Numerical Integration of Multiple Integrals using Taylor Polynomial

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Abstract: The paper concentrates on a new method of numerical computation of multiple integrals. Equations based on Taylor polynomial are derived. Multiple integral of a continuous function of n-variables is numerically integrated step by step by reducing its dimension. First, integration formulas for a function of two variables are derived. Formulas for function of n-variables are generalized using composition. Numerical derivatives for Taylor terms are repeatedly computed from simple integrals. Finally method is demonstrated on an exponential function of two-variables and a new approach to determine a number of Taylor terms is discussed.

1 INTRODUCTION

The aim of this paper is to show a way how to numerically integrate a function of n variables by transforming the integral into a differential equation. Differential equation is solved by Taylor polynomial where higher derivatives are numerically computed from simple integrals of previously solved differential equations.

To determine a number of Taylor terms for a general function known only by its discrete points and to achieve a given error is still an open problem. Upper bound of an absolute value of all terms can be investigated. Optimal order approach to determine an integration step and a degree of Taylor polynomial (Jordan, Maorong, 2005) and its modification (Abad, Barrio, Blesa, Rodriguez, 2012) can be found. In this paper terms are computed by dot product of rows of an inverse matrix and a vector of discrete points. Analysis of the product and partial approximation of the bound is given.

2 DOUBLE INTEGRAL

Let's have a double integral of general continuous function $\int_{y_0}^{y_1}\int_{x_0}^{x_1} f(x,y)\,\mathrm{d}x\,\mathrm{d}y$.

Translating f into origin, the equivalent integral can be obtained.

$$\int_0^{y_1-y_0}\int_0^{x_1-x_0} f_t(x,y)\,\mathrm{d}x\,\mathrm{d}y \tag{1}$$

$$f_t(x,y) = f(x_0+x, y_0+y) \tag{2}$$

Without any loss of generality all double integrals can be rewritten into the following form (lower bounds are equal to zero).

$$\int_0^{b_2}\int_0^{b_1} f(x,y)\,\mathrm{d}x\,\mathrm{d}y \tag{3}$$

Integration step for x-axis is denoted by h, for y-axis by k, $h=\frac{b_1}{p}$, $k=\frac{b_2}{q}$, where p, resp. q are number of partitions of x, resp. y axis.

2.1 Sampling by an Axis

Sampling the function f by y-axis, the following system is obtained.

$$\begin{aligned} g(x,0) &= f(x,0)\\ g(x,k) &= f(x,k)\\ g(x,2k) &= f(x,2k)\\ &\vdots\\ g(x,jk) &= f(x,jk)\\ &\vdots\\ g(x,qk) &= f(x,qk) \end{aligned} \tag{4}$$

From here on, $g_j(x)$ denotes $g(x,jk)$.

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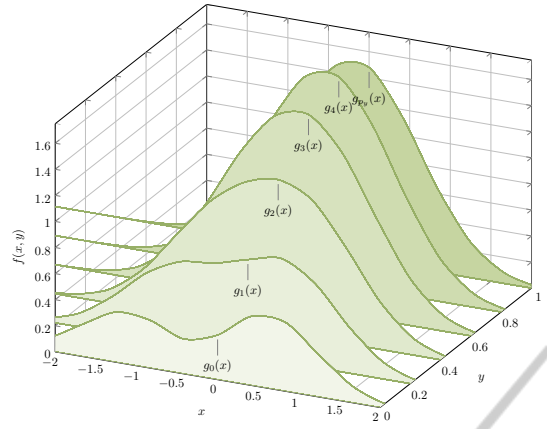

Figure 1: Graph of f(x,y).

2.2 First Integration

From the Figure 1.

$$\begin{aligned} I_0 &= \int_0^{b_1} g_0(x)\,\mathrm{d}x \\ I_1 &= \int_0^{b_1} g_1(x)\,\mathrm{d}x \\ &\vdots \\ I_j &= \int_0^{b_1} g_j(x)\,\mathrm{d}x \\ &\vdots \\ I_q &= \int_0^{b_1} g_q(x)\,\mathrm{d}x \end{aligned} \tag{5}$$

This system can be transformed into an equivalent system of differential equations.

$$\begin{aligned} I_0'(t) &= g_0(t), I_0(0) = 0 \\ I_1'(t) &= g_1(t), I_1(0) = 0 \\ &\vdots \\ I_j'(t) &= g_j(t), I_j(0) = 0 \\ &\vdots \\ I_q'(t) &= g_q(t), I_q(0) = 0 \end{aligned} \tag{6}$$

2.3 Numerical Solution of Differential Equations using Taylor Polynomial

Taylor polynomial is used to approximate functions $I_j(x)$ (system of equations (6)).

$$\begin{aligned} I_j(h) &= I_j(0) &&+ \sum_{m=1}^{n} DI_j(m,0) \\ I_j(2h) &= I_j(h) &&+ \sum_{m=1}^{n} DI_j(m,1) \\ &\vdots \\ I_j(sh) &= I_j((s-1)h) &&+ \sum_{m=1}^{n} DI_j(m,s-1) \\ &\vdots \\ I_j(ph) &= I_j((p-1)h) &&+ \sum_{m=1}^{n} DI_j(m,p-1) \end{aligned} \tag{7}$$

where $j = 1..q$, $DI_j(m,s)$ are members of Taylor expansion, $\forall j : I_j(0) = 0$ (area starts from 0).

2.4 Sampled Integrals

New function ψ (Figure 2) can be created from solutions of the system (7). Its domain consists of points given as multiples of k.

$$\begin{aligned} \psi(0) &= I_0 \\ \psi(k) &= I_1 \\ &\vdots \\ \psi(jk) &= I_j \\ &\vdots \\ \psi(qk) &= I_q \end{aligned} \tag{8}$$

Using ψ, double integral can be approximated using the following formula (supposing $\tilde{\psi}$ is continuous version of ψ).

$$\int_{y_0}^{b_2}\int_0^{b_1} f(x,y)\,\mathrm{d}x\,\mathrm{d}y \approx \int_0^{b_2} \tilde{\psi}(y)\,\mathrm{d}y \tag{9}$$

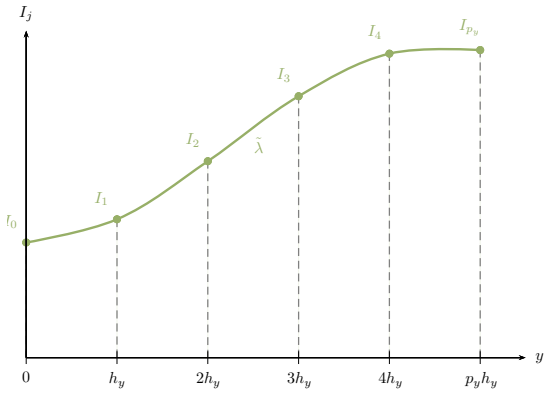

Figure 2: Graph of $\psi(y)$.

Let's denote integral (9) as function e.

$$e(y_1) = \int_0^{b_2} \psi(y)\,\mathrm{d}y \tag{10}$$

Now the process of differentiation can be repeated.

$$e'(t) = \psi(t) \quad e(0) = 0 \tag{11}$$

Again, function e can be solved using Taylor polynomial.

$$\begin{aligned}
e(k) &= e(0) &&+ \sum_{m=1}^{n} \frac{e^{(m)}(0)}{m!}k^m \\
e(2k) &= e(k) &&+ \sum_{m=1}^{n} \frac{e^{(m)}(k)}{m!}k^m \\
&\vdots \\
e(jk) &= e((j-1)k) &&+ \sum_{m=1}^{n} \frac{e^{(m)}((j-1)k)}{m!}k^m \\
&\vdots \\
e(qk) &= e((q-1)k) &&+ \sum_{m=1}^{n} \frac{e^{(m)}((q-1)k)}{m!}k^m
\end{aligned} \tag{12}$$

Using (11).

$$\frac{e^{(m)}(t)}{m!}k^m = \frac{\psi^{(m-1)}(t)}{m!}k^m \tag{13}$$

So the system of equations (12) can be rewritten.

$$\begin{aligned}
e(k) &= e(0) + \sum_{m=1}^{n} \frac{\psi^{(m-1)}(0)}{m!}k^m \\
e(2k) &= e(k) + \sum_{m=1}^{n} \frac{\psi^{(m-1)}(k)}{m!}k^m \\
&\vdots \\
e(jk) &= e((j-1)k) + \sum_{m=1}^{n} \frac{\psi^{(m-1)}((j-1)k)}{m!}k^m \\
&\vdots \\
e(qk) &= e((q-1)k) + \sum_{m=1}^{n} \frac{\psi^{(m-1)}((q-1)k)}{m!}k^m
\end{aligned} \tag{14}$$

However, in this case the derivatives of ψ are unknown, which means that a different approach is needed. Let's construct the following system,

$$\begin{aligned}
\psi(k) &= \psi(0) + \sum_{m=1}^{n} D\psi(m,0) \\
\psi(2k) &= \psi(0) + \sum_{m=1}^{n} 2^m D\psi(m,0) \\
&\vdots \\
\psi(jk) &= \psi(0) + \sum_{m=1}^{n} j^m D\psi(m,0) \\
&\vdots \\
\psi(nk) &= \psi(0) + \sum_{m=1}^{n} n^m D\psi(m,0)
\end{aligned} \tag{15}$$

where $D\psi(m,t) = \frac{\psi^{(m)}(t)}{m!}k^m$.
Because $\psi(y_0 + jk) = I_j$, the system (15) is equivalent to system (16).

$$\begin{aligned}
I_1 &= I_0 + \sum_{m=1}^{n} D\psi(m,0) \\
I_2 &= I_0 + \sum_{m=1}^{n} 2^m D\psi(m,0) \\
&\vdots \\
I_j &= I_0 + \sum_{m=1}^{n} j^m D\psi(m,0) \\
&\vdots \\
I_n &= I_0 + \sum_{m=1}^{n} n^m D\psi(m,0)
\end{aligned} \tag{16}$$

System can then be expressed in matrix form.

$$b = \begin{pmatrix} I_1 - I_0 \\ I_2 - I_0 \\ \cdots \\ I_j - I_0 \\ \cdots \\ I_n - I_0 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 1^2 & \ldots & 1^n \\ 2 & 2^2 & \ldots & 2^n \\ \cdots & & & \\ j & j^2 & \ldots & j^n \\ \cdots & & & \\ n & n^2 & \ldots & n^n \end{pmatrix}$$

$$x = \begin{pmatrix} D\psi(1,0) \\ D\psi(2,0) \\ \cdots \\ D\psi(j,0) \\ \cdots \\ D\psi(n,0) \end{pmatrix}$$

Putting it all together.

$$Ax = b \tag{17}$$

Solving a vector x terms $D\psi(j,0)$ are obtained. These terms can be used to solve $e(k)$ in system of equations (14). The remaining $e(jk)$ can be calculated analogically, the matrix A stays the same, matrix b needs to be recalculated for each $e(jk)$.

Matrix A is ill-conditioned, multiple precision arithmetic must be used for higher order. Block-wise inversion and twelfth-order convergence numerical method (Haghani and Soleymani, 2014) can be used for inverse matrix computation.

3 MULTIPLE INTEGRAL

Generally, a multiple integral is in the following form.

$$\int_{a_n}^{b_n} \dots \int_{a_1}^{b_1} f(x_1,\dots,x_n)\,\mathrm{d}x_1 \dots \mathrm{d}x_n \tag{18}$$

Translating function f to origin, equivalent integral is obtained.

$$\int_{0}^{b_n-a_n} \dots \int_{0}^{b_1-a_1} g(x_1,\dots,x_n)\,\mathrm{d}x_1 \dots \mathrm{d}x_n \tag{19}$$

$$g(x_1,\dots,x_n) = f(a_1+x_1,\dots,a_n+x_n) \tag{20}$$

3.1 Basic Definitions

Let $N = \{1,\dots,n\}$. Permutation from N to N is a bijective function $\pi : N \to N$. In order to eliminate an ordering of integration steps in (19), permutation is used.

$$\int_{0}^{b_{\pi(n)}} \dots \int_{0}^{b_{\pi(1)}} g(x_1,\dots,x_n)\,\mathrm{d}x_{\pi(1)} \dots \mathrm{d}x_{\pi(n)} \tag{21}$$

With no loss of generality, it can be written as follows.

$$\int_{0}^{b_n} \dots \int_{0}^{b_1} g(x_1,\dots,x_n)\,\mathrm{d}x_1 \dots \mathrm{d}x_n \tag{22}$$

Further, isomorphism of composition of tuples is implicitly taken, e.i. $((1,\dots,n),(n+1,\dots,m)) \approx (1,\dots,m)$. Integral's bounds define a closed space $\Pi_{i\in N}\langle 0,b_i\rangle$. Integration step for i-th interval is denoted as h_i, number of partitions as d_i. Then $h_i = \frac{b_i}{d_i}$.

3.2 First Integration

To simplify a notation of sampling.

$$g_1(x_1,i_2,\dots,i_n) = f(x_1,i_2h_2,\dots,i_nh_n) \tag{23}$$

The definition of g_1 is derived from the the number of simple integrals after sampling ($c = \times_{i\in N}(d_i+1)$). The set of indexes has to be defined.

$$\Phi(x_j) = \Pi_{i\in N-\{1,\dots,j\}}\{0,\dots,d_i\}, 1 \le j \le n \tag{24}$$

To compute all c integrals, for each $s \in \Phi(x_1)$.

$$I_s = I_s(b_1) = \int_0^{b_1} g_1(x_1,s)\mathrm{d}x_1 \tag{25}$$

Transformed into differential equation (initial value problem).

$$I_s'(t) = g_1(t,s), \quad I_s(0) = 0 \tag{26}$$

Solving using Taylor polynomial

$$\begin{aligned} I_s(h_1) &= I_s(0) + \sum_{m=1}^{\alpha(s,1)} \frac{h_1^m}{m!} I_s^{(m)}(0) \\ I_s(2h_1) &= I_s(h_1) + \sum_{m=1}^{\alpha(s,2)} \frac{h_1^m}{m!} I_s^{(m)}(h_1) \\ &\vdots \\ I_s(d_1h_1) &= I_s(d_ph_1) + \sum_{m=1}^{\alpha(s,d_1)} \frac{h_1^m}{m!} I_s^{(m)}(d_ph_1) \end{aligned} \tag{27}$$

where $d_p = d_1 - 1$ and $\alpha(s,j)$ stands for number of members of Taylor polynomial for integral s in time jh_1.

At the point d_1h_1.

$$I_s(d_1h_1) \approx \int_0^{b_1} g_1(x_1,s)\,\mathrm{d}x_1 \tag{28}$$

3.3 Composition

For each $p \in \Phi(x_2)$ where

$$s_i = (i,p), 0 \le i \le d_2 \tag{29}$$

new function is obtained (analogy to double integral depicted in Figure 2).

Multiple integral can be approximated

$$\begin{aligned} &\int_0^{b_n} \dots \int_0^{b_1} f(x_1,\dots,x_n)\mathrm{d}x_1 \dots \mathrm{d}x_n \approx \\ &\int_0^{b_n} \dots \int_0^{b_2} \tilde{\psi_2}(x_2,\dots,x_n)\,\mathrm{d}x_2 \dots \mathrm{d}x_n \end{aligned} \tag{30}$$

where $\tilde{\psi_2}$ is a continuous version of ψ_2.

$$\psi_2(i_2h_2,\dots,i_nh_n) = I_s, s = (i_2,\dots,i_n) \in \Phi(x_1) \tag{31}$$

Further

$$\begin{aligned} &\int_0^{b_n} \dots \int_0^{b_2} \tilde{\psi}(x_2,\dots,x_n)\,\mathrm{d}x_2 \dots \mathrm{d}x_n = \\ &\int_0^{b_n} \dots \int_0^{b_3} \sigma_2(x_3,\dots,x_n)\,\mathrm{d}x_3 \dots \mathrm{d}x_n \end{aligned} \tag{32}$$

where

$$\sigma_2(t_3,\dots,t_n) = \int_0^{b_2} \tilde{\psi}(x_2,t_3\dots,t_n)\,\mathrm{d}x_2 \tag{33}$$

3.4 Generalization

I_s was defined previously for $s \in \Phi(x_1)$. This can be generalized for $p_k \in \Phi(x_k), 2 \le k < n$:

$$I_{p_k} = I_{p_k}(b_k) = \int_0^{b_k} g_k(x_k, p_k) dx_k \tag{34}$$

$$g_k(x_k, p_k) = I_{p_{k-1}}, p_{k-1} = (\lceil \frac{x_k}{h_k} \rceil, p_k) \tag{35}$$

For integrals:

$$\int_0^{b_n} \dots \int_0^{b_k} \tilde{\psi}_k(x_k, \dots, x_n)\, \mathrm{d}x_k \dots \mathrm{d}x_n = \int_0^{b_n} \dots \int_0^{b_{k+1}} \sigma_k(x_{k+1}, \dots, x_n)\, \mathrm{d}x_{k+1} \dots \mathrm{d}x_n \tag{36}$$

where

$$\sigma_k(t_{k+1}, \dots, t_n) = \int_0^{b_k} \tilde{\psi}_k(x_k, t_{k+1} \dots, t_n)\, \mathrm{d}x_k \tag{37}$$

$$\psi_k(i_k h_k, \dots, i_n h_n) = I_{p_{k-1}}, p_{k-1} = (i_k, \dots, i_n) \tag{38}$$

Using formula (35) ψ_k can be expressed as

$$\psi_k(i_k h_k, \dots, i_n h_n) = g_k(i_k h_k, i_{k+1}, \dots, i_n) \tag{39}$$

3.5 Numerical Computation of Derivatives

From sampled values $f_{-l}, \dots, f_{-1}, f_0, f_1, \dots, f_k$ at points $-lh, \dots, -h, 0, h, \dots, kh$, $n = k + l + 1$, set of equations from Taylor polynomials can be constructed.

$$\forall i \in \{-l, -l+1, \dots, -1, 0, 1, \dots, k-1, k\}: \quad f_i = f_0 + \sum_{m=1}^{n} \frac{i^m h^m}{m!} f^{(m)}(0) \tag{40}$$

Written in matrix form:

$$b = \begin{pmatrix} f_{-l} - f_0 \\ f_{-l+1} - f_0 \\ \dots \\ f_{-1} - f_0 \\ f_1 - f_0 \\ \dots \\ f_{k-1} - f_0 \\ f_k - f_0 \end{pmatrix}$$

$$A = \begin{pmatrix} (-l)^1 & (-l)^2 & \dots & (-l)^n \\ (-l+1)^1 & (-l+1)^2 & \dots & (-l+1)^n \\ \dots \\ (-1)^1 & (-1)^2 & \dots & (-1)^n \\ (1)^1 & (1)^2 & \dots & (1)^n \\ \dots \\ (k-1)^1 & (k-1)^2 & \dots & (k-1)^n \\ (k)^1 & (k)^2 & \dots & (k)^n \end{pmatrix}$$

$$x = \begin{pmatrix} f^{(1)}(0) \\ \dots \\ f^{(n)}(0) \end{pmatrix}$$

$$x = A^{-1} b \tag{41}$$

Vector x represents a vector of Taylor terms. Terms in x can be transformed (transformation denoted by G) into derivatives.

For given numbers (l,k), matrix b changes, A^{-1} stays the same. From now on, this system is going to be denoted as $\kappa_h^{l,k}$ and its solution as $\kappa_h^{l,k}(f_{-l}, \dots, f_k) = G(A^{-1}b)$.

3.6 Approximation of I_k

The main task is to solve the following integral.

$$I_{p_k} = I_{p_k}(b_k) = \int_0^{b_k} g_k(x_k, p_k)\, \mathrm{d}x_k \tag{42}$$

It can be transformed into an initial value problem.

$$I'_{p_k}(t) = g_k(t, p_k) \quad I_{p_k}(0) = 0 \tag{43}$$

Solved by Taylor polynomial the system can be written as follows

$$\begin{aligned} I_{p_k}(h_k) &= I_{p_k}(0) + \sum_{m=1}^{\alpha(p_k,1)} \frac{h_k^m}{m!} I_{p_k}^{(m)}(0) \\ I_{p_k}(2h_k) &= I_{p_k}(h_k) + \sum_{m=1}^{\alpha(p_k,2)} \frac{h_k^m}{m!} I_{p_k}^{(m)}(h_k) \\ &\vdots \\ I_{p_k}(d_k h_k) &= I_{p_k}(d_k^b h_k) + \sum_{m=1}^{\alpha(p_k,d_k)} \frac{h_k^m}{m!} I_{p_k}^{(m)}(d_k^b h_k) \end{aligned} \tag{44}$$

where $d_k^b = d_k - 1$. Substituting (43) into (44):

$$I_{p_k}(h_k) = I_{p_k}(0) + \sum_{m=1}^{\alpha(p_k,1)} \frac{h_k^m}{m!} g_k^{(m-1)}(0, p_k) \tag{45}$$

$$I_{p_k}(2h_k) = I_{p_k}(h_k) + \sum_{m=1}^{\alpha(p_k,2)} \frac{h_k^m}{m!} g_k^{(m-1)}(h_k, p_k) \tag{46}$$

$$\vdots$$

$$I_{p_k}(d_k h_k) = I_{p_k}(d_k^b h_k) + \sum_{m=1}^{\alpha(p_k,d_k)} \frac{h_k^m}{m!} g_k^{(m-1)}(d_k^b h_k, p_k) \tag{47}$$

Derivatives of $g_k(t, p_k)$ cannot be computed analytically, because g_k is a sampled function. However, the sampled values can be used to compute numerical derivatives

$$g_k^{(m)}(0, p_k) = x_k^0(m), 1 \le m \le n \tag{48}$$

where

$$x_k^0 = \kappa_{h_k}^{0,n}(g_k(0,p_k), g_k(h_k,p_k), \ldots, g_k(nh_k,p_k)) \quad (49)$$

Generally for x^i

$$x_k^i = \kappa_{h_k}^{l,m}(g_k(i+j_1h_k,p_k), \ldots, g_k(i+j_{|\beta(i,k)|}h_k,p_k))$$

$$\beta(i,k) = \{-l,\ldots,m\}, j_1,\ldots,j_{|\beta(i,k)|} \in \beta(i,k),$$

$$j_1 < j_2 < \cdots < j_{|\beta(i,k)|}, i+j_1 \geq 0$$

where $\beta(i,k)$ stands for a set of indeces in time ih_k for p_k, x_k^i's first coordinate has index 1.

$$g_k^{(m)}(ih_k,p_k) = x_k^i(m), 1 \leq m \leq n \quad (50)$$

$$I_{p_k}(h_k) = E_k^0 + \sum_{m=2}^{\alpha(p_k,1)} \frac{h_k^m}{m!} x_k^0(m-1) \quad (51)$$

$$I_{p_k}(2h_k) = E_k^1 + \sum_{m=2}^{\alpha(p_k,2)} \frac{h_k^m}{m!} x_k^1(m-1) \quad (52)$$

$$\ldots$$

$$I_{p_k}(d_kh_k) = E_k^{d_k^b} + \sum_{m=2}^{\alpha(p_k,d_k)} \frac{h_k^m}{m!} x_k^{d_k^b}(m-1) \quad (53)$$

where $E_k^j = I_{p_k}(0) + g_k(jh_k,p_k)$, $d_k^b = d_k - 1$.
From (35)

$$x_k^i = \kappa_{h_k}^{l,m}(I_{(i+j_1,p_k)}, I_{(i+j_2,p_k)}, \ldots, I_{(i+|\beta(i,k)|,p_k)}) \quad (54)$$

4 INTEGRATION OF AN EXPONENTIAL FUNCTION

The method was tested on a computation of double integral of an exponential function, i.e. $\int_0^2 \int_0^2 e^{x+y} dydx$. Tests were run for a different lengths of an integration step, number of Taylor terms and arithmetic precision. Precision is determined by a number of bits used for the mantissa of each number. Numerical solution was compared with an analytical solution of the double integral. Error of computation for each test instance is shown in table 1. For fixed integration step and precision, only a number of the Taylor term with the smallest error is shown.

5 COMPUTATION OF TAYLOR TERMS

In order to compute a value of function $f(h)$ at h, the following equation is used.

$$f(h) = f(0) + \frac{f^{(1)}(0)}{1!}h + \frac{f^{(2)}(0)}{2!}h^2 + \ldots \quad (55)$$

Table 1: Error of computation of $\int_0^2 \int_0^2 e^{x+y} dydx$.

step	no. of terms	precision [bits]	error
0.2	11	400	1.13e-06
0.1	19	400	8.95e-20
0.05	39	400	6.88e-52
0.04	49	400	9.14e-70
0.02	99	800	8.02e-170
0.01	104	900	8.09e-208
0.005	94	900	4.59e-217
0.004	91	900	4.13e-220
0.002	85	900	3.49e-227

The equation can be expressed using Di terms.

$$f(h) = f(0) + D1 + D2 + D3 + \ldots \quad (56)$$

The terms can be computed using combined method, where I_i are samples from which terms (b column vector) can be computed. Matrix A is a special form of matrix and x is a vector of the corresponding terms.

$$b = \begin{pmatrix} I_1 - I_0 \\ I_2 - I_0 \\ \ldots \\ I_j - I_0 \\ \ldots \\ I_n - I_0 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 1^2 & \ldots & 1^n \\ 2 & 2^2 & \ldots & 2^n \\ \ldots & & & \\ j & j^2 & \ldots & j^n \\ \ldots & & & \\ n & n^2 & \ldots & n^n \end{pmatrix}$$

$$x = \begin{pmatrix} D1 \\ D2 \\ \ldots \\ Dj \\ \ldots \\ Dn \end{pmatrix}$$

In matrix equation form.

$$Ax = b \quad (57)$$

The terms can be then computed by solving the matrix.

$$x = A^{-1}b \quad (58)$$

In order to compute x, it is necessary to compute inverse of matrix A.

5.1 Rate of Change of Taylor Terms

The number of terms n to be computed depends on the precision, i.e. it is a function of ε, i.e. $n(\varepsilon)$. The ideal case is when the terms form a descending sequence.

$$|D_1| > |D_2| > |D_3| > \ldots \quad (59)$$

The function $n(\varepsilon)$ is then given.

$$n(\varepsilon) = sup\{i||D_i| < \varepsilon\} \tag{60}$$

Generally, this is not the case. In order to find $n(\varepsilon)$, it is necessary to find a decreasing sequence $U(i)$ such that $U(i) >= |Di|$.

$$n(\varepsilon) = sup\{i|U_i < \varepsilon\} \tag{61}$$

Looking at equation (58) i-th term is given as a dot product of i-th row of A^{-1} and x.

$$D_i = a_i^{-1}x \tag{62}$$

From (62) it can see the value of D_i depends only on the A^{-1} (which is independent of h) and x (dependent of h). In order to construct $n(\varepsilon)$, inverse matrix and x have to be analyzed.

5.2 Analysis of A^{-1} and x

Dot product (62) can be over-approximated. Vector x is given as a difference of samples I_i. Taking the maximum value M and the minimum value m value of samples, new vector $x' = (M-m, M-m, \ldots, M-m)$ is constructed. Then $x \leq x'$. For the product, if $x \geq 0$ the following holds.

$$|D_i| = |a_i^{-1}x| = \left|\sum_{j=1}^{n} a_{i,j}^{-1}x_j\right| = \left|\sum_{a_{i,j}^{-1}>0}^{n} a_{i,j}^{-1}x_j + \sum_{a_{i,j}^{-1}<0}^{n} a_{i,j}^{-1}x_j\right| \tag{63}$$

For sum of positive members.

$$\sum_{a_{i,j}^{-1}>0}^{n} a_{i,j}^{-1}x_j \leq \sum_{a_{i,j}^{-1}>0}^{n} a_{i,j}^{-1}(M-m) = (M-m)\sum_{a_{i,j}^{-1}>0}^{n} a_{i,j}^{-1} = (M-m)P(i) \tag{64}$$

For sum of negative members.

$$\sum_{a_{i,j}^{-1}<0}^{n} a_{i,j}^{-1}x_j \geq \sum_{a_{i,j}^{-1}<0}^{n} a_{i,j}^{-1}(M-m) = (M-m)\sum_{a_{i,j}^{-1}<0}^{n} a_{i,j}^{-1} = (M-m)N(i) \tag{65}$$

Figure 3 shows the progress of summing the positive and negative members of matrix with 93 samples taken from the left and 113 samples taken from the right of a sampled function f. The sum of all members can be seen in Figure 4. Multiplying positive members with $M-n$ implies greater sum. Thus the entire sum is the upper bound of the term. Multiplying negative members with $M-m$ implies smaller sum. Now the question is how big is the decrease. Expressing $M-n$ in powers of 10.

$$M-n = 10^e \tag{66}$$

If all components of x on index of negative members of row i are smaller than 1, their weighted sum $N(i)$ increases. Thus an upper bound is still obtained. If there is a component $c_j > 1$, the sum is decreased. If c_j is substited with $M-m$, the sum is decreased by $(M-m-c_j)a_{i,j}$. In general, the decrease is given as $\sum_{c_j>1}(M-m-c_j)a_{i,j}$, because $M-n > M-n-c_j$ (as $M-n > c_j$).

$$\Delta = \sum_{c_j>1}(M-m-c_j)|a_{i,j}^{-1}| \leq \sum_{c_j>1}(M-m)|a_{i,j}^{-1}| = (M-m)\sum_{c_j>1}|a_{i,j}^{-1}| \tag{67}$$

This decrease for i-th row will be denoted as $\Delta(i)$.

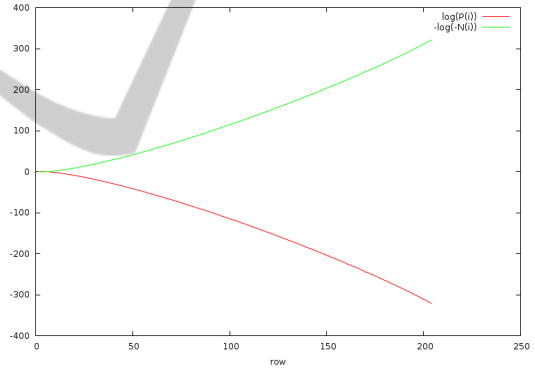

Figure 3: Sum of positive members as a function of row number.

To compensate for the decrease $\Delta(i)$, let's first take a look at Figure 3 again. Adding $-\Delta(i)$ to $N(i)$, $N(i)$ is decreased by a factor of 10^e. Assuming $N(i)$ and $P(i)$ are of the same order, the resulting sum is decreased by the same factor of 10^e. Denoting $\sum_{j=1}^{n} a_{i,j}^{-1}$ as $S(i)$ and putting it all together.

$$|Di| \leq |(M-n)P(i) + (M-n)N(i) + \Delta(i)| = |(M-m)S(i) + \Delta(i)| \tag{68}$$

Because $\Delta(i) \leq (M-m)|N(i)|$.

$$\begin{aligned} &|Di| \leq |(M-m)S(i) + \Delta(i)| \leq \\ &(M-m)|S(i)| + \Delta(i) \leq \\ &(M-m)|S(i)| + (M-m)|N(i)| = \\ &(M-m)(|S(i)| + |N(i)|) \end{aligned} \tag{69}$$

Assuming $|N(i)| \geq |S(i)|$ terms are upper bounded only by $N(i)$.

$$|Di| \leq (M-m)(|N(i)|+|N(i)|) = \\ = 2(M-m)|N(i)| \approx 10^{e}|N(i)| \quad (70)$$

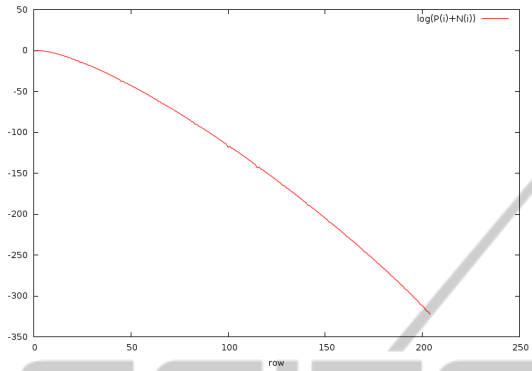

Figure 4: Sum of negative members as a function of row number.

Inequality (70) is valid because given values $N(i)$ and $P(i)$ are very similar and their sum $S(i)$ has smaller exponent. However any weighted sum with weights different from $w_i = 1$ changes mantissa of both $N(i)$ and $P(i)$ and their sum $D(i)$ therefore has bigger exponent.

Thus using $(M-m)S(i)$ for upper bound results in smaller value, but with a suitable vector x, weighted sum of Di can give greater value than $(M-m)S(i)$. From observation, $S(i)$ is always smaller than $N(i)$. The higher the dimension of A, the greater ratio $\frac{N(i)}{S(i)}$ (in positive powers of 10). For example, for dimension equal to 400, the ratio can be 10^{40}.

Figure 5 shows $N(n)$ of the last row for each k,l-matrix on $\langle 3;100\rangle \times \langle 3;100\rangle$ (k stands for a number of samples taken from left, l for a number of samples taken from the right). Symmetry in sums of negative members can be seen. For $k = l$ the greatest sum is obtained. If $k \neq l$, the sum gets smaller. To eliminate dependency on k and l, only cases, where $k = l$ need to be analyzed.

Figure 6 shows the case where $k+l = n$. It can be seen, that the graph decreases as n increases.

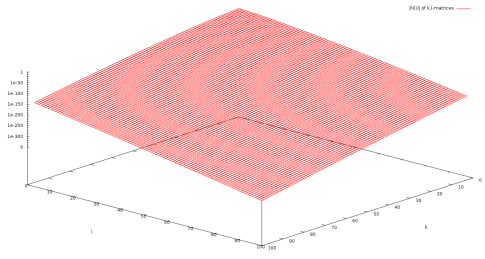

Figure 5: $|N(n)|$ of A^{-1} as a function of n.

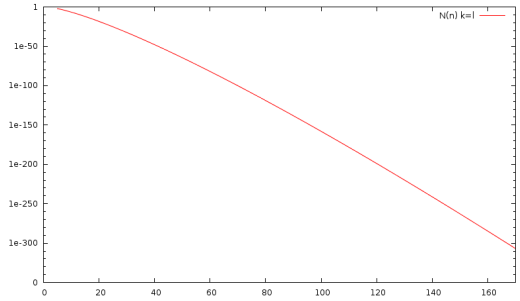

Figure 6: $|N(n)|$ for $k = l$ as a function of n.

6 COMPUTATION OF $N(I)$

The number of terms needed to compute the Taylor series depends on the quality of the approximation of the decreasing upper bound of an absolute value of the sum of the last row for each dimension.

6.1 Approximation of a $N(i)$ of the Last Rows

The better approximation of the upper bound the smaller number of Taylor terms needed. Denoting the upper bound as $s(n)$ minimization problem is to minimize error

$$err = \sum_{i=1}^{\infty} ||Di| - s(i)| \quad (71)$$

given the following equations:

$$|Di| \leq s(i) \forall i \in N \quad (72)$$
$$s(i) > s(i+1) \forall i \in N \quad (73)$$

6.2 State of the Art Approximation

Figure 7 shows an approximation of $\log(|N(n)|)$ by a function:

$$g(n) = -n\log_a(n) + b \quad (74)$$

where $a = 21.1$ and $b = 40$.

In the Figure 7 function $g(n)$ is an upper bound of the graph. Values a and b were determined experimentally.

Function $g(n)$ is given by multiplication of logarithmic and linear function. As logarithm is negative and determines a tangent, $g(n)$ is decreasing function of n.

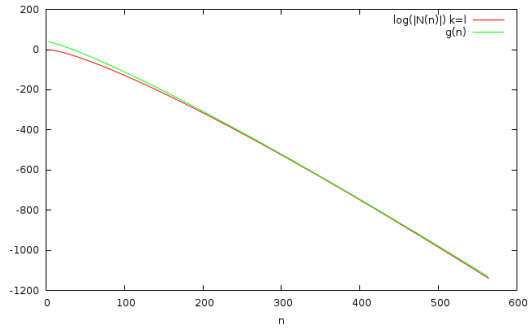

Figure 7: Approximation of $g(n)$.

Upper bound $s(n)$ is given by the following equation.

$$s(n) = (M-m)10^{40-n\log_{21.1}(n)} \quad (75)$$
$$= 10^{e+40-n\log_{21.1}(n)}$$

7 TEST OF THE UPPER BOUND

Figure 8 shows progress of computed value of Di of a function e^x for $k = l = \frac{n}{2}$. Graph e^x gives a value of e^x's $D(n)$. As we can see its values are upper bounded by $s(n)$. Bound $\log(|N(i)|)$ still holds for this case as $M-m$ is of order 10^0. Second function is $\frac{1}{x^2+10}$. It is upper bounded again, $M-m$ is of order 10^{-2}. Polynomial function $10^{85}x^{33} + 10^{10}x^{45}$ test a case where a range of first to 32-th terms are equal to 0, 33-th derivative is non-zero, 34-th to 44-th equal to 0, 45-th non-zero and the higher are equal to 0. Forward method *poly1 f* (Figure 8) is used for this case as it computes derivatives in time 0. The polynom is for the first 100 samples of order 10^{-9}. As it can be seen in the graph it is still upper bounded. However the bound is too high and multiplicator $(M-m)$ must be analyzed to give better upper bound. The same apply for *poly1 c* (Figure 8). It uses combined method with samples of order 10^{61}.

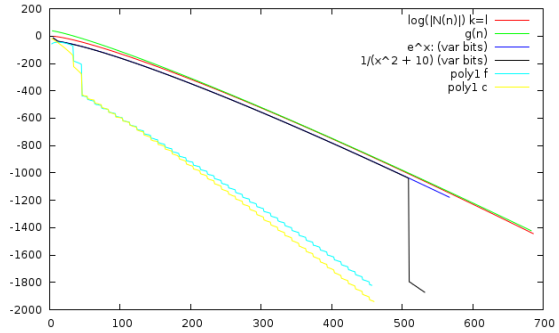

Figure 8: Approximation of Di by $g(n)$.

8 CONCLUSIONS

New method for numerical integration of a function of n-variables has been introduced. It is based on Taylor polynomials and computation of its terms from differential equations previously solved. Determining an optimal number of terms for each polynomial is still an open problem. Further analysis of a dot product is required.

The method has been tested on integrals with known analytical solution. Only hyper-cubical integration areas were explored so far.

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REFERENCES

F. Khaksar Haghani, F. Soleymani. (2014). *A New High-Order Stable Numerical Method for Matrix Inversion*. The Scientific World Journal, Volume 2014

A. Jordan, Z. Maorong. (2005). *A Software Package for the Numerical Integration of ODEs by Means of High-Order Taylor Methods*. Experimental Mathematics, Vol. 14, No. 1

A. Abad, R. Barrio, F. Blesa, M. Rodriguez. (2012). *Algorithm 924: TIDES, a Taylor Series Integrator for Differential EquationS*. ACM Transactions on Mathematical Software, Vol. 39, No. 1, Article 5.