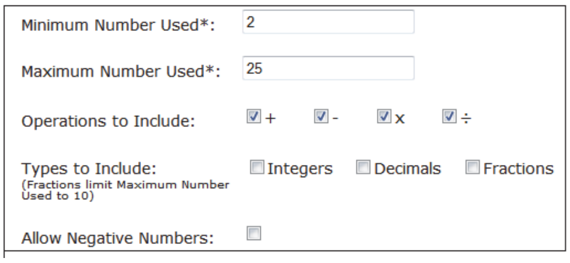
Exercise software often contains teacher tools for compiling homework and test task sets. The teacher can usually design a task chain by selecting for each task the task type and by entering 'difficulty level' or the values of generation parameters. Quite often the optional parameters describe mainly the *syntactic properties of generated expressions* and do not enable specification of tasks with desired *properties of solutions*. In this case the teacher can implement his/her pedagogical ideas only partially or has to use fixed tasks that are the same for all students.

This paper analyses random generation options of two environments that are implemented and used at our institute for algebraic exercises in Propositional Logic. Both environments use quite typical syntax-oriented formula generation algorithms where the main teacher-defined parameters are the number of variables and numbers of each of the logical operators. We identify what solution-oriented generation parameters are necessary to add to such generators and what computational engines or precomputed resources could produce the desired properties of tasks.

Section 2 of this paper describes related work. In Section 3 we attempt to present a list of typical pedagogical requirements for a random generation algorithm and for a set of teacher-definable generation parameters. Section 4 introduces our software for computerized propositional exercises. Sections 5 and 6 analyze random generation in our truth-table and formula manipulation environments in the framework of the requirements introduced in Section 3. Section 7 contains a summary.

2 RELATED WORKS

There are many websites that offer students a possibility to solve randomly generated tasks belonging to virtually all task types in Basic School algebra. Many such sites give teachers the possibility to compile worksheets consisting of tasks of one or more task types. For example, a web search with the phrase 'random generation algebra' gave us seven quite universal websites (beginning with Coolmath in References). The website of IXL.com lists hundreds of task types but offers no difficulty levels or specification of more detailed properties of expressions within the types. The website of Mathmix.com does not contain nearly as many types but the expression generation can be driven by a number of parameters (Figure 1).



Minimum Number Used*:	2
Maximum Number Used*:	25
Operations to Include:	☑ + ☑ - ☑ x ☑ ÷
Types to Include: (Fractions limit Maximum Number Used to 10)	☐ Integers ☐ Decimals ☐ Fractions
Allow Negative Numbers:	☐

Figure 1: Task generation parameters for two-step linear equations in Mathmix.com.

Available websites demonstrate the features that are currently available for students and teachers, the pedagogical and computational level of applications. It is clear that none of the websites enable the teacher to compile task chains which contain all the cases that are covered by exercises in textbooks. The menus simply do not contain enough options and usually the generated expressions contain only 2-4 operations (fractions, monomials etc).

Another relevant tool is Wolfram Problem Generator (Wolfram). It generates problems, checks answers, provides hints and displays step by step solutions for a small number of most relevant task types. The tasks have three difficulty levels, with the advanced tasks having much more complex expressions than the aforementioned websites.

Several authors have created syntactical tools that can be used for random generation of expressions from predefined operators and functions. Examples include: RANDPOLY (Wright,1994), Matlab Expression Generator, Math Expression Generator (see References).

There are only a few published papers about the use of random generation of expressions with pedagogical objectives. The best-known are the papers on major systems of Computer Aided Assessment. These systems are designed for task creators who are able to use some more powerful language for description of desired expressions. AIM is a well-known test system that uses computer algebra. N. Strickland states in (Strickland, 2002): “Currently all AIM questions are written using a kind of mark-up language incorporating elements of Maple code and elements of LaTeX”. STACK, a more recent system, allows describing random generation of expressions based on templates with variable coefficients and expressions for generation of each coefficient (Sangwin, 2006). Similar features of the ActiveMath system are described in (Mavrikis and Gonzales, 2004). In these systems pedagogical considerations are not so much concern of generating mechanism but concern of people who create the templates. The same is true for Basic Logic Tutor ORGANON (Dostalova and Lang, 2011) where the authors state "Therefore the database is not filled up by concrete exercises but by PATTERNS". Unfortunately the authors do not demonstrate the spectrum of exercises derived from one pattern.

T-algebra is an interactive exercise environment for Basic School Algebra (Prank et al, 2007). A system of task types and difficulty levels and the functioning of random generation at each level can be observed when trying to compose task files with the T-algebra teacher program (T-algebra).

A series of papers on random generation of tasks in different topics has been published by a group of researchers supported by Microsoft where S. Gulwani seems to be the key person. The paper (Andersen, Gulwani and Popovic, 2013) traces the use of operations in standard solutions of tasks. The received traces are used for comparison of problems, analysis and synthesis of problem sets.

3 DESIRED FEATURES OF A GENERATION ALGORITHM AND TEACHER CONTROL

Many task types in our package for Mathematical Logic (Prank, 1991) and all the 60+ task types in T-algebra (Prank et al, 2007) contain random generation of exercises. We have had many discussions with university colleagues and school teachers about their expectations in that regard. We now try to formulate general requirements for random generation of initial expressions of tasks. Requirements A and B describe mainly the features of generation algorithm while C-F indicate what should be possible to specify by selection of generation parameters.

- A. Sufficient variety of generated expressions. The set of possible initial expressions should cover initial expressions of usual textbook exercises.
- B. Exception of cases that are forbidden for particular task type or are trivial.
- C. Sufficiently precise specification of qualitative content of student’s work. In case of our propositional exercises it should specify:
  - what operations are performed by filling the truth-table,

— what patterns are to be used for finding the formula corresponding to given column of values,

— what conversion rules are necessary for formula transformation.

- D. Specification of opportunities to make certain errors/inappropriate steps.
- E. Specification of solution size (amount of student's work):
  - — general size,
  - — size of particular parts of the solution (stages of solution algorithm).
- F. Equal or teacher-definable probabilities of positive and negative answers about tautologicity, equivalence etc.

Requirement A is satisfied practically by any usual (string- or tree-based) syntax-oriented generation algorithm if the allowed number of variables and length of generated expressions is not too small.

In our course the only task types where some initial formulas are forbidden are conversions to normal form where tautologically false/true initial expressions do not have disjunctive/conjunctive NF. Our random generator takes the task type into account and does not generate forbidden initial formulas. The generators also do not generate formulas where an operation is to be performed between two copies of the same variable. We discuss type-specific trivial cases and requirements C-F in the context of concrete environment and task types.

4 OUR SOFTWARE FOR PROPOSITIONAL LOGIC

The content of the two following sections is based on our experience with the design, implementation and use of exercise environments for truth-table and formula manipulation exercises. In 1987-1993 we computerized ten 90-minute exercise labs together with homework and tests. The computerized exercise labs were introduced in our third-semester course Introduction to Mathematical Logic, beginning from 1991 (Prank, 1991).

Having been created many years ago, the software has required upgrades in task types, user interface, hint facilities, security, support for teacher's work, web access, etc. The programs have been reimplemented as Bachelor and Master projects. However, even implementation of the minimum necessary mechanisms of the exercise environments usually exceeded the normal size of a thesis. Next round in our work will be implementation of desired *new* features. In the next two Sections of the paper we describe what can be added to our existing computational tools and random generation schemes to achieve sufficient meeting of didactical needs.

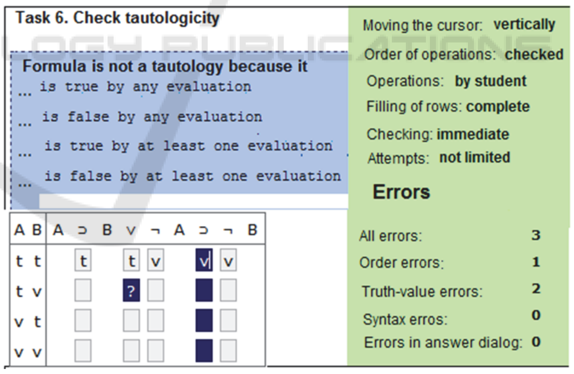
5 TRUTH-TABLE EXERCISES

Our truth-table environment contains the following exercise types:

1) Filling the truth-table,
2) Checking of tautology,
3) Checking of satisfiability,
4) Checking of equivalence of two formulas,
5) Checking whether the first formula implies the second,
6) Construction of formula having a given resulting truth-column.

The solution dialogs of tasks of types 2-5 begin with filling the rows of the table until it is possible to justify the decision (positive or negative).

Figure 2 depicts the solution window for tautology checking (original Estonian texts have been replaced with English). The student has filled one row and switched to answer dialog because the truth-value on this row was *v* (false). The student should now select from the menu the reason why the formula is not a tautology.



Task 6. Check tautologicity

Formula is not a tautology because it
... is true by any evaluation
... is false by any evaluation
... is true by at least one evaluation
... is false by at least one evaluation

A	B	A	⊃	B	∨	¬	A	⊃	¬	B
t	t		t		t	v		v	v	
t	v				?					
v	t									
v	v									

Moving the cursor: **vertically**
Order of operations: **checked**
Operations: **by student**
Filling of rows: **complete**
Checking: **immediate**
Attempts: **not limited**

Errors

All errors:	3
Order errors:	1
Truth-value errors:	2
Syntax erros:	0
Errors in answer dialog:	0

Figure 2: Solution window for tautology checking task.

For formula construction tasks the program displays the required column of truth-values. When the student enters the formula, the program automatically calculates its truth-table. In case of a wrong answer the student should correct the formula.

For random generation the teacher program enables to specify the number of variables, the numbers of each of the propositional connectives ¬, &, ∨, ⊃, ~ and the percentage of truth-values *true* in the column of values (0-25, 26-50, 51-75, 76-100 or unspecified). For example, initial formula of Figure

2 can be received when the teacher selects 2 variables, 1 disjunction, 2 implications and 2 negations.

Consider now our pedagogical aims for different task types and expression generation principles that allow implementing them. Our task sets contain for each exercise type some tasks with fixed formulas. The fixed tasks of types 2-5 contain classical examples and counterexamples of formulas with and without corresponding properties, and a collection of the main propositional equivalencies to be proved using a truth-table. In type 6 there are some fixed tasks that guide the student to recognize patterns of all binary connectives and their negations and a series of tasks that directs the student to discover the possibility to build the formula with given truth-values in disjunctive normal form (before introduction of NF in lectures). However, to ensure independent work of students in every task type, we include tasks with randomly generated formulas.

Propositional connectives are completely new material for the students. Weaker students need a considerable amount of training, with the program checking the truth-values and the order of operations. The available generating parameters allow adjusting the content of tasks and corresponding error opportunities (requirements C-D) by specifying the connectives in the formula. The amount of work (req. E) is sufficiently well determined by the number of variables and total number of connectives. The same is true for task types 2-5 where filling the table constitutes the bulk of the work and remaining part of the content is defined by the task type.

For checking of tautology, satisfiability etc it would be desirable to have nearly equal probabilities of positive and negative answers (requirement F). Our current control of parameters does not enable this. Currently we generate formulas with 76-100 percents of values *true* for tautology and 0-25 percents of *true* for satisfiability checking but in practice we only get a few tautological and a few non-satisfiable formulas. The generator could work in two stages, first choosing the answer (*tautology/not*) and then generating formulas until the suitable truth-column is received.

With our current generation control options, we generate for equivalence checking two formulas with the lowest or highest percentage of *true*. For inference checking we generate first a formula with 0-25 percent of *true* and second a formula with maximum number of *true*. This produces quite a satisfactory distribution of results for inference but not for equivalence. It is clear that the formulas for types 4-5 should actually be generated together, choosing first the answer *true/false* and then iterating the generation.

Consider now the tasks on construction of a formula for a given column of truth-values. After presenting and practicing the above-described ideas of pattern recognition and normal form we assign a series of randomly generated tasks with three variables and 3-6 values *true*. The student is asked to find a formula of ‘reasonable’ length. Having 3-6 values *true* guarantees that the normal forms are longer than the ‘reasonable’ formula.

Note here that our current task parameters do not enable to set an upper limit for the length of the formula (in terms of the number of connectives or symbols). This feature can easily be added but it should be used carefully. The task of finding the formula is quite difficult for weaker students and there should remain some opportunity to get partial credit for the long answer in DNF.

In our obligatory exercises we do not require that the shortest formula for given truth-values should be found. This task is too difficult. However, we have assigned such tasks for bonus points outside of computerized environment. Could we generate reasonable shortest formula tasks randomly? It proves to be easier than expected. The usual breadth-first algorithm produces the shortest formulas (containing all connectives or a specified subset of them) for three-variable columns very quickly and for four-variable columns within seconds (on a standard laptop or desktop computer). In the latter case we can also use pre-computed files with 64K formulas. This allows generation of training series with a growing length of optimal formulas, tasks with equal complexity for tests, tasks with maximum complexity, etc. Knowledge about the shortest formula can also be used for elimination of trivial cases (like *tftftftf*) by generating the truth-columns for standard tasks.

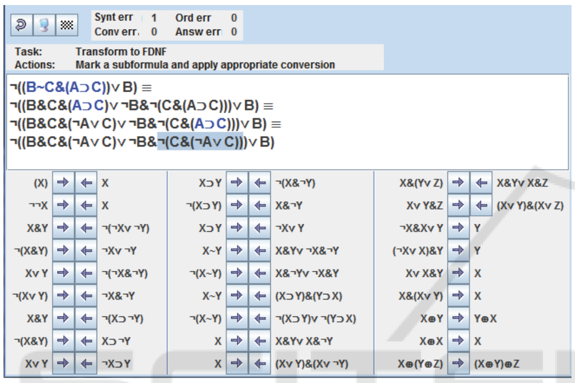
6 FORMULA MANIPULATION EXERCISES

Our environment for algebraic manipulation implements the following exercise types for propositional formulas:

1) Three task types on expression of formula using $\{\neg, \&\}, \{\neg, \vee\}, \{\neg, \supset\}$,
2) Conversion to disjunctive/conjunctive normal form (DNF/CNF),
3) Conversion to full disjunctive/conjunctive normal form (FDNF/FCNF),
4) Moving negations inside/outside,
5) Free conversion (the program checks only

equivalence at each step).

While working with our program, the student enters the solution step by step. Each conversion step is entered in two substeps. At the first substep the student marks a subformula to be changed. For the second substep the program has two different working modes. In Input mode the student enters a subformula that replaces the marked part. In Rule mode the student selects a conversion rule from the menu and the program applies it. Figure 3 shows the solution window in Rule mode. The student has eliminated biconditional and implications and intends now to move negation inside (it could be better to begin with the outermost negation).


Synt err 1 Ord err 0
Conv err. 0 Answ err 0

Task: Transform to FDNF
Actions: Mark a subformula and apply appropriate conversion

¬((B~C&(A⊃C))∨B) ≡
¬((B&C&(A⊃C)∨¬B&¬(C&(A⊃C)))∨B) ≡
¬((B&C&(¬A∨C)∨¬B&¬(C&(A⊃C)))∨B) ≡
¬((B&C&(¬A∨C)∨¬B&¬(C&(¬A∨C)))∨B)

(X)	→	←	X	X⊃Y	→	←	¬(X&¬Y)	X&(Y∨Z)	→	←	X&Y∨X&Z
¬¬X	→	←	X	¬(X⊃Y)	→	←	X&¬Y	X∨Y&Z	→	←	(X∨Y)&(X∨Z)
X&Y	→	←	¬(¬X∨¬Y)	X⊃Y	→	←	¬X∨Y	¬X&X∨Y	→		Y
¬(X&Y)	→	←	¬X∨¬Y	X~Y	→	←	X&Y∨¬X&¬Y	(¬X∨X)&Y	→		Y
X∨Y	→	←	¬(¬X&¬Y)	¬(X~Y)	→	←	X&¬Y∨¬X&Y	X∨X&Y	→		X
¬(X∨Y)	→	←	¬X&¬Y	X~Y	→	←	(X⊃Y)&(Y⊃X)	X&(X∨Y)	→		X
X&Y	→	←	¬(X⊃¬Y)	¬(X~Y)	→	←	¬(X⊃Y)∨¬(Y⊃X)	X⊕Y	→		Y⊕X
¬(X&Y)	→	←	X⊃¬Y	X	→	←	X&Y∨X&¬Y	X⊕X	→		X
X∨Y	→	←	¬X⊃Y	X	→	←	(X∨Y)&(X∨¬Y)	X⊕(Y⊕Z)	→		(X⊕Y)⊕Z

Figure 3: Solution window of Formula Manipulation Environment in Rule mode. The student has performed three solution steps and marked a subformula for the fourth step.

Our exercise labs in Elements of Discrete Mathematics contain exercises of types 1 and conversion to full disjunctive normal form.

The teacher program allows specification of the number of variables and the numbers of propositional connectives for random generation of a formula. Consider now our type-specific pedagogical aims and corresponding ideas for task generation.

In the first task types on expression of formulas using given connectives the solution algorithm is trivial: eliminate the undesired connectives one by one. The qualitative content of tasks and error opportunities mean then: what equivalencies (conversion rules) are necessary in the tasks. Requirements C and D are satisfied because the teacher can (roughly speaking) specify the left side of the conversion rule by placing the corresponding connective in the initial formula and the right side by specifying the task type (what connectives are allowed in the result).

Concerning requirement E we see that the size of solution can be very directly specified by connectives in the initial and target formula. Every conjunction, disjunction and implication can be eliminated in one step. Biconditional requires 2-4 steps depending on binary target connective. Additional steps for simplification of the result (mainly related to negations) are considerably easier and do not have great impact.

There is yet one issue that affects the difficulty of generated tasks (both on expression through given connectives and DNF). The conversion rules for elimination of biconditional multiply the arguments of biconditional into two copies and make the formula longer. For example, $X{\sim}Y \equiv X\&Y \vee \neg X\&\neg Y$. This impact is small if the arguments of biconditional are only variables or their negations, and is much larger if the arguments are complex formulas (see Fig. 3). It means that the amount and complexity of work on tasks to be solved by different students has a random component. For elimination of this randomness it could be desirable to add a generation parameter that defines the size(s) of arguments of biconditional or at least have some lower bound (for example, at least one argument/both arguments should contain a binary connective).

Consider now FDNF exercises. We present in our lectures the following version (Palm and Prank, 2004) of the usual algorithm for conversion of a formula to FDNF :

1) Eliminate implications and biconditionals from the formula;
2) Move negations inside;
3) Use distributive law to expand the conjunctions of disjunctions;
4) Exclude contradictory conjunctions and redundant copies of literals;
5) Add missing variables to conjunctions;
6) Order variables alphabetically, exclude redundant copies of conjunctions.

Requirements C-E mean in FDNF exercises mainly control over the numbers of steps to be made at stages 1-6 of the algorithm. The teacher program enables to generate formulas with the desired number of implications and biconditionals for stage 1. We can also get a high probability of nontrivial stage 2, generating formulas with 3-4 negations. Using existing generating parameters, we have limited control over stages 3-6. The use of reverse engineering does not show much promise. The 6-stage algorithm is too long for this. What can be added?

There is a quite simple opportunity for regulation of sizes of stages 5-6 and partially of stages 3-4. The number of necessary steps in these stages depends, to a great extent, on the number of members in full DNF. It means that the control over the number of truth-values *true* would improve the situation with regard to requirements C-E.

More promising seems to be to generate the initial formulas and pick out those that meet our pedagogical needs. For this we need an Automated Solver. How far are we from it? Our student program of 2013 already contains a solution engine that is used for giving hints for next step (what to mark and how to replace the marked part). It even creates full solutions but does not make them available outside. On the other hand, the student program saves in the solution file the whole student solution together with conversion rules used at each step (in case of Rule mode). This means that, combining these two mechanisms, we would not have to program any new engines for solving the task and checking what stages of the algorithm are active. Our experience with random generation for school algebra program T-algebra shows that it is also possible to solve the computation speed problems.

7 SUMMARY

In Sections 5 and 6 we found several possibilities for giving the teacher better control over the properties of generated tasks:

1. The teacher can specify precisely the acceptable numbers of values *true* for generated formulas.
2. The program first randomly chooses the desired answer (tautology/not, etc) and then generates the corresponding formula(s).
3. The program is able to find the shortest formula with a given column of values (using a corresponding program module or a precomputed list of formulas).
4. The teacher can specify the sizes of arguments of biconditionals in generated formulas.
5. The program computes the solution stages of generated normal form tasks, finds their sizes and chooses appropriate formulas.

Addition of features 1-4 to our programs can be implemented in a Bachelor thesis. Addition of feature 5 could result in a strong Master thesis. In fact, our program already contains an algorithm for finding the solution but it is necessary to experiment and find sufficiently quick ways of generation.

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HomeSchoolmath, http://www.homeschoolmath.net/worksheets/

IXL Maths, http://eu.ixl.com/math/

Math.com Algebra Worksheet Generator, http://www.math.com/students/worksheet/algebra_sp.htm.

Math Mix, http://www.mathmix.com/worksheets/Algebra/

StudyMaths.co.uk, http://studymaths.co.uk/workoutMenu.php?type=algebra.

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