An Order Hyperresolution Calculus for Gödel Logic with Truth Constants

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Abstract: In (Guller, 2012), we have generalised the well-known hyperresolution principle to the first-order Gödel logic for the general case. This paper is a continuation of our work. We propose a modification of the hyperresolution calculus suitable for automated deduction with explicit partial truth. We expand the first-order Gödel logic by a countable set of intermediate truth constants $\bar{c}$, $c \in (0,1)$. Our approach is based on translation of a formula to an equivalent satisfiable finite order clausal theory, consisting of order clauses. An order clause is a finite set of order literals of the form $\varepsilon_1 \diamond \varepsilon_2$ where $\diamond$ is a connective either $\eqcirc$ or $\prec$. $\eqcirc$ and $\prec$ are interpreted by the equality and standard strict linear order on $[0,1]$, respectively. We shall investigate the so-called canonical standard completeness, where the semantics of the first-order Gödel logic is given by the standard *G*-algebra and truth constants are interpreted by themselves. The modified hyperresolution calculus is refutation sound and complete for a countable order clausal theory under a certain condition for suprema and infima of sets of the truth constants occurring in the theory.

1 INTRODUCTION

Current research in many-valued logics is mainly concerned with left-continuous t-norm based logics including the three fundamental fuzzy logics: Gödel, Łukasiewicz, and Product ones. From a syntactical point of view, classical many-valued deduction calculi are widely studied, especially Hilbert-style ones. In addition, a perspective from automated deduction has received attractivity during the last two decades. A considerable effort has been made in development of *SAT* solvers for the problem of Boolean satisfiability. *SAT* solvers may exploit either complete solution methods (called complete or systematic *SAT* solvers) or incomplete or hybrid ones. Complete *SAT* solvers are mostly based on the Davis-Putnam-Logemann-Loveland procedure (*DPLL*) (Davis and Putnam, 1960; Davis et al., 1962) or resolution proof methods (Robinson, 1965b; Robinson, 1965a; Gallier, 1985), improved by various features, (Biere et al., 2009). t-norm based logics are logics of comparative truth: the residuum of a t-norm satisfies, for all $x,y \in [0,1]$, $x \rightarrow y = 1$ if and only if $x \leq y$. Since implication is interpreted by a residuum, in the propositional case, a formula of the form $\phi \rightarrow \psi$ is a consequence of a theory if $\|\phi\|^{\mathfrak{A}} \leq \|\psi\|^{\mathfrak{A}}$ for every model $\mathfrak{A}$ of the theory. Most explorations of t-norm based logics are focused on tautologies and deduction calculi with the only distinguished truth degree 1, (Hájek, 2001). However, in many real-world applications, one may be interested in representation and inference with explicit partial truth; besides the truth constants *0*, *1*, intermediate truth constants are involved in. In the literature, two main approaches to expansions with truth constants, are described. Historically, first one has been introduced in (Pavelka, 1979), where the propositional Łukasiewicz logic is augmented by truth constants $\bar{r}$, $r \in [0,1]$, Pavelka's logic (*PL*). A formula of the form $\bar{r} \rightarrow \phi$ evaluated to 1 expresses that the truth value of ϕ is greater than or equal to r. In (Novák et al., 1999), further development of evaluated formulae, and in (Hájek, 2001), Rational Pavelka's logic (*RPL*) - a simplification of *PL*, are described. Another approach relies on traditional algebraic semantics. Various completeness results for expansions of t-norm based logics with countably many truth constants are investigated, among others, in (Esteva et al., 2001; Savický et al., 2006; Esteva et al., 2007b; Esteva et al., 2007a; Esteva et al., 2009; Esteva et al., 2010a; Esteva et al., 2010b).

Concerning the three fundamental first-order fuzzy logics, the set of logically valid formulae is Π_2-complete for Łukasiewicz logic, Π_2-hard for Product

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logic, and Σ_1-complete for Gödel logic, as with classical first-order logic. Among these fuzzy logics, only Gödel logic is recursively axiomatisable. Hence, it was necessary to provide a proof method suitable for automated deduction, as one has done for classical logic. In contrast to classical logic, we cannot make shifts of quantifiers arbitrarily and translate a formula to an equivalent (satisfiable) prenex form. In (Guller, 2012), we have generalised the well-known hyperresolution principle to the first-order Gödel logic for the general case. Our approach is based on translation of a formula of Gödel logic to an equivalent satisfiable finite order clausal theory, consisting of order clauses. We have introduced a notion of quantified atom: a formula a is a quantified atom if $a = Qx\, p(t_0,\dots,t_\tau)$ where Q is a quantifier ($\forall$, $\exists$); $p(t_0,\dots,t_\tau)$ is an atom; x is a variable occurring in $p(t_0,\dots,t_\tau)$; for all $i \leq \tau$, either $t_i = x$ or x does not occur in t_i. An order clause is a finite set of order literals of the form $\varepsilon_1 \diamond \varepsilon_2$ where ε_i is either an atom or a quantified atom; and $\diamond$ is a connective either $\eqcirc$ or $\prec$. $\eqcirc$ and $\prec$ are interpreted by the equality and standard strict linear order on $[0,1]$, respectively. For an input theory of Gödel logic, the proposed translation produces a so-called admissible order clausal theory. On the basis of the hyperresolution principle, a calculus operating over admissible order clausal theories, has been devised. The calculus is proved to be refutation sound and complete for the countable case with respect to the standard G-algebra $G = ([0,1],\leq,\vee,\wedge,\Rightarrow,\overline{},\eqcirc,\prec,0,1)$ augmented by binary operators $\eqcirc$ and $\prec$ for $\eqcirc$ and $\prec$, respectively, cf. Section 2. As another step, one may incorporate a countable set of intermediate truth constants $\bar{c}$, $c \in (0,1)$, to get a modification of our hyperresolution calculus suitable for automated deduction with explicit partial truth. We shall investigate the so-called canonical standard completeness, where the semantics of the first-order Gödel logic is given by the standard G-algebra G and truth constants are interpreted by themselves. Note that the Hilbert-style calculus for the first-order Gödel logic introduced in (Hájek, 2001) is not suitable for expansion with truth constants. We have $\phi \vdash \psi$ if and only if $\phi \models \psi$ (wrt. G). However, that cannot be preserved after adding truth constants. Let $c \in (0,1)$ and a be an atom different from a constant. Then $\bar{c} \models a$ ($\bar{c}$ is unsatisfiable) but $\not\models \bar{c} \to a$, $\not\vdash \bar{c} \to a$, $\bar{c} \not\vdash a$ (from the soundness and the deduction-detachment theorem for this calculus). So, we cannot achieve a strict canonical standard completeness after expansion with truth constants. On the other side, such a completeness can be feasible for our hyperresolution calculus under certain condition. We say that a set X of truth constants is admissible with respect to suprema and infima if, for all $Y_1, Y_2 \subseteq X \cup \{\mathit{0},\mathit{1}\}$ and $\bigvee Y_1 = \bigwedge Y_2$, either $\bigvee Y_1 \in Y_1$, $\bigwedge Y_2 \in Y_2$, or $\bigvee Y_1 \notin Y_1$, $\bigwedge Y_2 \notin Y_2$ (constants are interpreted by themselves). Then the hyperresolution calculus is refutation sound and complete for a countable order clausal theory if the set of all truth constants occurring in the theory is admissible with respect to suprema and infima. This condition obviously covers the case of finite order clausal theories.

The paper is organised as follows. Section 2 gives the basic notions and notation concerning the first-order Gödel logic. Section 3 deals with clause form translation. In Section 4, we propose a hyperresolution calculus with truth constants and prove its refutational soundness, completeness. Section 5 brings conclusions.

2 FIRST-ORDER GÖDEL LOGIC

By L we denote a first-order language. Var_L | $Func_L$ | $Pred_L$ | $Term_L$ | $GTerm_L$ | $Atom_L$ | $GAtom_L$ denotes the set of all variables | function symbols | predicate symbols | terms | ground terms | atoms | ground atoms of L. $ar_L : Func_L \cup Pred_L \longrightarrow \mathbb{N}$ denotes the mapping assigning an arity to every function and predicate symbol of L. We assume nullary predicate symbols $\mathit{0},\mathit{1} \in Pred_L$, $ar_L(\mathit{0}) = ar_L(\mathit{1}) = 0$; $\mathit{0}$ denotes the false and $\mathit{1}$ the true in L. In addition, we assume a countable set of nullary predicate symbols $\overline{C}_L = \{\bar{c} \mid \bar{c} \in Pred_L, ar_L(\bar{c}) = 0, c \in (0,1)\} \subseteq Pred_L$. $\mathit{0}$, $\mathit{1}$, $\bar{c} \in \overline{C}_L$ are called truth constants. We denote $Tcons_L = \{\mathit{0},\mathit{1}\} \cup \overline{C}_L \subseteq Pred_L$. Let $X \subseteq Tcons_L$. We denote $\overline{X} = \{0 \mid \mathit{0} \in X\} \cup \{1 \mid \mathit{1} \in X\} \cup \{c \mid \bar{c} \in \overline{C}_L\} \subseteq [0,1]$. By $Form_L$ we designate the set of all formulae of L built up from $Atom_L$ and Var_L using the connectives: $\neg$, negation, $\wedge$, conjunction, $\vee$, disjunction, $\to$, implication, and the quantifiers: $\forall$, the universal quantifier, $\exists$, the existential one. In addition, we introduce new binary connectives $\eqcirc$, equality, and $\prec$, strict order. We denote $Con = \{\neg,\wedge,\vee,\to,\eqcirc,\prec\}$. By $OrdForm_L$ we designate the set of all so-called order formulae of L built up from $Atom_L$ and Var_L using the connectives in Con and the quantifiers: $\forall$, $\exists$.[1] Note that $OrdForm_L \supseteq Form_L$. In the paper, we shall assume that L is a countable first-order language; hence, all the above mentioned sets of symbols and expressions are countable. Let ε | ε_i, $1 \leq i \leq m$ | υ_i, $1 \leq i \leq n$, be either an expression or a set of expressions or a set of sets of expressions of L, in general. By $vars(\varepsilon_1,\dots,\varepsilon_m) \subseteq Var_L$ | $freevars(\varepsilon_1,\dots,\varepsilon_m) \subseteq Var_L$ | $boundvars(\varepsilon_1,\dots,\varepsilon_m) \subseteq Var_L$ |

[1] We assume a decreasing connective and quantifier precedence: $\forall$, $\exists$, $\neg$, $\wedge$, $\to$, $\eqcirc$, $\prec$, $\vee$.

$preds(\varepsilon_1, \dots, \varepsilon_m) \subseteq Pred_{\mathcal{L}} \mid atoms(\varepsilon_1, \dots, \varepsilon_m) \subseteq Atom_{\mathcal{L}}$ we denote the set of all variables | free variables | bound variables | predicate symbols | atoms of $\mathcal{L}$ occurring in $\varepsilon_1, \dots, \varepsilon_m$. ε is closed iff $freevars(\varepsilon) = \emptyset$. By ℓ we denote the empty sequence. By $|\varepsilon_1, \dots, \varepsilon_m| = m$ we denote the length of a sequence $\varepsilon_1, \dots, \varepsilon_m$. We define the concatenation of sequences $\varepsilon_1, \dots, \varepsilon_m$ and $\upsilon_1, \dots, \upsilon_n$ as $(\varepsilon_1, \dots, \varepsilon_m), (\upsilon_1, \dots, \upsilon_n) = \varepsilon_1, \dots, \varepsilon_m, \upsilon_1, \dots, \upsilon_n$. Note that concatenation of sequences is associative.

Let X, Y, Z be sets, $Z \subseteq X$; $f : X \longrightarrow Y$ be a mapping. By $\|X\|$ we denote the set-theoretic cardinality of X. X being a finite subset of Y is denoted as $X \subseteq_{\mathcal{F}} Y$. We designate $\mathcal{P}(X) = \{x \mid x \subseteq X\}$; $\mathcal{P}(X)$ is the power set of X; $\mathcal{P}_{\mathcal{F}}(X) = \{x \mid x \subseteq_{\mathcal{F}} X\}$; $\mathcal{P}_{\mathcal{F}}(X)$ is the set of all finite subsets of X; $f[Z] = \{f(z) \mid z \in Z\}$; $f[Z]$ is the image of Z under f; $f|_Z = \{(z, f(z)) \mid z \in Z\}$; $f|_Z$ is the restriction of f onto Z. Let $\gamma \leq \omega$. A sequence δ of X is a bijection $\delta : \gamma \longrightarrow X$. X is countable if and only if there exists a sequence of X. Let I be a set and $S_i \neq \emptyset$, $i \in I$, be sets. A selector $\mathcal{S}$ over $\{S_i \mid i \in I\}$ is a mapping $\mathcal{S} : I \longrightarrow \bigcup\{S_i \mid i \in I\}$ such that for all $i \in I$, $\mathcal{S}(i) \in S_i$. We denote $\mathcal{S}el(\{S_i \mid i \in I\}) = \{\mathcal{S} \mid \mathcal{S} \textit{ is a selector over } \{S_i \mid i \in I\}\}$. Let $c \in \mathbb{R}^+$. $\log c$ denotes the binary logarithm of c. Let $f, g : \mathbb{N} \longrightarrow \mathbb{R}_0^+$. f is of the order of g, in symbols $f \in O(g)$, iff there exist $n_0 \in \mathbb{N}$ and $c^* \in \mathbb{R}_0^+$ such that for all $n \geq n_0$, $f(n) \leq c^* \cdot g(n)$.

Let $t \in Term_{\mathcal{L}}$, $\phi \in OrdForm_{\mathcal{L}}$, $T \subseteq_{\mathcal{F}} OrdForm_{\mathcal{L}}$. The size of $t \mid \phi$, in symbols $|t| \mid |\phi|$, is defined as the number of nodes of its standard tree representation. We define the size of T as $|T| = \sum_{\phi \in T} |\phi|$. By $varseq(\phi)$, $vars(varseq(\phi)) \subseteq Var_{\mathcal{L}}$, we denote the sequence of all variables of $\mathcal{L}$ occurring in ϕ which is built up via the left-right preorder traversal of ϕ. For example, $varseq(\exists w (\forall x\, p(x,x,z) \vee \exists y\, q(x,y,z))) = w,x,x,x,z,y,x,y,z$ and $|w,x,x,x,z,y,x,y,z| = 9$. Let $Q \in \{\forall, \exists\}$ and $\bar{x} = x_1, \dots, x_n$ be a sequence of variables of $\mathcal{L}$. By $Q\bar{x}\phi$ we denote $Qx_1 \dots Qx_n \phi$.

Gödel logic is interpreted by the standard G-algebra augmented by binary operators $\eqcirc$ and $\prec$ for $\eqcirc$ and $\prec$, respectively.

$$G = ([0,1], \leq, \vee, \wedge, \Rightarrow, \overline{}, \eqcirc, \prec, 0, 1)$$

where $\vee \mid \wedge$ denotes the supremum | infimum operator on $[0,1]$;

$$a \Rightarrow b = \begin{cases} 1 \textit{ if } a \leq b, \\ b \textit{ else}; \end{cases} \qquad \overline{a} = \begin{cases} 1 \textit{ if } a = 0, \\ 0 \textit{ else}; \end{cases}$$

$$a \eqcirc b = \begin{cases} 1 \textit{ if } a = b, \\ 0 \textit{ else}; \end{cases} \qquad a \prec b = \begin{cases} 1 \textit{ if } a < b, \\ 0 \textit{ else}. \end{cases}$$

We recall that G is a complete linearly ordered lattice algebra; $\vee \mid \wedge$ is commutative, associative, idempotent, monotone; $0 \mid 1$ is its neutral element; the residuum operator $\Rightarrow$ of $\wedge$ satisfies the condition of residuation:

$$\text{for all } a,b,c \in G, a \wedge b \leq c \Longleftrightarrow a \leq b \Rightarrow c; \tag{1}$$

Gödel negation $\overline{}$ satisfies the condition:

$$\text{for all } a \in G, \overline{a} = a \Rightarrow 0; \tag{2}$$

the following properties, which will be exploited later, hold:[2]

for all $a, b, c \in G$,

$$a \vee b \wedge c = (a \vee b) \wedge (a \vee c), \quad \text{(distributivity of } \vee \text{ over } \wedge) \tag{3}$$

$$a \wedge (b \vee c) = a \wedge b \vee a \wedge c, \quad \text{(distributivity of } \wedge \text{ over } \vee) \tag{4}$$

$$a \Rightarrow (b \vee c) = a \Rightarrow b \vee a \Rightarrow c, \tag{5}$$

$$a \Rightarrow b \wedge c = (a \Rightarrow b) \wedge (a \Rightarrow c), \tag{6}$$

$$(a \vee b) \Rightarrow c = (a \Rightarrow c) \wedge (b \Rightarrow c), \tag{7}$$

$$a \wedge b \Rightarrow c = a \Rightarrow c \vee b \Rightarrow c, \tag{8}$$

$$a \Rightarrow (b \Rightarrow c) = a \wedge b \Rightarrow c, \tag{9}$$

$$((a \Rightarrow b) \Rightarrow b) \Rightarrow b = a \Rightarrow b, \tag{10}$$

$$(a \Rightarrow b) \Rightarrow c = ((a \Rightarrow b) \Rightarrow b) \wedge (b \Rightarrow c) \vee c, \tag{11}$$

$$(a \Rightarrow b) \Rightarrow 0 = ((a \Rightarrow 0) \Rightarrow 0) \wedge (b \Rightarrow 0). \tag{12}$$

An interpretation I for $\mathcal{L}$ is a triple $\big(\mathcal{U}_I, \{f^I \mid f \in Func_{\mathcal{L}}\}, \{p^I \mid p \in Pred_{\mathcal{L}}\}\big)$ defined as follows: $\mathcal{U}_I \neq \emptyset$ is the universum of I; every $f \in Func_{\mathcal{L}}$ is interpreted as a function $f^I : \mathcal{U}_I^{ar(f)} \longrightarrow \mathcal{U}_I$; every $p \in Pred_{\mathcal{L}}$ is interpreted as a $[0,1]$-relation $p^I : \mathcal{U}_I^{ar(p)} \longrightarrow [0,1]$; particularly, $0^I = 0$, $1^I = 1$, for all $\bar{c} \in \overline{C}_{\mathcal{L}}$, $\bar{c}^I = c$. A variable assignment in I is a mapping $Var_{\mathcal{L}} \longrightarrow \mathcal{U}_I$. We denote the set of all variable assignments in I as $\mathcal{S}_I$. Let $e \in \mathcal{S}_I$ and $u \in \mathcal{U}_I$. A variant $e[x/u] \in \mathcal{S}_I$ of e with respect to x and u is defined as

$$e[x/u](z) = \begin{cases} u & \textit{if } z = x, \\ e(z) & \textit{else}. \end{cases}$$

Let $t \in Term_{\mathcal{L}}$, $\bar{x}$ be a sequence of variables of $\mathcal{L}$, $\phi \in OrdForm_{\mathcal{L}}$. In I with respect to e, we define the value $\|t\|_e^I \in \mathcal{U}_I$ of t by recursion on the structure of t, the value $\|\bar{x}\|_e^I \in \mathcal{U}_I^{|\bar{x}|}$ of $\bar{x}$, the truth value $\|\phi\|_e^I \in [0,1]$ of ϕ by recursion on the structure of ϕ, as usual. Let ϕ be closed. Then, for all $e, e' \in \mathcal{S}_I$, $\|\phi\|_e^I = \|\phi\|_{e'}^I$. Let $e \in \mathcal{S}_I \neq \emptyset$. We denote $\|\phi\|^I = \|\phi\|_e^I$.

Let $\mathcal{L} \mid \mathcal{L}'$ be a first-order language and $I \mid I'$ be an interpretation for $\mathcal{L} \mid \mathcal{L}'$. $\mathcal{L}'$ is an expansion of $\mathcal{L}$

[2] We assume a decreasing operator precedence: $\overline{}$, $\wedge$, $\Rightarrow$, $\eqcirc$, $\prec$, $\vee$.

iff $Func_{\mathcal{L}'} \supseteq Func_{\mathcal{L}}$ and $Pred_{\mathcal{L}'} \supseteq Pred_{\mathcal{L}}$; on the other side, we say $\mathcal{L}$ is a reduct of $\mathcal{L}'$. I' is an expansion of I to $\mathcal{L}'$ iff $\mathcal{L}'$ is an expansion of $\mathcal{L}$, $\mathcal{U}_{I'} = \mathcal{U}_I$, for all $f \in Func_{\mathcal{L}}$, $f^{I'} = f^I$, for all $p \in Pred_{\mathcal{L}}$, $p^{I'} = p^I$; on the other side, we say I is a reduct of I' to $\mathcal{L}$, in symbols $I = I'|_{\mathcal{L}}$.

A theory of $\mathcal{L}$ is a set of formulae of $\mathcal{L}$. An order theory of $\mathcal{L}$ is a set of order formulae of $\mathcal{L}$. Let $\phi, \phi' \in OrdForm_{\mathcal{L}}$, $T \subseteq OrdForm_{\mathcal{L}}$, $e \in \mathcal{S}_I$. ϕ is true in I with respect to e, written as $I \models_e \phi$, iff $\|\phi\|^I_e = 1$. I is a model of ϕ, in symbols $I \models \phi$, iff, for all $e \in \mathcal{S}_I$, $I \models_e \phi$. I is a model of T, in symbols $I \models T$, iff, for all $\phi \in T$, $I \models \phi$. ϕ is a logically valid formula iff, for every interpretation I for $\mathcal{L}$, $I \models \phi$. ϕ is equivalent to ϕ', in symbols $\phi \equiv \phi'$, iff, for every interpretation I for $\mathcal{L}$ and $e \in \mathcal{S}_I$, $\|\phi\|^I_e = \|\phi'\|^I_e$. We denote $tcons(\phi) = \{0,1\} \cup (preds(\phi) \cap \overline{C}_{\mathcal{L}}) \subseteq Tcons_{\mathcal{L}}$ and $tcons(T) = \{0,1\} \cup (preds(T) \cap \overline{C}_{\mathcal{L}}) \subseteq Tcons_{\mathcal{L}}$.

3 TRANSLATION TO CLAUSAL FORM

In the propositional case (Guller, 2010), we have proposed some translation of a formula to an equivalent *CNF* containing literals of the form either a or $a \rightarrow b$ or $(a \rightarrow b) \rightarrow b$ where a is a propositional atom and b is either a propositional atom or the propositional constant 0. An output equivalent *CNF* may be of exponential size with respect to the input formula; we had laid no restrictions on use of the distributivity law (3) during translation to conjunctive normal form. To avoid this disadvantage, we have devised translation to *CNF* via interpolation using new atoms, which produces an output *CNF* of linear size at the cost of being only equisatisfiable to the input formula. A similar approach exploiting the renaming subformulae technique can be found in (Plaisted and Greenbaum, 1986; de la Tour, 1992; Hähnle, 1994; Nonnengart et al., 1998). A *CNF* is further translated to a finite set of order clauses. An order clause is a finite set of order literals of the form $\varepsilon_1 \diamond \varepsilon_2$ where ε_i is either a propositional atom or a propositional constant, 0, 1, and $\diamond \in \{\eqcirc, \prec\}$.

We have described some generalisation of the mentioned translation to the first-order case in (Guller, 2012). At first, we recall the notion of quantified atom. Let $a \in Form_{\mathcal{L}}$. a is a quantified atom of $\mathcal{L}$ iff $a = Qx\, p(t_0, \dots, t_\tau)$ where $p(t_0, \dots, t_\tau) \in Atom_{\mathcal{L}}$, $x \in vars(p(t_0, \dots, t_\tau))$, either $t_i = x$ or $x \notin vars(t_i)$. $QAtom_{\mathcal{L}} \subseteq Form_{\mathcal{L}}$ denotes the set of all quantified atoms of $\mathcal{L}$. $QAtom^Q_{\mathcal{L}} \subseteq QAtom_{\mathcal{L}}$, $Q \in \{\forall, \exists\}$, denotes the set of all quantified atoms of $\mathcal{L}$ of the form $Qx\,a$. Let $\varepsilon \mid \varepsilon_i,\ 1 \le i \le m \mid \upsilon_i,\ 1 \le i \le n$, be either an expression or a set of expressions or a set of sets of expressions of $\mathcal{L}$, in general. By $qatoms(\varepsilon_1, \dots, \varepsilon_m) \subseteq QAtom_{\mathcal{L}}$ we denote the set of all quantified atoms of $\mathcal{L}$ occurring in $\varepsilon_1, \dots, \varepsilon_m$. We denote $qatoms^Q(\varepsilon_1, \dots, \varepsilon_m) = qatoms(\varepsilon_1, \dots, \varepsilon_m) \cap QAtom^Q_{\mathcal{L}}$, $Q \in \{\forall, \exists\}$. Let $Qx\, p(t_0, \dots, t_\tau) \in QAtom_{\mathcal{L}}$ and $p(t'_0, \dots, t'_\tau) \in Atom_{\mathcal{L}}$. Let $I = \{i \,|\, i \le \tau, x \notin vars(t_i)\}$ and $r_1, \dots, r_k$, $r_i \le \tau$, $k \le \tau$, for all $1 \le i < i' \le k$, $r_i < r_{i'}$, be a sequence such that $\{r_i \,|\, 1 \le i \le k\} = I$. We denote

$$freetermseq(Qx\, p(t_0, \dots, t_\tau)) = t_{r_1}, \dots, t_{r_k},$$
$$freetermseq(p(t'_0, \dots, t'_\tau)) = t'_0, \dots, t'_\tau.$$

We further introduce conjunctive normal form (*CNF*) in Gödel logic. In contrast to two-valued logic, we have to consider an augmented set of literals appearing in *CNF* formulae. Let $l, \phi \in Form_{\mathcal{L}}$. l is a literal of $\mathcal{L}$ iff either $l = a$ or $l = b \rightarrow c$ or $l = (a \rightarrow d) \rightarrow d$ or $l = a \rightarrow e$ or $l = e \rightarrow a$, $a \in Atom_{\mathcal{L}} - Tcons_{\mathcal{L}}$, $b \in Atom_{\mathcal{L}} - \{0,1\}$, $c \in Atom_{\mathcal{L}} - \{1\}$, $d \in (Atom_{\mathcal{L}} - Tcons_{\mathcal{L}}) \cup \{0\}$, $e \in QAtom_{\mathcal{L}}$, $\{b, c\} \not\subseteq Tcons_{\mathcal{L}}$. The set of all literals of $\mathcal{L}$ is designated as $Lit_{\mathcal{L}} \subseteq Form_{\mathcal{L}}$. ϕ is a conjunctive $\mid$ disjunctive normal form of $\mathcal{L}$, in symbols *CNF* $\mid$ *DNF*, iff either $\phi \in Tcons_{\mathcal{L}}$ or $\phi = \bigwedge_{i \le n} \bigvee_{j \le m_i} l^i_j \mid \phi = \bigvee_{i \le n} \bigwedge_{j \le m_i} l^i_j$, $l^i_j \in Lit_{\mathcal{L}}$. Let $D = l_1 \vee \cdots \vee l_n \in Form_{\mathcal{L}}$, $l_i \in Lit_{\mathcal{L}}$. We denote $lits(D) = \{l_1, \dots, l_n\} \subseteq Lit_{\mathcal{L}}$. D is a factor iff, for all $1 \le i < i' \le n$, $l_i \ne l_{i'}$.

We finally introduce order clauses in Gödel logic. Let $l \in OrdForm_{\mathcal{L}}$. l is an order literal of $\mathcal{L}$ iff $l = \varepsilon_1 \diamond \varepsilon_2$, $\varepsilon_i \in Atom_{\mathcal{L}} \cup QAtom_{\mathcal{L}}$, $\diamond \in \{\eqcirc, \prec\}$. The set of all order literals of $\mathcal{L}$ is designated as $OrdLit_{\mathcal{L}} \subseteq OrdForm_{\mathcal{L}}$. An order clause of $\mathcal{L}$ is a finite set of order literals of $\mathcal{L}$; since $=$ is commutative, we identify, for all $\varepsilon_1 \eqcirc \varepsilon_2 \in OrdLit_{\mathcal{L}}$, $\varepsilon_1 \eqcirc \varepsilon_2$ and $\varepsilon_2 \eqcirc \varepsilon_1 \in OrdLit_{\mathcal{L}}$ with respect to order clauses. An order clause $\{l_1, \dots, l_n\}$ is written in the form $l_1 \vee \cdots \vee l_n$. The order clause $\emptyset$ is called the empty order clause and denoted as $\square$. An order clause $\{l\}$ is called a unit order clause and denoted as l; if it does not cause the ambiguity with the denotation of the single order literal l in given context. We designate the set of all order clauses of $\mathcal{L}$ as $OrdCl_{\mathcal{L}}$. Let $l, l_0, \dots, l_n \in OrdLit_{\mathcal{L}}$ and $C, C' \in OrdCl_{\mathcal{L}}$. We define the size of C as $|C| = \sum_{l \in C} |l|$. By $l \vee C$ we denote $\{l\} \cup C$ where $l \notin C$. Analogously, by $l_0 \vee \cdots \vee l_n \vee C$ we denote $\{l_0\} \cup \cdots \cup \{l_n\} \cup C$ where, for all $i, i' \le n$, $i \ne i'$, $l_i \notin C$ and $l_i \ne l_{i'}$. By $C \vee C'$ we denote $C \cup C'$. C is a subclause of C', in symbols $C \sqsubseteq C'$, iff $C \subseteq C'$. An order clausal theory of $\mathcal{L}$ is a set of order clauses of $\mathcal{L}$. A unit order clausal theory is a set of unit order clauses.

Let $\phi, \phi' \in OrdForm_{\mathcal{L}}$, $T, T' \subseteq OrdForm_{\mathcal{L}}$, $S, S' \subseteq OrdCl_{\mathcal{L}}$, I be an interpretation for $\mathcal{L}$, $e \in \mathcal{S}_I$. Note that

$I \models_e l$ if and only if either $l = \varepsilon_1 \eqcirc \varepsilon_2$, $\|\varepsilon_1 \eqcirc \varepsilon_2\|_e^I = 1$, $\|\varepsilon_1\|_e^I = \|\varepsilon_2\|_e^I$; or $l = \varepsilon_1 \prec \varepsilon_2$, $\|\varepsilon_1 \prec \varepsilon_2\|_e^I = 1$, $\|\varepsilon_1\|_e^I < \|\varepsilon_2\|_e^I$. C is true in I with respect to e, written as $I \models_e C$, iff there exists $l^* \in C$ such that $I \models_e l^*$. I is a model of C, in symbols $I \models C$, iff, for all $e \in \mathcal{S}_I$, $I \models_e C$. I is a model of S, in symbols $I \models S$, iff, for all $C \in S$, $I \models C$. $\phi' \mid T' \mid C' \mid S'$ is a logical consequence of $\phi \mid T \mid C \mid S$, in symbols $\phi \mid T \mid C \mid S \models \phi' \mid T' \mid C' \mid S'$, iff, for every model I of $\phi \mid T \mid C \mid S$ for $\mathcal{L}$, $I \models \phi' \mid T' \mid C' \mid S'$. $\phi \mid T \mid C \mid S$ is satisfiable iff there exists a model of $\phi \mid T \mid C \mid S$ for $\mathcal{L}$. Note that both $\square$ and $\square \in S$ are unsatisfiable. $\phi \mid T \mid C \mid S$ is equisatisfiable to $\phi' \mid T' \mid C' \mid S'$ iff $\phi \mid T \mid C \mid S$ is satisfiable if and only if $\phi' \mid T' \mid C' \mid S'$ is satisfiable. We denote $tcons(S) = \{0,1\} \cup (preds(S) \cap \overline{C}_{\mathcal{L}}) \subseteq Tcons_{\mathcal{L}}$. Let $S \subseteq_{\mathcal{F}} OrdCl_{\mathcal{L}}$. We define the size of S as $|S| = \sum_{C \in S} |C|$. l is a simplified order literal of $\mathcal{L}$ iff $l = \varepsilon_1 \diamond \varepsilon_2$, $\{\varepsilon_1, \varepsilon_2\} \not\subseteq Tcons_{\mathcal{L}}$, $\{\varepsilon_1, \varepsilon_2\} \not\subseteq QAtom_{\mathcal{L}}$. The set of all simplified order literals of $\mathcal{L}$ is designated as $SimOrdLit_{\mathcal{L}} \subseteq OrdLit_{\mathcal{L}}$. We denote $SimOrdCl_{\mathcal{L}} = \{C \mid C \in OrdCl_{\mathcal{L}}, C \subseteq SimOrdLit_{\mathcal{L}}\} \subseteq OrdCl_{\mathcal{L}}$. Let $\tilde{f}_0 \notin Func_{\mathcal{L}}$; $\tilde{f}_0$ is a new function symbol. Let $\mathbb{I} = \mathbb{N} \times \mathbb{N}$; $\mathbb{I}$ is an infinite countable set of indices. Let $\tilde{\mathbb{P}} = \{\tilde{p}_{\mathbb{i}} \mid \mathbb{i} \in \mathbb{I}\}$ such that $\tilde{\mathbb{P}} \cap Pred_{\mathcal{L}} = \emptyset$; $\tilde{\mathbb{P}}$ is an infinite countable set of new predicate symbols.

From a computational point of view, the worst case time and space complexity will be estimated using the logarithmic cost measurement. Let $\mathcal{A}$ be an algorithm. $\#O_{\mathcal{A}}(In) \geq 1$ denotes the number of all elementary operations executed by $\mathcal{A}$ on an input In.

3.1 Substitutions

We assume the reader to be familiar with the standard notions and notation of substitutions. We introduce a few definitions and denotations; some of them are slightly different from the standard ones, but found to be more convenient. Let $X = \{x_i \mid 1 \leq i \leq n\} \subseteq Var_{\mathcal{L}}$. A substitution ϑ of $\mathcal{L}$ is a mapping $\vartheta : X \longrightarrow Term_{\mathcal{L}}$. ϑ may be written in the form $x_1/\vartheta(x_1), \ldots, x_n/\vartheta(x_n)$. We denote $dom(\vartheta) = X \subseteq_{\mathcal{F}} Var_{\mathcal{L}}$ and $range(\vartheta) = \bigcup_{x \in X} vars(\vartheta(x)) \subseteq_{\mathcal{F}} Var_{\mathcal{L}}$. The set of all substitutions of $\mathcal{L}$ is designated as $Subst_{\mathcal{L}}$. Let $\vartheta, \vartheta' \in Subst_{\mathcal{L}}$. ϑ is a variable renaming of $\mathcal{L}$ iff $\vartheta : dom(\vartheta) \longrightarrow Var_{\mathcal{L}}$, and for all $x, x' \in dom(\vartheta)$, $x \neq x'$, $\vartheta(x) \neq \vartheta(x')$. We define $id_{\mathcal{L}} : Var_{\mathcal{L}} \longrightarrow Var_{\mathcal{L}}$, $id_{\mathcal{L}}(x) = x$. Let $t \in Term_{\mathcal{L}}$. ϑ is applicable to t iff $dom(\vartheta) \supseteq vars(t)$. Let ϑ be applicable to t. We define the application $t\vartheta \in Term_{\mathcal{L}}$ of ϑ to t by recursion on the structure of t in the standard manner. Let $range(\vartheta) \subseteq dom(\vartheta')$. We define the composition of ϑ and ϑ' as $\vartheta \circ \vartheta' : dom(\vartheta) \longrightarrow Term_{\mathcal{L}}$, $\vartheta \circ \vartheta'(x) = \vartheta(x)\vartheta'$, $\vartheta \circ \vartheta' \in Subst_{\mathcal{L}}$, $dom(\vartheta \circ \vartheta') = dom(\vartheta)$, $range(\vartheta \circ \vartheta') = range(\vartheta'|_{range(\vartheta)})$. Note that composition of substitutions is associative. ϑ' is a regular extension of ϑ iff $dom(\vartheta') \supseteq dom(\vartheta)$, $\vartheta'|_{dom(\vartheta)} = \vartheta$, $\vartheta'|_{dom(\vartheta') - dom(\vartheta)}$ is a variable renaming such that $range(\vartheta'|_{dom(\vartheta') - dom(\vartheta)}) \cap range(\vartheta) = \emptyset$. Let $a \in Atom_{\mathcal{L}}$. ϑ is applicable to a iff $dom(\vartheta) \supseteq vars(a)$. Let ϑ be applicable to a and $a = p(t_1, \ldots, t_\tau)$. We define the application of ϑ to a as $a\vartheta = p(t_1\vartheta, \ldots, t_\tau\vartheta) \in Atom_{\mathcal{L}}$. Let $Qx\,a \in QAtom_{\mathcal{L}}$. ϑ is applicable to $Qx\,a$ iff $dom(\vartheta) \supseteq freevars(Qx\,a)$ and $x \notin range(\vartheta|_{freevars(Qx\,a)})$. Let ϑ be applicable to $Qx\,a$. We define the application of ϑ to $Qx\,a$ as $(Qx\,a)\vartheta = Qx\,a(\vartheta|_{freevars(Qx\,a)} \cup x/x) \in QAtom_{\mathcal{L}}$. Let $\varepsilon_1 \diamond \varepsilon_2 \in OrdLit_{\mathcal{L}}$. ϑ is applicable to $\varepsilon_1 \diamond \varepsilon_2$ iff, for both i, ϑ is applicable to ε_i. Let ϑ be applicable to $\varepsilon_1 \diamond \varepsilon_2$. Then, for both i, ϑ is applicable to ε_i, $dom(\vartheta) \supseteq freevars(\varepsilon_i)$, $dom(\vartheta) \supseteq freevars(\varepsilon_1) \cup freevars(\varepsilon_2) = freevars(\varepsilon_1 \diamond \varepsilon_2)$. We define the application of ϑ to $\varepsilon_1 \diamond \varepsilon_2$ as $(\varepsilon_1 \diamond \varepsilon_2)\vartheta = \varepsilon_1\vartheta \diamond \varepsilon_2\vartheta \in OrdLit_{\mathcal{L}}$. Let $E \subseteq \mathsf{A}$, $\mathsf{A} = Term_{\mathcal{L}} \mid \mathsf{A} = Atom_{\mathcal{L}} \mid \mathsf{A} = QAtom_{\mathcal{L}} \mid \mathsf{A} = OrdLit_{\mathcal{L}}$. ϑ is applicable to E iff, for all $\varepsilon \in E$, ϑ is applicable to ε. Let ϑ be applicable to E. Then, for all $\varepsilon \in E$, ϑ is applicable to ε, $dom(\vartheta) \supseteq freevars(\varepsilon)$, $dom(\vartheta) \supseteq \bigcup_{\varepsilon \in E} freevars(\varepsilon) = freevars(E)$. We define the application of ϑ to E as $E\vartheta = \{\varepsilon\vartheta \mid \varepsilon \in E\} \subseteq \mathsf{A}$. Let $\varepsilon, \varepsilon' \in \mathsf{A} \mid \varepsilon, \varepsilon' \in OrdCl_{\mathcal{L}}$. ε' is an instance of ε of $\mathcal{L}$ iff there exists $\vartheta^* \in Subst_{\mathcal{L}}$ such that $\varepsilon' = \varepsilon\vartheta^*$. ε' is a variant of ε of $\mathcal{L}$ iff there exists a variable renaming $\rho^* \in Subst_{\mathcal{L}}$ such that $\varepsilon' = \varepsilon\rho^*$. Let $C \in OrdCl_{\mathcal{L}}$ and $S \subseteq OrdCl_{\mathcal{L}}$. C is an instance $\mid$ a variant of S of $\mathcal{L}$ iff there exists $C^* \in S$ such that C is an instance $\mid$ a variant of C^* of $\mathcal{L}$. We denote $Inst_{\mathcal{L}}(S) = \{C \mid C \textit{ is an instance of } S \textit{ of } \mathcal{L}\} \subseteq OrdCl_{\mathcal{L}}$ and $Vrnt_{\mathcal{L}}(S) = \{C \mid C \textit{ is a variant of } S \textit{ of } \mathcal{L}\} \subseteq OrdCl_{\mathcal{L}}$.

ϑ is a unifier of $\mathcal{L}$ for E iff $E\vartheta$ is a singleton set. Note that there does not exist a unifier for $\emptyset$. Let $\theta \in Subst_{\mathcal{L}}$. θ is a most general unifier of $\mathcal{L}$ for E iff θ is a unifier of $\mathcal{L}$ for E, and for every unifier ϑ of $\mathcal{L}$ for E, there exists $\gamma^* \in Subst_{\mathcal{L}}$ such that $\vartheta|_{freevars(E)} = \theta|_{freevars(E)} \circ \gamma^*$. By $mgu_{\mathcal{L}}(E) \subseteq Subst_{\mathcal{L}}$ we denote the set of all most general unifiers of $\mathcal{L}$ for E. Let $\overline{E} = E_0, \ldots, E_n$, $E_i \subseteq \mathsf{A}_i$, either $\mathsf{A}_i = Term_{\mathcal{L}}$ or $\mathsf{A}_i = Atom_{\mathcal{L}}$ or $\mathsf{A}_i = QAtom_{\mathcal{L}}$ or $\mathsf{A}_i = OrdLit_{\mathcal{L}}$. ϑ is applicable to $\overline{E}$ iff, for all $i \leq n$, ϑ is applicable to E_i. Let ϑ be applicable to $\overline{E}$. Then, for all $i \leq n$, ϑ is applicable to E_i, $dom(\vartheta) \supseteq freevars(E_i)$, $dom(\vartheta) \supseteq \bigcup_{i \leq n} freevars(E_i) = freevars(\overline{E})$. We define the application of ϑ to $\overline{E}$ as $\overline{E}\vartheta = E_0\vartheta, \ldots, E_n\vartheta$, $E_i\vartheta \subseteq \mathsf{A}_i$. ϑ is a unifier of $\mathcal{L}$ for $\overline{E}$ iff, for all $i \leq n$, ϑ is a unifier of $\mathcal{L}$ for E_i. Note that if there exists $i^* \leq n$ and $E_{i^*} = \emptyset$, then there does not exist a unifier for $\overline{E}$. θ is a most general unifier of $\mathcal{L}$ for $\overline{E}$ iff θ is a unifier of $\mathcal{L}$ for $\overline{E}$, and for every unifier ϑ of $\mathcal{L}$ for $\overline{E}$, there exists $\gamma^* \in Subst_{\mathcal{L}}$ such that $\vartheta|_{freevars(\overline{E})} = \theta|_{freevars(\overline{E})} \circ \gamma^*$.

By $mgu_{\mathcal{L}}(\overline{E}) \subseteq Subst_{\mathcal{L}}$ we denote the set of all most general unifiers of $\mathcal{L}$ for $\overline{E}$.

Theorem 3.1 (Unification Theorem)**.** *Let $\overline{E} = E_0, \dots, E_n$, either $E_i \subseteq_{\mathcal{F}} Term_{\mathcal{L}}$ or $E_i \subseteq_{\mathcal{F}} Atom_{\mathcal{L}}$. If there exists a unifier of $\mathcal{L}$ for $\overline{E}$, then there exists $\theta^* \in mgu_{\mathcal{L}}(\overline{E})$ such that $range(\theta^*|_{vars(\overline{E})}) \subseteq vars(\overline{E})$.*

Proof. By induction on $\|vars(\overline{E})\|$. □

Theorem 3.2 (Extended Unification Theorem)**.** *Let $\overline{E} = E_0, \dots, E_n$, either $E_i \subseteq_{\mathcal{F}} Term_{\mathcal{L}}$ or $E_i \subseteq_{\mathcal{F}} Atom_{\mathcal{L}}$ or $E_i \subseteq_{\mathcal{F}} QAtom_{\mathcal{L}}$ or $E_i \subseteq_{\mathcal{F}} OrdLit_{\mathcal{L}}$, and $boundvars(\overline{E}) \subseteq V \subseteq_{\mathcal{F}} Var_{\mathcal{L}}$. If there exists a unifier of $\mathcal{L}$ for $\overline{E}$, then there exists $\theta^* \in mgu_{\mathcal{L}}(\overline{E})$ such that $range(\theta^*|_{freevars(\overline{E})}) \cap V = \emptyset$.*

Proof. A straightforward consequence of Theorem 3.1. □

3.2 A Formal Treatment

Translation of a formula or a theory to *CNF* and clausal form, is based on the following lemma:

Lemma 3.3. *Let $n_\phi, n_0 \in \mathbb{N}$, $\phi \in Form_{\mathcal{L}}$, $T \subseteq Form_{\mathcal{L}}$.*

(I) *There exist either $J_\phi = \emptyset$ or $J_\phi = \{(n_\phi, j) \,|\, j \leq n_{J_\phi}\}$, $J_\phi \subseteq \{(n_\phi, j) \,|\, j \in \mathbb{N}\}$; a CNF $\psi \in Form_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_\phi\}}$, $S_\phi \subseteq_{\mathcal{F}} SimOrdCl_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_\phi\}}$ such that*

- (a) *$\|J_\phi\| \leq 2 \cdot |\phi|$;*
- (b) *there exists an interpretation $\mathfrak{A}$ for $\mathcal{L}$ and $\mathfrak{A} \models \phi$ if and only if there exists an interpretation $\mathfrak{A}'$ for $\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_\phi\}$ and $\mathfrak{A}' \models \psi$, satisfying $\mathfrak{A} = \mathfrak{A}'|_{\mathcal{L}}$;*
- (c) *there exists an interpretation $\mathfrak{A}$ for $\mathcal{L}$ and $\mathfrak{A} \models \phi$ if and only if there exists an interpretation $\mathfrak{A}'$ for $\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_\phi\}$ and $\mathfrak{A}' \models S_\phi$, satisfying $\mathfrak{A} = \mathfrak{A}'|_{\mathcal{L}}$;*
- (d) *$|\psi| \in O(|\phi|^2)$; the number of all elementary operations of the translation of ϕ to ψ, is in $O(|\phi|^2)$; the time and space complexity of the translation of ϕ to ψ, is in $O(|\phi|^2 \cdot (\log(1 + n_\phi) + \log|\phi|))$;*
- (e) *$|S_\phi| \in O(|\phi|^2)$; the number of all elementary operations of the translation of ϕ to S_ϕ, is in $O(|\phi|^2)$; the time and space complexity of the translation of ϕ to S_ϕ, is in $O(|\phi|^2 \cdot (\log(1 + n_\phi) + \log|\phi|))$;*
- (f) *for all $a \in qatoms(\psi)$, there exists $\mathbb{j}^* \in J_\phi$ and $preds(a) = \{\tilde{p}_{\mathbb{j}^*}\}$;*
- (g) *for all $\mathbb{j} \in J_\phi$, there exist a sequence $\overline{x}$ of variables of $\mathcal{L}$ and $\tilde{p}_{\mathbb{j}}(\overline{x}) \in atoms(\psi)$ satisfying, for all $a \in atoms(\psi)$ and $preds(a) = \{\tilde{p}_{\mathbb{j}}\}$, $a = \tilde{p}_{\mathbb{j}}(\overline{x})$; if there exists $a^* \in qatoms(\psi)$ and $preds(a^*) = \{\tilde{p}_{\mathbb{j}}\}$, then there exists $Qx\, \tilde{p}_{\mathbb{j}}(\overline{x}) \in qatoms(\psi)$ satisfying, for all $a \in qatoms(\psi)$ and $preds(a) = \{\tilde{p}_{\mathbb{j}}\}$, $a = Qx\, \tilde{p}_{\mathbb{j}}(\overline{x})$;*
- (h) *for all $a \in qatoms(S_\phi)$, there exists $\mathbb{j}^* \in J_\phi$ and $preds(a) = \{\tilde{p}_{\mathbb{j}^*}\}$;*
- (i) *for all $\mathbb{j} \in J_\phi$, there exist a sequence $\overline{x}$ of variables of $\mathcal{L}$ and $\tilde{p}_{\mathbb{j}}(\overline{x}) \in atoms(S_\phi)$ satisfying, for all $a \in atoms(S_\phi)$ and $preds(a) = \{\tilde{p}_{\mathbb{j}}\}$, $a = \tilde{p}_{\mathbb{j}}(\overline{x})$; if there exists $a^* \in qatoms(S_\phi)$ and $preds(a^*) = \{\tilde{p}_{\mathbb{j}}\}$, then there exists $Qx\, \tilde{p}_{\mathbb{j}}(\overline{x}) \in qatoms(S_\phi)$ satisfying, for all $a \in qatoms(S_\phi)$ and $preds(a) = \{\tilde{p}_{\mathbb{j}}\}$, $a = Qx\, \tilde{p}_{\mathbb{j}}(\overline{x})$;*
- (j) *$tcons(S_\phi) \subseteq tcons(\phi)$.*

(II) *There exist $J_T \subseteq \{(i, j) \,|\, i \geq n_0\}$ and $S_T \subseteq SimOrdCl_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_T\}}$ such that*

- (a) *there exists an interpretation $\mathfrak{A}$ for $\mathcal{L}$ and $\mathfrak{A} \models T$ if and only if there exists an interpretation $\mathfrak{A}'$ for $\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_T\}$ and $\mathfrak{A}' \models S_T$, satisfying $\mathfrak{A} = \mathfrak{A}'|_{\mathcal{L}}$;*
- (b) *if $T \subseteq_{\mathcal{F}} Form_{\mathcal{L}}$, then $J_T \subseteq_{\mathcal{F}} \{(i, j) \,|\, i \geq n_0\}$, $\|J_T\| \leq 2 \cdot |T|$; $S_T \subseteq_{\mathcal{F}} SimOrdCl_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_T\}}$, $|S_T| \in O(|T|^2)$; the number of all elementary operations of the translation of T to S_T, is in $O(|T|^2)$; the time and space complexity of the translation of T to S_T, is in $O(|T|^2 \cdot \log(1 + n_0 + |T|))$;*
- (c) *for all $a \in qatoms(S_T)$, there exists $\mathbb{j}^* \in J_T$ and $preds(a) = \{\tilde{p}_{\mathbb{j}^*}\}$;*
- (d) *for all $\mathbb{j} \in J_T$, there exist a sequence $\overline{x}$ of variables of $\mathcal{L}$ and $\tilde{p}_{\mathbb{j}}(\overline{x}) \in atoms(S_T)$ satisfying, for all $a \in atoms(S_T)$ and $preds(a) = \{\tilde{p}_{\mathbb{j}}\}$, $a = \tilde{p}_{\mathbb{j}}(\overline{x})$; if there exists $a^* \in qatoms(S_T)$ and $preds(a^*) = \{\tilde{p}_{\mathbb{j}}\}$, then there exists $Qx\, \tilde{p}_{\mathbb{j}}(\overline{x}) \in qatoms(S_T)$ satisfying, for all $a \in qatoms(S_T)$ and $preds(a) = \{\tilde{p}_{\mathbb{j}}\}$, $a = Qx\, \tilde{p}_{\mathbb{j}}(\overline{x})$;*
- (e) *$tcons(S_T) \subseteq tcons(T)$.*

Proof. Technical using interpolation.

Let $n_\theta \in \mathbb{N}$ and $\theta \in Form_{\mathcal{L}}$. There exists $\theta' \in Form_{\mathcal{L}}$ such that (13)

(a) $\theta' \equiv \theta$;
(b) $|\theta'| \leq 2 \cdot |\theta|$; θ' can be built up via a postorder traversal of θ with $\#O(\theta) \in O(|\theta|)$ and the time, space complexity in $O(|\theta| \cdot (\log(1+n_\theta) + \log|\theta|))$;
(c) θ' does not contain $\neg$;
(d) $\theta' \in Tcons_{\mathcal{L}}$; or $\mathit{1}$ is not a subformula of θ'; for every subformula of θ' of the form $\varepsilon_1 \diamond \varepsilon_2$, $\diamond \in \{\wedge, \vee\}$, $\varepsilon_i \neq \mathit{0}, \mathit{1}$, $\{\varepsilon_1, \varepsilon_2\} \not\subseteq Tcons_{\mathcal{L}}$; for every subformula of θ' of the form $\varepsilon_1 \rightarrow \varepsilon_2$, $\varepsilon_1 \neq \mathit{0}, \mathit{1}$, $\varepsilon_2 \neq \mathit{1}$, $\{\varepsilon_1, \varepsilon_2\} \not\subseteq Tcons_{\mathcal{L}}$;
(e) $tcons(\theta') \subseteq tcons(\theta)$.

The proof is by induction on the structure of θ.

Let $l \in Lit_{\mathcal{L}}$. There exists $C \in SimOrdCl_{\mathcal{L}}$ such that (14)

(a) for every interpretation $\mathfrak{A}$ for $\mathcal{L}$, for all $e \in \mathcal{S}_{\mathfrak{A}}$, $\mathfrak{A} \models_e l$ if and only if $\mathfrak{A} \models_e C$;
(b) $|C| \leq 3 \cdot |l|$, C can be built up from l with $\#O(l) \in O(|l|)$.

In Table 1, for every form of l, C is assigned so that for every interpretation $\mathfrak{A}$ for $\mathcal{L}$, for all $e \in \mathcal{S}_{\mathfrak{A}}$, $\mathfrak{A} \models_e l$ if and only if $\mathfrak{A} \models_e C$.

Let $n_\theta \in \mathbb{N}$, $\theta \in Form_{\mathcal{L}} - Tcons_{\mathcal{L}}$, (13c–e) hold for θ; $\bar{x}$ be a sequence of variables, $vars(\theta) \subseteq vars(\bar{x}) \subseteq Var_{\mathcal{L}}$; $\mathbb{i} = (n_\theta, j_{\mathbb{i}}) \in \{(n_\theta, j) \mid j \in \mathbb{N}\}$, $\tilde{p}_{\mathbb{i}} \in \tilde{\mathbb{P}}$, $ar(\tilde{p}_{\mathbb{i}}) = |\bar{x}|$. There exist $J = \{(n_\theta, j) \mid j_{\mathbb{i}} + 1 \leq j \leq n_J\} \subseteq \{(n_\theta, j) \mid j \in \mathbb{N}\}$, $j_{\mathbb{i}} \leq n_J$, $\mathbb{i} \notin J$; a *CNF* $\psi^s \in Form_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{j}} \mid \mathbb{j} \in J\}}$, $S^s \subseteq_{\mathcal{F}} SimOrdCl_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{j}} \mid \mathbb{j} \in J\}}$, $s = +, -$, such that for both s, (15)

(a) $\|J\| \leq |\theta| - 1$;
(b) there exists an interpretation $\mathfrak{A}$ for $\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\}$ and $\mathfrak{A} \models \tilde{p}_{\mathbb{i}}(\bar{x}) \rightarrow \theta \in Form_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\}}$ if and only if there exists an interpretation $\mathfrak{A}'$ for $\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{j}} \mid \mathbb{j} \in J\}$ and $\mathfrak{A}' \models \psi^+$, satisfying $\mathfrak{A} = \mathfrak{A}'|_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\}}$;
(c) there exists an interpretation $\mathfrak{A}$ for $\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\}$ and $\mathfrak{A} \models \theta \rightarrow \tilde{p}_{\mathbb{i}}(\bar{x}) \in Form_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\}}$ if and only if there exists an interpretation $\mathfrak{A}'$ for $\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{j}} \mid \mathbb{j} \in J\}$ and $\mathfrak{A}' \models \psi^-$, satisfying $\mathfrak{A} = \mathfrak{A}'|_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\}}$;
(d) there exists an interpretation $\mathfrak{A}$ for $\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\}$ and $\mathfrak{A} \models \tilde{p}_{\mathbb{i}}(\bar{x}) \rightarrow \theta \in Form_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\}}$ if and only if there exists an interpretation $\mathfrak{A}'$ for $\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{j}} \mid \mathbb{j} \in J\}$ and $\mathfrak{A}' \models S^+$, satisfying $\mathfrak{A} = \mathfrak{A}'|_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\}}$;
(e) there exists an interpretation $\mathfrak{A}$ for $\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\}$ and $\mathfrak{A} \models \theta \rightarrow \tilde{p}_{\mathbb{i}}(\bar{x}) \in Form_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\}}$ if and only if there exists an interpretation $\mathfrak{A}'$ for $\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{j}} \mid \mathbb{j} \in J\}$ and $\mathfrak{A}' \models S^-$, satisfying $\mathfrak{A} = \mathfrak{A}'|_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\}}$;
(f) $|\psi^s| \leq 15 \cdot |\theta| \cdot (1 + |\bar{x}|)$, ψ^s can be built up from θ and $\tilde{f}_0(\bar{x})$ via a preorder traversal of θ with $\#O(\theta, \tilde{f}_0(\bar{x})) \in O(|\theta| \cdot (1 + |\bar{x}|))$;
(g) $|S^s| \leq 15 \cdot |\theta| \cdot (1 + |\bar{x}|)$, S^s can be built up from θ and $\tilde{f}_0(\bar{x})$ via a preorder traversal of θ with $\#O(\theta, \tilde{f}_0(\bar{x})) \in O(|\theta| \cdot (1 + |\bar{x}|))$;
(h) for all $a \in qatoms(\psi^s)$, there exists $\mathbb{j}^* \in J$ and $preds(a) = \{\tilde{p}_{\mathbb{j}^*}\}$;
(i) for all $\mathbb{j} \in \{\mathbb{i}\} \cup J$, $\tilde{p}_{\mathbb{j}}(\bar{x}) \in atoms(\psi^s)$ satisfying, for all $a \in atoms(\psi^s)$ and $preds(a) = \{\tilde{p}_{\mathbb{j}}\}$, $a = \tilde{p}_{\mathbb{j}}(\bar{x})$; $\tilde{p}_{\mathbb{i}} \notin preds(qatoms(\psi^s))$, for all $\mathbb{j} \in J$, if there exists $a^* \in qatoms(\psi^s)$ and $preds(a^*) = \{\tilde{p}_{\mathbb{j}}\}$, then there exists $Qx\, \tilde{p}_{\mathbb{j}}(\bar{x}) \in qatoms(\psi^s)$ satisfying, for all $a \in qatoms(\psi^s)$ and $preds(a) = \{\tilde{p}_{\mathbb{j}}\}$, $a = Qx\, \tilde{p}_{\mathbb{j}}(\bar{x})$;
(j) for all $a \in qatoms(S^s)$, there exists $\mathbb{j}^* \in J$ and $preds(a) = \{\tilde{p}_{\mathbb{j}^*}\}$;
(k) for all $\mathbb{j} \in \{\mathbb{i}\} \cup J$, $\tilde{p}_{\mathbb{j}}(\bar{x}) \in atoms(S^s)$ satisfying, for all $a \in atoms(S^s)$ and $preds(a) = \{\tilde{p}_{\mathbb{j}}\}$, $a = \tilde{p}_{\mathbb{j}}(\bar{x})$; $\tilde{p}_{\mathbb{i}} \notin preds(qatoms(S^s))$, for all $\mathbb{j} \in J$, if there exists $a^* \in qatoms(S^s)$ and $preds(a^*) = \{\tilde{p}_{\mathbb{j}}\}$, then there exists $Qx\, \tilde{p}_{\mathbb{j}}(\bar{x}) \in qatoms(S^s)$ satisfying, for all $a \in qatoms(S^s)$ and $preds(a) = \{\tilde{p}_{\mathbb{j}}\}$, $a = Qx\, \tilde{p}_{\mathbb{j}}(\bar{x})$;
(l) $tcons(\theta) = tcons(\psi^s) = tcons(S^s)$.

The proof is by induction on the structure of θ using the interpolation rules in Tables 2–5.

(I) By (13) for n_ϕ, ϕ, there exists $\phi' \in Form_{\mathcal{L}}$

Table 1: Translation of l to C, $a, b \in Atom_{\mathcal{L}} - Tcons_{\mathcal{L}}$, $\bar{c} \in \overline{C}_{\mathcal{L}}$, $d \in QAtom_{\mathcal{L}}$.

Case	l	C	$\lvert l \rvert$	$\lvert C \rvert$
1	a	$a \eqcirc \mathit{1}$	$\lvert a \rvert$	$\lvert a \rvert + 2 \leq 3 \cdot \lvert l \rvert$
2	$a \rightarrow \mathit{0}$	$a \eqcirc \mathit{0}$	$\lvert a \rvert + 2$	$\lvert a \rvert + 2 \leq 3 \cdot \lvert l \rvert$
3	$\bar{c} \rightarrow b$	$\bar{c} \prec b \vee \bar{c} \eqcirc b$	$\lvert b \rvert + 2$	$2 \cdot \lvert b \rvert + 4 \leq 3 \cdot \lvert l \rvert$
4	$a \rightarrow \bar{c}$	$a \prec \bar{c} \vee a \eqcirc \bar{c}$	$\lvert a \rvert + 2$	$2 \cdot \lvert a \rvert + 4 \leq 3 \cdot \lvert l \rvert$
5	$a \rightarrow b$	$a \prec b \vee a \eqcirc b$	$\lvert a \rvert + \lvert b \rvert + 1$	$2 \cdot \lvert a \rvert + 2 \cdot \lvert b \rvert + 2 \leq 3 \cdot \lvert l \rvert$
6	$(a \rightarrow \mathit{0}) \rightarrow \mathit{0}$	$\mathit{0} \prec a$	$\lvert a \rvert + 4$	$\lvert a \rvert + 2 \leq 3 \cdot \lvert l \rvert$
7	$(a \rightarrow b) \rightarrow b$	$b \prec a \vee b \eqcirc \mathit{1}$	$\lvert a \rvert + 2 \cdot \lvert b \rvert + 2$	$\lvert a \rvert + 2 \cdot \lvert b \rvert + 3 \leq 3 \cdot \lvert l \rvert$
8	$a \rightarrow d$	$a \prec d \vee a \eqcirc d$	$\lvert a \rvert + \lvert d \rvert + 1$	$2 \cdot \lvert a \rvert + 2 \cdot \lvert d \rvert + 2 \leq 3 \cdot \lvert l \rvert$
9	$d \rightarrow a$	$d \prec a \vee d \eqcirc a$	$\lvert a \rvert + \lvert d \rvert + 1$	$2 \cdot \lvert a \rvert + 2 \cdot \lvert d \rvert + 2 \leq 3 \cdot \lvert l \rvert$

Table 2: Binary interpolation rules for $\wedge$ and $\vee$.

Case		Laws	
$\theta = \theta_1 \wedge \theta_2$			
Positive interpolation	$\dfrac{\tilde{p}_{\mathbb{i}}(\overline{x}) \to \theta_1 \wedge \theta_2}{(\tilde{p}_{\mathbb{i}}(\overline{x}) \to \tilde{p}_{\mathbb{i}_1}(\overline{x})) \wedge (\tilde{p}_{\mathbb{i}}(\overline{x}) \to \tilde{p}_{\mathbb{i}_2}(\overline{x})) \wedge (\tilde{p}_{\mathbb{i}_1}(\overline{x}) \to \theta_1) \wedge (\tilde{p}_{\mathbb{i}_2}(\overline{x}) \to \theta_2)}$	(6)	(16)
$\lvert\text{Consequent}\rvert = 9+4\cdot\lvert\overline{x}\rvert + \lvert\tilde{p}_{\mathbb{i}_1}(\overline{x}) \to \theta_1\rvert + \lvert\tilde{p}_{\mathbb{i}_2}(\overline{x}) \to \theta_2\rvert \le 13\cdot(1+\lvert\overline{x}\rvert) + \lvert\tilde{p}_{\mathbb{i}_1}(\overline{x}) \to \theta_1\rvert + \lvert\tilde{p}_{\mathbb{i}_2}(\overline{x}) \to \theta_2\rvert$			
Positive interpolation	$\dfrac{\tilde{p}_{\mathbb{i}}(\overline{x}) \to \theta_1 \wedge \theta_2}{\{\tilde{p}_{\mathbb{i}}(\overline{x}) \prec \tilde{p}_{\mathbb{i}_1}(\overline{x}) \vee \tilde{p}_{\mathbb{i}}(\overline{x}) \eqcirc \tilde{p}_{\mathbb{i}_1}(\overline{x}), \tilde{p}_{\mathbb{i}}(\overline{x}) \prec \tilde{p}_{\mathbb{i}_2}(\overline{x}) \vee \tilde{p}_{\mathbb{i}}(\overline{x}) \eqcirc \tilde{p}_{\mathbb{i}_2}(\overline{x}), \tilde{p}_{\mathbb{i}_1}(\overline{x}) \to \theta_1, \tilde{p}_{\mathbb{i}_2}(\overline{x}) \to \theta_2\}}$		(17)
$\lvert\text{Consequent}\rvert = 12+8\cdot\lvert\overline{x}\rvert + \lvert\tilde{p}_{\mathbb{i}_1}(\overline{x}) \to \theta_1\rvert + \lvert\tilde{p}_{\mathbb{i}_2}(\overline{x}) \to \theta_2\rvert \le 15\cdot(1+\lvert\overline{x}\rvert) + \lvert\tilde{p}_{\mathbb{i}_1}(\overline{x}) \to \theta_1\rvert + \lvert\tilde{p}_{\mathbb{i}_2}(\overline{x}) \to \theta_2\rvert$			
Negative interpolation	$\dfrac{\theta_1 \wedge \theta_2 \to \tilde{p}_{\mathbb{i}}(\overline{x})}{(\tilde{p}_{\mathbb{i}_1}(\overline{x}) \to \tilde{p}_{\mathbb{i}}(\overline{x}) \vee \tilde{p}_{\mathbb{i}_2}(\overline{x}) \to \tilde{p}_{\mathbb{i}}(\overline{x})) \wedge (\theta_1 \to \tilde{p}_{\mathbb{i}_1}(\overline{x})) \wedge (\theta_2 \to \tilde{p}_{\mathbb{i}_2}(\overline{x}))}$	(8)	(18)
$\lvert\text{Consequent}\rvert = 9+4\cdot\lvert\overline{x}\rvert + \lvert\theta_1 \to \tilde{p}_{\mathbb{i}_1}(\overline{x})\rvert + \lvert\theta_2 \to \tilde{p}_{\mathbb{i}_2}(\overline{x})\rvert \le 13\cdot(1+\lvert\overline{x}\rvert) + \lvert\theta_1 \to \tilde{p}_{\mathbb{i}_1}(\overline{x})\rvert + \lvert\theta_2 \to \tilde{p}_{\mathbb{i}_2}(\overline{x})\rvert$			
Negative interpolation	$\dfrac{\theta_1 \wedge \theta_2 \to \tilde{p}_{\mathbb{i}}(\overline{x})}{\{\tilde{p}_{\mathbb{i}_1}(\overline{x}) \prec \tilde{p}_{\mathbb{i}}(\overline{x}) \vee \tilde{p}_{\mathbb{i}_1}(\overline{x}) \eqcirc \tilde{p}_{\mathbb{i}}(\overline{x}), \tilde{p}_{\mathbb{i}_2}(\overline{x}) \prec \tilde{p}_{\mathbb{i}}(\overline{x}) \vee \tilde{p}_{\mathbb{i}_2}(\overline{x}) \eqcirc \tilde{p}_{\mathbb{i}}(\overline{x}), \theta_1 \to \tilde{p}_{\mathbb{i}_1}(\overline{x}), \theta_2 \to \tilde{p}_{\mathbb{i}_2}(\overline{x})\}}$		(19)
$\lvert\text{Consequent}\rvert = 12+8\cdot\lvert\overline{x}\rvert + \lvert\theta_1 \to \tilde{p}_{\mathbb{i}_1}(\overline{x})\rvert + \lvert\theta_2 \to \tilde{p}_{\mathbb{i}_2}(\overline{x})\rvert \le 15\cdot(1+\lvert\overline{x}\rvert) + \lvert\theta_1 \to \tilde{p}_{\mathbb{i}_1}(\overline{x})\rvert + \lvert\theta_2 \to \tilde{p}_{\mathbb{i}_2}(\overline{x})\rvert$			
$\theta = \theta_1 \vee \theta_2$			
Positive interpolation	$\dfrac{\tilde{p}_{\mathbb{i}}(\overline{x}) \to (\theta_1 \vee \theta_2)}{(\tilde{p}_{\mathbb{i}}(\overline{x}) \to \tilde{p}_{\mathbb{i}_1}(\overline{x}) \vee \tilde{p}_{\mathbb{i}}(\overline{x}) \to \tilde{p}_{\mathbb{i}_2}(\overline{x})) \wedge (\tilde{p}_{\mathbb{i}_1}(\overline{x}) \to \theta_1) \wedge (\tilde{p}_{\mathbb{i}_2}(\overline{x}) \to \theta_2)}$	(5)	(20)
$\lvert\text{Consequent}\rvert = 9+4\cdot\lvert\overline{x}\rvert + \lvert\tilde{p}_{\mathbb{i}_1}(\overline{x}) \to \theta_1\rvert + \lvert\tilde{p}_{\mathbb{i}_2}(\overline{x}) \to \theta_2\rvert \le 13\cdot(1+\lvert\overline{x}\rvert) + \lvert\tilde{p}_{\mathbb{i}_1}(\overline{x}) \to \theta_1\rvert + \lvert\tilde{p}_{\mathbb{i}_2}(\overline{x}) \to \theta_2\rvert$			
Positive interpolation	$\dfrac{\tilde{p}_{\mathbb{i}}(\overline{x}) \to (\theta_1 \vee \theta_2)}{\{\tilde{p}_{\mathbb{i}}(\overline{x}) \prec \tilde{p}_{\mathbb{i}_1}(\overline{x}) \vee \tilde{p}_{\mathbb{i}}(\overline{x}) \eqcirc \tilde{p}_{\mathbb{i}_1}(\overline{x}) \vee \tilde{p}_{\mathbb{i}}(\overline{x}) \prec \tilde{p}_{\mathbb{i}_2}(\overline{x}) \vee \tilde{p}_{\mathbb{i}}(\overline{x}) \eqcirc \tilde{p}_{\mathbb{i}_2}(\overline{x}), \tilde{p}_{\mathbb{i}_1}(\overline{x}) \to \theta_1, \tilde{p}_{\mathbb{i}_2}(\overline{x}) \to \theta_2\}}$		(21)
$\lvert\text{Consequent}\rvert = 12+8\cdot\lvert\overline{x}\rvert + \lvert\tilde{p}_{\mathbb{i}_1}(\overline{x}) \to \theta_1\rvert + \lvert\tilde{p}_{\mathbb{i}_2}(\overline{x}) \to \theta_2\rvert \le 15\cdot(1+\lvert\overline{x}\rvert) + \lvert\tilde{p}_{\mathbb{i}_1}(\overline{x}) \to \theta_1\rvert + \lvert\tilde{p}_{\mathbb{i}_2}(\overline{x}) \to \theta_2\rvert$			
Negative interpolation	$\dfrac{(\theta_1 \vee \theta_2) \to \tilde{p}_{\mathbb{i}}(\overline{x})}{(\tilde{p}_{\mathbb{i}_1}(\overline{x}) \to \tilde{p}_{\mathbb{i}}(\overline{x})) \wedge (\tilde{p}_{\mathbb{i}_2}(\overline{x}) \to \tilde{p}_{\mathbb{i}}(\overline{x})) \wedge (\theta_1 \to \tilde{p}_{\mathbb{i}_1}(\overline{x})) \wedge (\theta_2 \to \tilde{p}_{\mathbb{i}_2}(\overline{x}))}$	(7)	(22)
$\lvert\text{Consequent}\rvert = 9+4\cdot\lvert\overline{x}\rvert + \lvert\theta_1 \to \tilde{p}_{\mathbb{i}_1}(\overline{x})\rvert + \lvert\theta_2 \to \tilde{p}_{\mathbb{i}_2}(\overline{x})\rvert \le 13\cdot(1+\lvert\overline{x}\rvert) + \lvert\theta_1 \to \tilde{p}_{\mathbb{i}_1}(\overline{x})\rvert + \lvert\theta_2 \to \tilde{p}_{\mathbb{i}_2}(\overline{x})\rvert$			
Negative interpolation	$\dfrac{(\theta_1 \vee \theta_2) \to \tilde{p}_{\mathbb{i}}(\overline{x})}{\{\tilde{p}_{\mathbb{i}_1}(\overline{x}) \prec \tilde{p}_{\mathbb{i}}(\overline{x}) \vee \tilde{p}_{\mathbb{i}_1}(\overline{x}) \eqcirc \tilde{p}_{\mathbb{i}}(\overline{x}), \tilde{p}_{\mathbb{i}_2}(\overline{x}) \prec \tilde{p}_{\mathbb{i}}(\overline{x}) \vee \tilde{p}_{\mathbb{i}_2}(\overline{x}) \eqcirc \tilde{p}_{\mathbb{i}}(\overline{x}), \theta_1 \to \tilde{p}_{\mathbb{i}_1}(\overline{x}), \theta_2 \to \tilde{p}_{\mathbb{i}_2}(\overline{x})\}}$		(23)
$\lvert\text{Consequent}\rvert = 12+8\cdot\lvert\overline{x}\rvert + \lvert\theta_1 \to \tilde{p}_{\mathbb{i}_1}(\overline{x})\rvert + \lvert\theta_2 \to \tilde{p}_{\mathbb{i}_2}(\overline{x})\rvert \le 15\cdot(1+\lvert\overline{x}\rvert) + \lvert\theta_1 \to \tilde{p}_{\mathbb{i}_1}(\overline{x})\rvert + \lvert\theta_2 \to \tilde{p}_{\mathbb{i}_2}(\overline{x})\rvert$			

such that (13a–e) hold for n_ϕ, ϕ, ϕ'. We distinguish three cases for ϕ'. Case 1: $\phi' \in \mathit{Tcons}_{\mathcal{L}} - \{\mathit{1}\}$. We put $J_\phi = \emptyset \subseteq \{(n_\phi, j) \mid j \in \mathbb{N}\}$, $\psi = \phi' \in \mathit{Form}_{\mathcal{L}}$, $S_\phi = \{\square\} \subseteq_{\mathcal{F}} \mathit{SimOrdCl}_{\mathcal{L}}$. Case 2: $\phi' = \mathit{1}$. We put $J_\phi = \emptyset \subseteq \{(n_\phi, j) \mid j \in \mathbb{N}\}$, $\psi = \mathit{1} \in \mathit{Form}_{\mathcal{L}}$, $S_\phi = \emptyset \subseteq_{\mathcal{F}} \mathit{SimOrdCl}_{\mathcal{L}}$. Case 3: $\phi' \not\in \mathit{Tcons}_{\mathcal{L}}$. We put $\overline{x} = \mathit{varseq}(\phi')$, $j_{\mathbb{i}} = 0$, $\mathbb{i} = (n_\phi, j_{\mathbb{i}})$, $\mathit{ar}(\tilde{p}_{\mathbb{i}}) = \lvert\overline{x}\rvert$. We get by (15) for n_ϕ, ϕ', $\overline{x}$, $\mathbb{i}$, $\tilde{p}_{\mathbb{i}}$ that there exist $J = \{(n_\phi, j) \mid 1 \le j \le n_J\} \subseteq \{(n_\phi, j) \mid j \in \mathbb{N}\}$, $j_{\mathbb{i}} \le n_J$, $\mathbb{i} \not\in J$, a *CNF* $\psi^+ \in \mathit{Form}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{j}} \mid \mathbb{j} \in J\}}$, $S^+ \subseteq_{\mathcal{F}} \mathit{SimOrdCl}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{j}} \mid \mathbb{j} \in J\}}$, and (15a,b,d,f–l) hold for ϕ', $\overline{x}$, $\tilde{p}_{\mathbb{i}}$, J, ψ^+, S^+. We put $n_{J_\phi} = n_J$, $J_\phi = \{(n_\phi, j) \mid j \le n_{J_\phi}\} \subseteq \{(n_\phi, j) \mid j \in \mathbb{N}\}$, $\psi = \tilde{p}_{\mathbb{i}}(\overline{x}) \wedge \psi^+ \in \mathit{Form}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \mid \mathbb{j} \in J_\phi\}}$, $S_\phi = \{\tilde{p}_{\mathbb{i}}(\overline{x}) \eqcirc \mathit{1}\} \cup S^+ \subseteq_{\mathcal{F}} \mathit{SimOrdCl}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \mid \mathbb{j} \in J_\phi\}}$. (II) straightforwardly follows from (I). $\square$

The described translation produces order clausal theories in some restrictive form, which will be utilised in inference using our order hyperresolution calculus to get shorter deductions in average case. Let $P \subseteq \tilde{\mathbb{P}}$ and $S \subseteq \mathit{OrdCl}_{\mathcal{L} \cup P}$. S is admissible iff

(a) for all $a \in \mathit{qatoms}(S)$, $\mathit{preds}(a) \subseteq P$;

(b) for all $\tilde{p} \in P$, there exist a sequence $\overline{x}$ of vari-

Table 3: Binary interpolation rules for $\rightarrow$.

Case		Laws	
$\theta = \theta_1 \rightarrow \theta_2, \theta_2 \neq \mathit{0}$			
Positive interpolation	$\dfrac{\tilde{p}_{\mathbb{i}}(\bar{x}) \rightarrow (\theta_1 \rightarrow \theta_2)}{(\tilde{p}_{\mathbb{i}}(\bar{x}) \rightarrow \tilde{p}_{\mathbb{i}_2}(\bar{x}) \vee \tilde{p}_{\mathbb{i}_1}(\bar{x}) \rightarrow \tilde{p}_{\mathbb{i}_2}(\bar{x})) \wedge (\theta_1 \rightarrow \tilde{p}_{\mathbb{i}_1}(\bar{x})) \wedge (\tilde{p}_{\mathbb{i}_2}(\bar{x}) \rightarrow \theta_2)}$	(9), (8)	(24)
$\lvert\text{Consequent}\rvert = 9 + 4 \cdot \lvert\bar{x}\rvert + \lvert\theta_1 \rightarrow \tilde{p}_{\mathbb{i}_1}(\bar{x})\rvert + \lvert\tilde{p}_{\mathbb{i}_2}(\bar{x}) \rightarrow \theta_2\rvert \leq 13 \cdot (1 + \lvert\bar{x}\rvert) + \lvert\theta_1 \rightarrow \tilde{p}_{\mathbb{i}_1}(\bar{x})\rvert + \lvert\tilde{p}_{\mathbb{i}_2}(\bar{x}) \rightarrow \theta_2\rvert$			
Positive interpolation	$\dfrac{\tilde{p}_{\mathbb{i}}(\bar{x}) \rightarrow (\theta_1 \rightarrow \theta_2)}{\{\tilde{p}_{\mathbb{i}}(\bar{x}) \prec \tilde{p}_{\mathbb{i}_2}(\bar{x}) \vee \tilde{p}_{\mathbb{i}}(\bar{x}) \eqcirc \tilde{p}_{\mathbb{i}_2}(\bar{x}) \vee \tilde{p}_{\mathbb{i}_1}(\bar{x}) \prec \tilde{p}_{\mathbb{i}_2}(\bar{x}) \vee \tilde{p}_{\mathbb{i}_1}(\bar{x}) \eqcirc \tilde{p}_{\mathbb{i}_2}(\bar{x}), \theta_1 \rightarrow \tilde{p}_{\mathbb{i}_1}(\bar{x}), \tilde{p}_{\mathbb{i}_2}(\bar{x}) \rightarrow \theta_2\}}$		(25)
$\lvert\text{Consequent}\rvert = 12 + 8 \cdot \lvert\bar{x}\rvert + \lvert\theta_1 \rightarrow \tilde{p}_{\mathbb{i}_1}(\bar{x})\rvert + \lvert\tilde{p}_{\mathbb{i}_2}(\bar{x}) \rightarrow \theta_2\rvert \leq 15 \cdot (1 + \lvert\bar{x}\rvert) + \lvert\theta_1 \rightarrow \tilde{p}_{\mathbb{i}_1}(\bar{x})\rvert + \lvert\tilde{p}_{\mathbb{i}_2}(\bar{x}) \rightarrow \theta_2\rvert$			
Negative interpolation	$\dfrac{(\theta_1 \rightarrow \theta_2) \rightarrow \tilde{p}_{\mathbb{i}}(\bar{x})}{((\tilde{p}_{\mathbb{i}_1}(\bar{x}) \rightarrow \tilde{p}_{\mathbb{i}_2}(\bar{x})) \rightarrow \tilde{p}_{\mathbb{i}_2}(\bar{x}) \vee \tilde{p}_{\mathbb{i}}(\bar{x})) \wedge (\tilde{p}_{\mathbb{i}_2}(\bar{x}) \rightarrow \tilde{p}_{\mathbb{i}}(\bar{x})) \wedge (\tilde{p}_{\mathbb{i}_1}(\bar{x}) \rightarrow \theta_1) \wedge (\theta_2 \rightarrow \tilde{p}_{\mathbb{i}_2}(\bar{x}))}$	(11), (3), (1)	(26)
$\lvert\text{Consequent}\rvert = 13 + 6 \cdot \lvert\bar{x}\rvert + \lvert\tilde{p}_{\mathbb{i}_1}(\bar{x}) \rightarrow \theta_1\rvert + \lvert\theta_2 \rightarrow \tilde{p}_{\mathbb{i}_2}(\bar{x})\rvert \leq 13 \cdot (1 + \lvert\bar{x}\rvert) + \lvert\tilde{p}_{\mathbb{i}_1}(\bar{x}) \rightarrow \theta_1\rvert + \lvert\theta_2 \rightarrow \tilde{p}_{\mathbb{i}_2}(\bar{x})\rvert$			
Negative interpolation	$\dfrac{(\theta_1 \rightarrow \theta_2) \rightarrow \tilde{p}_{\mathbb{i}}(\bar{x})}{\{\tilde{p}_{\mathbb{i}_2}(\bar{x}) \prec \tilde{p}_{\mathbb{i}_1}(\bar{x}) \vee \tilde{p}_{\mathbb{i}_2}(\bar{x}) \eqcirc \mathit{1} \vee \tilde{p}_{\mathbb{i}}(\bar{x}) \eqcirc \mathit{1}, \tilde{p}_{\mathbb{i}_2}(\bar{x}) \prec \tilde{p}_{\mathbb{i}}(\bar{x}) \vee \tilde{p}_{\mathbb{i}_2}(\bar{x}) \eqcirc \tilde{p}_{\mathbb{i}}(\bar{x}), \tilde{p}_{\mathbb{i}_1}(\bar{x}) \rightarrow \theta_1, \theta_2 \rightarrow \tilde{p}_{\mathbb{i}_2}(\bar{x})\}}$		(27)
$\lvert\text{Consequent}\rvert = 15 + 8 \cdot \lvert\bar{x}\rvert + \lvert\tilde{p}_{\mathbb{i}_1}(\bar{x}) \rightarrow \theta_1\rvert + \lvert\theta_2 \rightarrow \tilde{p}_{\mathbb{i}_2}(\bar{x})\rvert \leq 15 \cdot (1 + \lvert\bar{x}\rvert) + \lvert\tilde{p}_{\mathbb{i}_1}(\bar{x}) \rightarrow \theta_1\rvert + \lvert\theta_2 \rightarrow \tilde{p}_{\mathbb{i}_2}(\bar{x})\rvert$			

Table 4: Unary interpolation rules for $\rightarrow$.

Case		Laws	
$\theta = \theta_1 \rightarrow \mathit{0}$			
Positive interpolation	$\dfrac{\tilde{p}_{\mathbb{i}}(\bar{x}) \rightarrow (\theta_1 \rightarrow \mathit{0})}{(\tilde{p}_{\mathbb{i}}(\bar{x}) \rightarrow \mathit{0} \vee \tilde{p}_{\mathbb{i}_1}(\bar{x}) \rightarrow \mathit{0}) \wedge (\theta_1 \rightarrow \tilde{p}_{\mathbb{i}_1}(\bar{x}))}$	(9), (8)	(28)
$\lvert\text{Consequent}\rvert = 8 + 2 \cdot \lvert\bar{x}\rvert + \lvert\theta_1 \rightarrow \tilde{p}_{\mathbb{i}_1}(\bar{x})\rvert \leq 13 \cdot (1 + \lvert\bar{x}\rvert) + \lvert\theta_1 \rightarrow \tilde{p}_{\mathbb{i}_1}(\bar{x})\rvert$			
Positive interpolation	$\dfrac{\tilde{p}_{\mathbb{i}}(\bar{x}) \rightarrow (\theta_1 \rightarrow \mathit{0})}{\{\tilde{p}_{\mathbb{i}}(\bar{x}) \eqcirc \mathit{0} \vee \tilde{p}_{\mathbb{i}_1}(\bar{x}) \eqcirc \mathit{0}, \theta_1 \rightarrow \tilde{p}_{\mathbb{i}_1}(\bar{x})\}}$		(29)
$\lvert\text{Consequent}\rvert = 6 + 2 \cdot \lvert\bar{x}\rvert + \lvert\theta_1 \rightarrow \tilde{p}_{\mathbb{i}_1}(\bar{x})\rvert \leq 15 \cdot (1 + \lvert\bar{x}\rvert) + \lvert\theta_1 \rightarrow \tilde{p}_{\mathbb{i}_1}(\bar{x})\rvert$			
Negative interpolation	$\dfrac{(\theta_1 \rightarrow \mathit{0}) \rightarrow \tilde{p}_{\mathbb{i}}(\bar{x})}{((\tilde{p}_{\mathbb{i}_1}(\bar{x}) \rightarrow \mathit{0}) \rightarrow \mathit{0} \vee \tilde{p}_{\mathbb{i}}(\bar{x})) \wedge (\tilde{p}_{\mathbb{i}_1}(\bar{x}) \rightarrow \theta_1)}$	(11)	(30)
$\lvert\text{Consequent}\rvert = 8 + 2 \cdot \lvert\bar{x}\rvert + \lvert\tilde{p}_{\mathbb{i}_1}(\bar{x}) \rightarrow \theta_1\rvert \leq 13 \cdot (1 + \lvert\bar{x}\rvert) + \lvert\tilde{p}_{\mathbb{i}_1}(\bar{x}) \rightarrow \theta_1\rvert$			
Negative interpolation	$\dfrac{(\theta_1 \rightarrow \mathit{0}) \rightarrow \tilde{p}_{\mathbb{i}}(\bar{x})}{\{\mathit{0} \prec \tilde{p}_{\mathbb{i}_1}(\bar{x}) \vee \tilde{p}_{\mathbb{i}}(\bar{x}) \eqcirc \mathit{1}, \tilde{p}_{\mathbb{i}_1}(\bar{x}) \rightarrow \theta_1\}}$		(31)
$\lvert\text{Consequent}\rvert = 6 + 2 \cdot \lvert\bar{x}\rvert + \lvert\tilde{p}_{\mathbb{i}_1}(\bar{x}) \rightarrow \theta_1\rvert \leq 15 \cdot (1 + \lvert\bar{x}\rvert) + \lvert\tilde{p}_{\mathbb{i}_1}(\bar{x}) \rightarrow \theta_1\rvert$			

ables of $\mathcal{L}$ and $\tilde{p}(\bar{x}) \in atoms(S)$ satisfying, for all $a \in atoms(S)$ and $preds(a) = \{\tilde{p}\}$, a is an instance of $\tilde{p}(\bar{x})$ of $\mathcal{L} \cup P$; if there exists $a^* \in qatoms(S)$ and $preds(a^*) = \{\tilde{p}\}$, then there exists $Qx\,\tilde{p}(\bar{x}) \in qatoms(S)$ satisfying, for all $a \in qatoms(S)$ and $preds(a) = \{\tilde{p}\}$, a is an instance of $Qx\,\tilde{p}(\bar{x})$ of $\mathcal{L} \cup P$.

(a) and (b) imply that for all $Qx\,a, Q'x'\,a' \in qatoms(S)$, if $preds(a) = preds(a')$, then $Q = Q'$, $x = x'$, $boundindset(Qx\,a) = boundindset(Q'x'\,a')$.

Theorem 3.4. *Let $n_0 \in \mathbb{N}$, $\phi \in Form_{\mathcal{L}}$, $T \subseteq Form_{\mathcal{L}}$. There exist $J_T^{\phi} \subseteq \{(i,j) \,|\, i \geq n_0\}$ and $S_T^{\phi} \subseteq SimOrdCl_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_T^{\phi}\}}$ such that*

(i) *there exists an interpretation $\mathfrak{A}$ for $\mathcal{L}$ and $\mathfrak{A} \models T$, $\mathfrak{A} \not\models \phi$ if and only if there exists an interpretation $\mathfrak{A}'$ for $\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_T^{\phi}\}$ and $\mathfrak{A}' \models S_T^{\phi}$, satisfying $\mathfrak{A} = \mathfrak{A}'|_{\mathcal{L}}$;*

(ii) *if $T \subseteq_{\mathcal{F}} Form_{\mathcal{L}}$, then $J_T^{\phi} \subseteq_{\mathcal{F}} \{(i,j) \,|\, i \geq n_0\}$, $\|J_T^{\phi}\| \in O(|T| + |\phi|)$; $S_T^{\phi} \subseteq_{\mathcal{F}} SimOrdCl_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_T^{\phi}\}}$, $|S_T^{\phi}| \in O(|T|^2 + |\phi|^2)$; the number of all elementary operations of the translation of T and ϕ to S_T^{ϕ}, is in $O(|T|^2 + |\phi|^2)$; the time and space complexity of the translation of T and ϕ to S_T^{ϕ}, is in $O(|T|^2 \cdot \log(1 + n_0 + |T|) + |\phi|^2 \cdot (\log(1 + n_0) + \log|\phi|))$;*

(iii) *S_T^{ϕ} is admissible;*

(iv) *$tcons(S_T^{\phi}) \subseteq tcons(\phi) \cup tcons(T)$.*

Proof. We get by Lemma 3.3(II) for $n_0 + 1$, T that there exist $J_T \subseteq \{(i,j) \,|\, i \geq n_0 + 1\}$, $S_T \subseteq SimOrdCl_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_T\}}$, and 3.3(II a–e) hold for $n_0 + 1$, T, J_T, S_T. By (13) for n_0, ϕ, there exists $\phi' \in Form_{\mathcal{L}}$ such that (13a–e) hold for n_0, ϕ, ϕ'. We distinguish three cases for ϕ'. Case 1: $\phi' \in Tcons_{\mathcal{L}} - \{\mathit{1}\}$. We

Table 5: Unary interpolation rules for $\forall$ and $\exists$.

Case		
$\forall x\,\theta_1$		
Positive interpolation	$\dfrac{\tilde{p}_{\mathbb{i}}(\bar{x}) \rightarrow \forall x\,\theta_1}{(\tilde{p}_{\mathbb{i}}(\bar{x}) \rightarrow \forall x\,\tilde{p}_{\mathbb{i}_1}(\bar{x})) \wedge (\tilde{p}_{\mathbb{i}_1}(\bar{x}) \rightarrow \theta_1)}$	(32)
$\lvert\text{Consequent}\rvert = 6+2\cdot\lvert\bar{x}\rvert+\lvert\tilde{p}_{\mathbb{i}_1}(\bar{x}) \rightarrow \theta_1\rvert \le 13\cdot(1+\lvert\bar{x}\rvert)+\lvert\tilde{p}_{\mathbb{i}_1}(\bar{x}) \rightarrow \theta_1\rvert$		
Positive interpolation	$\dfrac{\tilde{p}_{\mathbb{i}}(\bar{x}) \rightarrow \forall x\,\theta_1}{\{\tilde{p}_{\mathbb{i}}(\bar{x}) \prec \forall x\,\tilde{p}_{\mathbb{i}_1}(\bar{x}) \vee \tilde{p}_{\mathbb{i}}(\bar{x}) \eqcirc \forall x\,\tilde{p}_{\mathbb{i}_1}(\bar{x}), \tilde{p}_{\mathbb{i}_1}(\bar{x}) \rightarrow \theta_1\}}$	(33)
$\lvert\text{Consequent}\rvert = 10+4\cdot\lvert\bar{x}\rvert+\lvert\tilde{p}_{\mathbb{i}_1}(\bar{x}) \rightarrow \theta_1\rvert \le 15\cdot(1+\lvert\bar{x}\rvert)+\lvert\tilde{p}_{\mathbb{i}_1}(\bar{x}) \rightarrow \theta_1\rvert$		
Negative interpolation	$\dfrac{\forall x\,\theta_1 \rightarrow \tilde{p}_{\mathbb{i}}(\bar{x})}{(\forall x\,\tilde{p}_{\mathbb{i}_1}(\bar{x}) \rightarrow \tilde{p}_{\mathbb{i}}(\bar{x})) \wedge (\theta_1 \rightarrow \tilde{p}_{\mathbb{i}_1}(\bar{x}))}$	(34)
$\lvert\text{Consequent}\rvert = 6+2\cdot\lvert\bar{x}\rvert+\lvert\theta_1 \rightarrow \tilde{p}_{\mathbb{i}_1}(\bar{x})\rvert \le 13\cdot(1+\lvert\bar{x}\rvert)+\lvert\theta_1 \rightarrow \tilde{p}_{\mathbb{i}_1}(\bar{x})\rvert$		
Negative interpolation	$\dfrac{\forall x\,\theta_1 \rightarrow \tilde{p}_{\mathbb{i}}(\bar{x})}{\{\forall x\,\tilde{p}_{\mathbb{i}_1}(\bar{x}) \prec \tilde{p}_{\mathbb{i}}(\bar{x}) \vee \forall x\,\tilde{p}_{\mathbb{i}_1}(\bar{x}) \eqcirc \tilde{p}_{\mathbb{i}}(\bar{x}), \theta_1 \rightarrow \tilde{p}_{\mathbb{i}_1}(\bar{x})\}}$	(35)
$\lvert\text{Consequent}\rvert = 10+4\cdot\lvert\bar{x}\rvert+\lvert\theta_1 \rightarrow \tilde{p}_{\mathbb{i}_1}(\bar{x})\rvert \le 15\cdot(1+\lvert\bar{x}\rvert)+\lvert\theta_1 \rightarrow \tilde{p}_{\mathbb{i}_1}(\bar{x})\rvert$		
$\exists x\,\theta_1$		
Positive interpolation	$\dfrac{\tilde{p}_{\mathbb{i}}(\bar{x}) \rightarrow \exists x\,\theta_1}{(\tilde{p}_{\mathbb{i}}(\bar{x}) \rightarrow \exists x\,\tilde{p}_{\mathbb{i}_1}(\bar{x})) \wedge (\tilde{p}_{\mathbb{i}_1}(\bar{x}) \rightarrow \theta_1)}$	(36)
$\lvert\text{Consequent}\rvert = 6+2\cdot\lvert\bar{x}\rvert+\lvert\tilde{p}_{\mathbb{i}_1}(\bar{x}) \rightarrow \theta_1\rvert \le 13\cdot(1+\lvert\bar{x}\rvert)+\lvert\tilde{p}_{\mathbb{i}_1}(\bar{x}) \rightarrow \theta_1\rvert$		
Positive interpolation	$\dfrac{\tilde{p}_{\mathbb{i}}(\bar{x}) \rightarrow \exists x\,\theta_1}{\{\tilde{p}_{\mathbb{i}}(\bar{x}) \prec \exists x\,\tilde{p}_{\mathbb{i}_1}(\bar{x}) \vee \tilde{p}_{\mathbb{i}}(\bar{x}) \eqcirc \exists x\,\tilde{p}_{\mathbb{i}_1}(\bar{x}), \tilde{p}_{\mathbb{i}_1}(\bar{x}) \rightarrow \theta_1\}}$	(37)
$\lvert\text{Consequent}\rvert = 10+4\cdot\lvert\bar{x}\rvert+\lvert\tilde{p}_{\mathbb{i}_1}(\bar{x}) \rightarrow \theta_1\rvert \le 15\cdot(1+\lvert\bar{x}\rvert)+\lvert\tilde{p}_{\mathbb{i}_1}(\bar{x}) \rightarrow \theta_1\rvert$		
Negative interpolation	$\dfrac{\exists x\,\theta_1 \rightarrow \tilde{p}_{\mathbb{i}}(\bar{x})}{(\exists x\,\tilde{p}_{\mathbb{i}_1}(\bar{x}) \rightarrow \tilde{p}_{\mathbb{i}}(\bar{x})) \wedge (\theta_1 \rightarrow \tilde{p}_{\mathbb{i}_1}(\bar{x}))}$	(38)
$\lvert\text{Consequent}\rvert = 6+2\cdot\lvert\bar{x}\rvert+\lvert\theta_1 \rightarrow \tilde{p}_{\mathbb{i}_1}(\bar{x})\rvert \le 13\cdot(1+\lvert\bar{x}\rvert)+\lvert\theta_1 \rightarrow \tilde{p}_{\mathbb{i}_1}(\bar{x})\rvert$		
Negative interpolation	$\dfrac{\exists x\,\theta_1 \rightarrow \tilde{p}_{\mathbb{i}}(\bar{x})}{\{\exists x\,\tilde{p}_{\mathbb{i}_1}(\bar{x}) \prec \tilde{p}_{\mathbb{i}}(\bar{x}) \vee \exists x\,\tilde{p}_{\mathbb{i}_1}(\bar{x}) \eqcirc \tilde{p}_{\mathbb{i}}(\bar{x}), \theta_1 \rightarrow \tilde{p}_{\mathbb{i}_1}(\bar{x})\}}$	(39)
$\lvert\text{Consequent}\rvert = 10+4\cdot\lvert\bar{x}\rvert+\lvert\theta_1 \rightarrow \tilde{p}_{\mathbb{i}_1}(\bar{x})\rvert \le 15\cdot(1+\lvert\bar{x}\rvert)+\lvert\theta_1 \rightarrow \tilde{p}_{\mathbb{i}_1}(\bar{x})\rvert$		

put $J_T^\phi = J_T \subseteq \{(i,j) \,|\, i \ge n_0+1\} \subseteq \{(i,j) \,|\, i \ge n_0\}$ and $S_T^\phi = S_T \subseteq \mathit{SimOrdCl}_{\mathcal{L}\cup\{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j}\in J_T^\phi\}}$. Case 2: $\phi' = \mathit{1}$. We put $J_T^\phi = \emptyset \subseteq \{(i,j) \,|\, i \ge n_0\}$ and $S_T^\phi = \{\square\} \subseteq \mathit{SimOrdCl}_{\mathcal{L}}$. Case 3: $\phi' \notin \mathit{Tcons}_{\mathcal{L}}$. We put $\bar{x} = \mathit{varseq}(\phi')$, $j_{\mathbb{i}} = 0$, $\mathbb{i} = (n_0, j_{\mathbb{i}})$, $\mathit{ar}(\tilde{p}_{\mathbb{i}}) = |\bar{x}|$. We get by (15) for n_0, $\forall\bar{x}\phi'$, $\bar{x}$, $\mathbb{i}$, $\tilde{p}_{\mathbb{i}}$ that there exist $J = \{(n_0,j) \,|\, 1 \le j \le n_J\} \subseteq \{(n_0,j) \,|\, j \in \mathbb{N}\}$, $j_{\mathbb{i}} \le n_J$, $\mathbb{i} \notin J$, $S^- \subseteq_{\mathcal{F}} \mathit{SimOrdCl}_{\mathcal{L}\cup\{\tilde{p}_{\mathbb{i}}\}\cup\{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j}\in J\}}$, and (15e,g,j–l) hold for $\forall\bar{x}\phi'$, $\bar{x}$, $\tilde{p}_{\mathbb{i}}$, J, S^-. We put $J_T^\phi = J_T \cup \{\mathbb{i}\} \cup J \subseteq \{(i,j) \,|\, i \ge n_0\}$ and $S_T^\phi = S_T \cup \{\tilde{p}_{\mathbb{i}}(\bar{x}) \prec \mathit{1}\} \cup S^- \subseteq \mathit{SimOrdCl}_{\mathcal{L}\cup\{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j}\in J_T^\phi\}}$. $\square$

Corollary 3.5. *Let $n_0 \in \mathbb{N}$, $\phi \in \mathit{Form}_{\mathcal{L}}$, $T \subseteq \mathit{Form}_{\mathcal{L}}$. There exist $J_T^\phi \subseteq \{(i,j) \,|\, i \ge n_0\}$ and $S_T^\phi \subseteq \mathit{SimOrdCl}_{\mathcal{L}\cup\{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j}\in J_T^\phi\}}$ such that*

(i) *$T \models \phi$ if and only if S_T^ϕ is unsatisfiable;*

(ii) *if $T \subseteq_{\mathcal{F}} \mathit{Form}_{\mathcal{L}}$, then $J_T^\phi \subseteq_{\mathcal{F}} \{(i,j) \,|\, i \ge n_0\}$, $\|J_T^\phi\| \in O(|T|+|\phi|)$; $S_T^\phi \subseteq_{\mathcal{F}} \mathit{SimOrdCl}_{\mathcal{L}\cup\{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j}\in J_T^\phi\}}$, $|S_T^\phi| \in O(|T|^2+|\phi|^2)$; the number of all elementary operations of the translation of T and ϕ to S_T^ϕ, is in $O(|T|^2+|\phi|^2)$; the time and space complexity of the translation of T and ϕ to S_T^ϕ, is in $O(|T|^2\cdot\log(1+n_0+|T|)+|\phi|^2\cdot(\log(1+n_0)+\log|\phi|))$;*

(iii) *S_T^ϕ is admissible;*

(iv) *$\mathit{tcons}(S_T^\phi) \subseteq \mathit{tcons}(\phi) \cup \mathit{tcons}(T)$.*

Proof. A straightforward consequence of Theorem 3.4. $\square$

4 ORDER HYPERRESOLUTION RULES

At first, we introduce some basic notions and notation concerning chains of order literals. A chain Ξ of $\mathcal{L}$ is a sequence $\Xi = \varepsilon_0 \diamond_0 \upsilon_0, \dots, \varepsilon_n \diamond_n \upsilon_n$, $\varepsilon_i \diamond_i \upsilon_i \in \mathit{OrdLit}_{\mathcal{L}}$, such that for all $i < n$, $\upsilon_i = \varepsilon_{i+1}$. ε_0 is the beginning element of Ξ and υ_n the ending element of Ξ. $\varepsilon_0\,\Xi\,\upsilon_n$ denotes Ξ together with its respective beginning and ending element. Let $\Xi = \varepsilon_0 \diamond_0 \upsilon_0, \dots, \varepsilon_n \diamond_n \upsilon_n$ be a chain of $\mathcal{L}$. Ξ is an equality chain of $\mathcal{L}$ iff, for all $i \le n$, $\diamond_i = \eqcirc$. Ξ is an increasing chain of $\mathcal{L}$ iff there exists $i^* \le n$ such that $\diamond_{i^*} = \prec$. Ξ is a contradiction of $\mathcal{L}$ iff Ξ is an increasing chain of $\mathcal{L}$ of the form $\varepsilon_0\,\Xi\,\mathit{0}$ or $\mathit{1}\,\Xi\,\upsilon_n$ or $\varepsilon_0\,\Xi\,\varepsilon_0$. Let $S \subseteq \mathit{OrdCl}_{\mathcal{L}}$ be unit and $\Xi = \varepsilon_0 \diamond_0 \upsilon_0, \dots, \varepsilon_n \diamond_n \upsilon_n$ be a chain | an equality chain | an increasing chain | a contradiction of $\mathcal{L}$. Ξ is a chain | an equality chain | an increasing chain | a contradiction of S iff, for all $i \le n$, $\varepsilon_i \diamond_i \upsilon_i \in S$.

Let $\tilde{\mathbb{W}} = \{\tilde{w}_{\mathbb{i}} \,|\, \mathbb{i} \in \mathbb{I}\}$ such that $\tilde{\mathbb{W}} \cap (\mathit{Func}_{\mathcal{L}} \cup \{\tilde{f}_0\}) = \emptyset$; $\tilde{\mathbb{W}}$ is an infinite countable set of new function symbols. Let $\mathcal{L}$ contain a constant (nullary function) symbol. Let $P \subseteq \tilde{\mathbb{P}}$ and $S \subseteq \mathit{OrdCl}_{\mathcal{L}\cup P}$. We denote $\mathit{GOrdCl}_{\mathcal{L}} = \{C \,|\, C \in \mathit{OrdCl}_{\mathcal{L}}$ *is closed*$\} \subseteq \mathit{OrdCl}_{\mathcal{L}}$, $\mathit{GInst}_{\mathcal{L}}(S) = \{C \,|\, C \in \mathit{GOrdCl}_{\mathcal{L}}$ *is an instance of S of $\mathcal{L}$*$\} \subseteq \mathit{GOrdCl}_{\mathcal{L}}$, $\mathit{ordtcons}(S) = \{\bar{c_1} \prec \bar{c_2} \,|\, \bar{c_1}, \bar{c_2} \in \mathit{tcons}(S), c_1 < c_2\} \subseteq \mathit{GOrdCl}_{\mathcal{L}}$. A basic order hyperresolution calculus is defined in Table 6. The basic order hyperresolution calculus can be generalised to an order hyperresolution one in Table 7. Let $\mathcal{L}_0 = \mathcal{L} \cup P$, a reduct of $\mathcal{L} \cup \tilde{\mathbb{W}} \cup P$, and $S_0 = \emptyset \subseteq \mathit{GOrdCl}_{\mathcal{L}_0} \mid \mathit{OrdCl}_{\mathcal{L}_0}$. Let

Table 6: Basic order hyperresolution rules.

(*Basic order hyperresolution rule*) (40)

$$\frac{l_0 \vee C_0, \dots, l_n \vee C_n \in S_{\kappa-1}}{\bigvee_{i=0}^{n} C_i \in S_\kappa};$$

$$l_0, \dots, l_n \textit{ is a contradiction of } \mathcal{L}_{\kappa-1}.$$

$\bigvee_{i=0}^{n} C_i$ is a basic order hyperresolvent of $l_0 \vee C_0, \dots, l_n \vee C_n$.

(*Basic order trichotomy rule*) (41)

$$\frac{a, b \in atoms(S_{\kappa-1}), \{a, b\} \not\subseteq Tcons_{\mathcal{L}}, qatoms(S) \neq \emptyset}{a \prec b \vee a \eqcirc b \vee b \prec a \in S_\kappa}.$$

$a \prec b \vee a \eqcirc b \vee b \prec a$ is a basic order trichotomy resolvent of a and b.

(*Basic order $\forall$-quantification rule*) (42)

$$\frac{\forall x\, a \in qatoms^{\forall}(S_{\kappa-1})}{\forall x\, a \prec a\gamma \vee \forall x\, a \eqcirc a\gamma \in S_\kappa};$$

$$t \in GTerm_{\mathcal{L}_{\kappa-1}}, \gamma = x/t \in Subst_{\mathcal{L}_{\kappa-1}}, dom(\gamma) = \{x\} = vars(a).$$

$\forall x\, a \prec a\gamma \vee \forall x\, a \eqcirc a\gamma$ is a basic order $\forall$-quantification resolvent of $\forall x\, a$.

(*Basic order $\exists$-quantification rule*) (43)

$$\frac{\exists x\, a \in qatoms^{\exists}(S_{\kappa-1})}{a\gamma \prec \exists x\, a \vee a\gamma \eqcirc \exists x\, a \in S_\kappa};$$

$$t \in GTerm_{\mathcal{L}_{\kappa-1}}, \gamma = x/t \in Subst_{\mathcal{L}_{\kappa-1}}, dom(\gamma) = \{x\} = vars(a).$$

$a\gamma \prec \exists x\, a \vee a\gamma \eqcirc \exists x\, a$ is a basic order $\exists$-quantification resolvent of $\exists x\, a$.

(*Basic order $\forall$-witnessing rule*) (44)

$$\frac{\forall x\, a \in qatoms^{\forall}(S_{\kappa-1}), b \in atoms(S_{\kappa-1}) \cup qatoms(S_{\kappa-1})}{a\gamma \prec b \vee b \eqcirc \forall x\, a \vee b \prec \forall x\, a \in S_\kappa};$$

$$\tilde{w} \in \tilde{\mathbb{W}} - Func_{\mathcal{L}_{\kappa-1}}, ar(\tilde{w}) = |freetermseq(\forall x\, a), freetermseq(b)|, \gamma = x/\tilde{w}(freetermseq(\forall x\, a), freetermseq(b)) \in Subst_{\mathcal{L}_\kappa}, dom(\gamma) = \{x\} = vars(a).$$

$a\gamma \prec b \vee b \eqcirc \forall x\, a \vee b \prec \forall x\, a$ is a basic order $\forall$-witnessing resolvent of $\forall x\, a$ and b.

(*Basic order $\exists$-witnessing rule*) (45)

$$\frac{\exists x\, a \in qatoms^{\exists}(S_{\kappa-1}), b \in atoms(S_{\kappa-1}) \cup qatoms(S_{\kappa-1})}{b \prec a\gamma \vee \exists x\, a \eqcirc b \vee \exists x\, a \prec b \in S_\kappa};$$

$$\tilde{w} \in \tilde{\mathbb{W}} - Func_{\mathcal{L}_{\kappa-1}}, ar(\tilde{w}) = |freetermseq(\exists x\, a), freetermseq(b)|, \gamma = x/\tilde{w}(freetermseq(\exists x\, a), freetermseq(b)) \in Subst_{\mathcal{L}_\kappa}, dom(\gamma) = \{x\} = vars(a).$$

$b \prec a\gamma \vee \exists x\, a \eqcirc b \vee \exists x\, a \prec b$ is a basic order $\exists$-witnessing resolvent of $\exists x\, a$ and b.

$\mathcal{D} = C_1, \dots, C_n$, $C_\kappa \in GOrdCl_{\mathcal{L} \cup \tilde{\mathbb{W}} \cup P} \mid OrdCl_{\mathcal{L} \cup \tilde{\mathbb{W}} \cup P}$, $n \geq 1$. $\mathcal{D}$ is a deduction of C_n from S by basic order hyperresolution iff, for all $1 \leq \kappa \leq n$, $C_\kappa \in ordtcons(S) \cup GInst_{\mathcal{L}_{\kappa-1}}(S)$, or there exist $1 \leq j_k^* \leq \kappa - 1$, $k = 1, \dots, m$, such that C_κ is a basic order resolvent of $C_{j_1^*}, \dots, C_{j_m^*} \in S_{\kappa-1}$ using Rule (40)–(45) with respect to $\mathcal{L}_{\kappa-1}$ and $S_{\kappa-1}$; $\mathcal{D}$ is a deduction of C_n from S by order hyperresolution iff, for all $1 \leq \kappa \leq n$, $C_\kappa \in ordtcons(S) \cup S$, or there exist $1 \leq j_k^* \leq \kappa - 1$, $k = 1, \dots, m$, such that C_κ is an order resolvent of $C'_{j_1^*}, \dots, C'_{j_m^*} \in S_{\kappa-1}^{Vr}$ using Rule (46)–(51) with respect to $\mathcal{L}_{\kappa-1}$ and $S_{\kappa-1}$ where $C'_{j_k^*}$ is a variant of $C_{j_k^*} \in S_{\kappa-1}$ of $\mathcal{L}_{\kappa-1}$; $\mathcal{L}_\kappa$ and S_κ are defined by recursion on $1 \leq \kappa \leq n$ as follows:

$$\mathcal{L}_\kappa = \begin{cases} \mathcal{L}_{\kappa-1} \cup \{\tilde{w}\} \textit{ in case of Rule } (44), (45) \mid \\ \qquad\qquad\qquad\qquad\qquad (50), (51), \\ \mathcal{L}_{\kappa-1} \qquad \textit{else}; \end{cases}$$

$$S_\kappa = S_{\kappa-1} \cup \{C_\kappa\} \subseteq GOrdCl_{\mathcal{L}_\kappa} \mid OrdCl_{\mathcal{L}_\kappa},$$

$$S_\kappa^{Vr} = Vrnt_{\mathcal{L}_\kappa}(S_\kappa) \subseteq OrdCl_{\mathcal{L}_\kappa}.$$

$\mathcal{D}$ is a refutation of S iff $C_n = \square$. We denote

$$clo^{\mathcal{BH}}(S) = \{C \mid \textit{there exists a deduction of } C \textit{ from } S \textit{ by basic order hyperresolution}\} \subseteq GOrdCl_{\mathcal{L} \cup \tilde{\mathbb{W}} \cup P},$$

$$clo^{\mathcal{H}}(S) = \{C \mid \textit{there exists a deduction of } C \textit{ from } S \textit{ by order hyperresolution}\} \subseteq OrdCl_{\mathcal{L} \cup \tilde{\mathbb{W}} \cup P}.$$

Lemma 4.1 (Lifting Lemma)**.** *Let $\mathcal{L}$ contain a constant symbol. Let $P \subseteq \tilde{\mathbb{P}}$ and $S \subseteq OrdCl_{\mathcal{L} \cup P}$. Let $C \in clo^{\mathcal{BH}}(S)$. There exists $C^* \in clo^{\mathcal{H}}(S)$ such that C is an instance of C^* of $\mathcal{L} \cup \tilde{\mathbb{W}} \cup P$.*

Proof. Straightforward. $\square$

Lemma 4.2 (Reduction Lemma)**.** *Let $\mathcal{L}$ contain a constant symbol. Let $P \subseteq \tilde{\mathbb{P}}$ and $S \subseteq OrdCl_{\mathcal{L} \cup P}$. Let $\{\bigvee_{j=0}^{k_i} \varepsilon_j^i \diamond_j^i \upsilon_j^i \vee C_i \mid i \leq n\} \subseteq clo^{\mathcal{BH}}(S)$ such that for all $\mathcal{S} \in \mathcal{S}el(\{\{j \mid j \leq k_i\}_i \mid i \leq n\})$, there exists a contradiction of $\{\varepsilon_{\mathcal{S}(i)}^i \diamond_{\mathcal{S}(i)}^i \upsilon_{\mathcal{S}(i)}^i \mid i \leq n\} \subseteq$*

Table 7: Order hyperresolution rules.

(*Order hyperresolution rule*) (46)

$$\frac{\bigvee_{j=0}^{k_0} \varepsilon_j^0 \diamond_j^0 \upsilon_j^0 \vee \bigvee_{j=1}^{m_0} l_j^0, \dots, \bigvee_{j=0}^{k_n} \varepsilon_j^n \diamond_j^n \upsilon_j^n \vee \bigvee_{j=1}^{m_n} l_j^n \in S_{\kappa-1}^{Vr}}{\left(\bigvee_{i=0}^{n} \bigvee_{j=1}^{m_i} l_j^i\right)\theta \in S_\kappa};$$

$$\textit{for all } i < i' \leq n, \mathit{freevars}(\bigvee_{j=0}^{k_i} \varepsilon_j^i \diamond_j^i \upsilon_j^i \vee \bigvee_{j=1}^{m_i} l_j^i) \cap \mathit{freevars}(\bigvee_{j=0}^{k_{i'}} \varepsilon_j^{i'} \diamond_j^{i'} \upsilon_j^{i'} \vee \bigvee_{j=1}^{m_{i'}} l_j^{i'}) = \emptyset,$$
$$\theta \in \mathit{mgu}_{\mathcal{L}_{\kappa-1}}\left(\bigvee_{j=0}^{k_0} \varepsilon_j^0 \diamond_j^0 \upsilon_j^0, l_1^0, \dots, l_{m_0}^0, \dots, \bigvee_{j=0}^{k_n} \varepsilon_j^n \diamond_j^n \upsilon_j^n, l_1^n, \dots, l_{m_n}^n, \{\upsilon_0^0, \varepsilon_0^1\}, \dots, \{\upsilon_0^{n-1}, \varepsilon_0^n\}, \{a, b\}\right),$$
$$\mathit{dom}(\theta) = \mathit{freevars}\left(\{\varepsilon_j^i \diamond_j^i \upsilon_j^i \,|\, j \leq k_i, i \leq n\}, \{l_j^i \,|\, 1 \leq j \leq m_i, i \leq n\}\right), a = \varepsilon_0^0, b = \mathit{1} \textit{ or } a = \upsilon_0^n, b = \mathit{0} \textit{ or } a = \varepsilon_0^0, b = \upsilon_0^n, \textit{ there exists } i^* \leq n \textit{ such that } \diamond_0^{i^*} = \prec.$$

$\left(\bigvee_{i=0}^{n} \bigvee_{j=1}^{m_i} l_j^i\right)\theta$ is an order hyperresolvent of $\bigvee_{j=0}^{k_0} \varepsilon_j^0 \diamond_j^0 \upsilon_j^0 \vee \bigvee_{j=1}^{m_0} l_j^0, \dots, \bigvee_{j=0}^{k_n} \varepsilon_j^n \diamond_j^n \upsilon_j^n \vee \bigvee_{j=1}^{m_n} l_j^n$.

(*Order trichotomy rule*) (47)

$$\frac{a, b \in \mathit{atoms}(S_{\kappa-1}^{Vr}), \{a, b\} \not\subseteq \mathit{Tcons}_{\mathcal{L}}, \mathit{qatoms}(S) \neq \emptyset}{a \prec b \vee a \eqcirc b \vee b \prec a \in S_\kappa};$$

$$\mathit{vars}(a) \cap \mathit{vars}(b) = \emptyset.$$

$a \prec b \vee a \eqcirc b \vee b \prec a$ is an order trichotomy resolvent of a and b.

(*Order $\forall$-quantification rule*) (48)

$$\frac{\forall x\, a \in \mathit{qatoms}^{\forall}(S_{\kappa-1})}{\forall x\, a \prec a \vee \forall x\, a \eqcirc a \in S_\kappa}.$$

$\forall x\, a \prec a \vee \forall x\, a \eqcirc a$ is an order $\forall$-quantification resolvent of $\forall x\, a$.

(*Order $\exists$-quantification rule*) (49)

$$\frac{\exists x\, a \in \mathit{qatoms}^{\exists}(S_{\kappa-1})}{a \prec \exists x\, a \vee a \eqcirc \exists x\, a \in S_\kappa}.$$

$a \prec \exists x\, a \vee a \eqcirc \exists x\, a$ is an order $\exists$-quantification resolvent of $\exists x\, a$.

(*Order $\forall$-witnessing rule*) (50)

$$\frac{\forall x\, a \in \mathit{qatoms}^{\forall}(S_{\kappa-1}^{Vr}), b \in \mathit{atoms}(S_{\kappa-1}^{Vr}) \cup \mathit{qatoms}(S_{\kappa-1}^{Vr})}{a\gamma \prec b \vee b \eqcirc \forall x\, a \vee b \prec \forall x\, a \in S_\kappa};$$

$$\mathit{freevars}(\forall x\, a) \cap \mathit{freevars}(b) = \emptyset, \tilde{w} \in \tilde{\mathbb{W}} - \mathit{Func}_{\mathcal{L}_{\kappa-1}}, \mathit{ar}(\tilde{w}) = |\mathit{freetermseq}(\forall x\, a), \mathit{freetermseq}(b)|,$$
$$\gamma = x/\tilde{w}(\mathit{freetermseq}(\forall x\, a), \mathit{freetermseq}(b)) \cup \mathit{id}|_{\mathit{vars}(a)-\{x\}} \in \mathit{Subst}_{\mathcal{L}_\kappa}, \mathit{dom}(\gamma) = \{x\} \cup (\mathit{vars}(a) - \{x\}) = \mathit{vars}(a).$$

$a\gamma \prec b \vee b \eqcirc \forall x\, a \vee b \prec \forall x\, a$ is an order $\forall$-witnessing resolvent of $\forall x\, a$ and b.

(*Order $\exists$-witnessing rule*) (51)

$$\frac{\exists x\, a \in \mathit{qatoms}^{\exists}(S_{\kappa-1}^{Vr}), b \in \mathit{atoms}(S_{\kappa-1}^{Vr}) \cup \mathit{qatoms}(S_{\kappa-1}^{Vr})}{b \prec a\gamma \vee \exists x\, a \eqcirc b \vee \exists x\, a \prec b \in S_\kappa};$$

$$\mathit{freevars}(\exists x\, a) \cap \mathit{freevars}(b) = \emptyset, \tilde{w} \in \tilde{\mathbb{W}} - \mathit{Func}_{\mathcal{L}_{\kappa-1}}, \mathit{ar}(\tilde{w}) = |\mathit{freetermseq}(\exists x\, a), \mathit{freetermseq}(b)|,$$
$$\gamma = x/\tilde{w}(\mathit{freetermseq}(\exists x\, a), \mathit{freetermseq}(b)) \cup \mathit{id}|_{\mathit{vars}(a)-\{x\}} \in \mathit{Subst}_{\mathcal{L}_\kappa}, \mathit{dom}(\gamma) = \{x\} \cup (\mathit{vars}(a) - \{x\}) = \mathit{vars}(a).$$

$b \prec a\gamma \vee \exists x\, a \eqcirc b \vee \exists x\, a \prec b$ is an order $\exists$-witnessing resolvent of $\exists x\, a$ and b.

$GOrdCl_{\mathcal{L}\cup\tilde{\mathbb{W}}\cup P}$. There exists $\emptyset \neq I^* \subseteq \{i \,|\, i \leq n\}$ such that $\bigvee_{i\in I^*} C_i \in clo^{\mathcal{BH}}(S)$.

Proof. Straightforward. ☐

Lemma 4.3 (Unit Lemma)**.** *Let $\mathcal{L}$ contain a constant symbol. Let $P \subseteq \tilde{\mathbb{P}}$ and $S \subseteq OrdCl_{\mathcal{L}\cup P}$. Let $\square \notin clo^{\mathcal{BH}}(S) = \{\bigvee_{j=0}^{k_\iota} \varepsilon_j^\iota \diamond_j^\iota \upsilon_j^\iota \,|\, \iota < \gamma\}, \gamma \leq \omega$. There exists $\mathcal{S}^* \in \mathcal{S}el(\{\{j \,|\, j \leq k_\iota\}_\iota \,|\, \iota < \gamma\})$ such that there does not exist a contradiction of $\{\varepsilon^\iota_{\mathcal{S}^*(\iota)} \diamond^\iota_{\mathcal{S}^*(\iota)} \upsilon^\iota_{\mathcal{S}^*(\iota)} \,|\, \iota < \gamma\} \subseteq GOrdCl_{\mathcal{L}\cup\tilde{\mathbb{W}}\cup P}$.*

Proof. An immediate consequence of König's Lemma and Lemma 4.2. ☐

We are in position to prove the refutational soundness and completeness of the order hyperresolution calculus. Let $\{\mathit{0}, \mathit{1}\} \subseteq X \subseteq Tcons_{\mathcal{L}}$. X is admissible with respect to suprema and infima iff, for all $Y_1, Y_2 \subseteq X$ and $\bigvee \overline{Y_1} = \bigwedge \overline{Y_2}$, either $\bigvee \overline{Y_1} \in \overline{Y_1}$, $\bigwedge \overline{Y_2} \in \overline{Y_2}$, or $\bigvee \overline{Y_1} \notin \overline{Y_1}$, $\bigwedge \overline{Y_2} \notin \overline{Y_2}$.

Theorem 4.4 (Refutational Soundness and Completeness)**.** *Let $\mathcal{L}$ contain a constant symbol. Let $P \subseteq \tilde{\mathbb{P}}$, $S \subseteq OrdCl_{\mathcal{L}\cup P}$, $tcons(S)$ be admissible with respect to* *suprema and infima.* $\square \in clo^{\mathcal{H}}(S)$ *if and only if* S *is unsatisfiable.*

Proof. ($\Longrightarrow$) Let $\mathfrak{A}$ be a model of S for $\mathcal{L} \cup P$ and $C \in clo^{\mathcal{H}}(S) \subseteq OrdCl_{\mathcal{L}\cup\tilde{\mathbb{W}}\cup P}$. Then there exists an expansion $\mathfrak{A}'$ of $\mathfrak{A}$ to $\mathcal{L} \cup \tilde{\mathbb{W}} \cup P$ such that $\mathfrak{A}' \models C$. The proof is by complete induction on the length of a deduction of C from S by order hyperresolution. Let $\square \in clo^{\mathcal{H}}(S)$ and $\mathfrak{A}$ be a model of S for $\mathcal{L} \cup P$. Hence, there exists an expansion $\mathfrak{A}'$ of $\mathfrak{A}$ to $\mathcal{L} \cup \tilde{\mathbb{W}} \cup P$ such that $\mathfrak{A}' \models \square$, which is a contradiction; S is unsatisfiable.

($\Longleftarrow$) Let $\square \notin clo^{\mathcal{H}}(S)$. Then, by Lemma 4.1 for S, $\square$, $\square \notin clo^{\mathcal{BH}}(S)$; we have $\mathcal{L}$, $\tilde{\mathbb{P}}$, $\tilde{\mathbb{W}}$ are countable, $P \subseteq \tilde{\mathbb{P}}$, $S \subseteq OrdCl_{\mathcal{L}\cup P}$, $clo^{\mathcal{BH}}(S) \subseteq GOrdCl_{\mathcal{L}\cup\tilde{\mathbb{W}}\cup P}$; P, $\mathcal{L} \cup P$, $OrdCl_{\mathcal{L}\cup P}$, S, $\mathcal{L} \cup \tilde{\mathbb{W}} \cup P$, $GOrdCl_{\mathcal{L}\cup\tilde{\mathbb{W}}\cup P}$, $clo^{\mathcal{BH}}(S)$ are countable; there exists $\gamma_1 \leq \omega$ and $\square \notin clo^{\mathcal{BH}}(S) = \{\bigvee_{j=0}^{k_\iota} \varepsilon_j^\iota \diamond_j^\iota \upsilon_j^\iota \mid \iota < \gamma_1\}$; by Lemma 4.3 for S, there exists $\mathcal{S}^* \in \mathcal{S}el(\{\{j \mid j \leq k_\iota\}_\iota \mid \iota < \gamma_1\})$ and there does not exist a contradiction of $\{\varepsilon^\iota_{\mathcal{S}^*(\iota)} \diamond^\iota_{\mathcal{S}^*(\iota)} \upsilon^\iota_{\mathcal{S}^*(\iota)} \mid \iota < \gamma_1\} \subseteq GOrdCl_{\mathcal{L}\cup\tilde{\mathbb{W}}\cup P}$. We put $\mathbb{S} = \{\varepsilon^\iota_{\mathcal{S}^*(\iota)} \diamond^\iota_{\mathcal{S}^*(\iota)} \upsilon^\iota_{\mathcal{S}^*(\iota)} \mid \iota < \gamma_1\} \subseteq GOrdCl_{\mathcal{L}\cup\tilde{\mathbb{W}}\cup P}$. Then $\mathbb{S} \supseteq ordtcons(S)$ is countable, unit, $(q)atoms(\mathbb{S}) \subseteq (q)atoms(clo^{\mathcal{BH}}(S))$. We put

$$\mathcal{U}_{\mathfrak{A}} = \begin{cases} GTerm_{\mathcal{L}\cup P} & \text{if } qatoms(S) = \emptyset, \\ GTerm_{\mathcal{L}\cup\tilde{\mathbb{W}}\cup P} & \text{else}, \end{cases} \qquad \mathcal{U}_{\mathfrak{A}} \neq \emptyset,$$

and $\mathcal{B} = atoms(\mathbb{S}) \cup qatoms(\mathbb{S}) \subseteq GAtom_{\mathcal{L}\cup\tilde{\mathbb{W}}\cup P} \cup QAtom_{\mathcal{L}\cup\tilde{\mathbb{W}}\cup P}$. We have $\mathbb{S}$ is countable. Then $tcons(S) \subseteq atoms(ordtcons(S)) \subseteq atoms(\mathbb{S}) \subseteq \mathcal{B}$, $\mathcal{B} = tcons(S) \cup (\mathcal{B} - tcons(S))$, $tcons(S) \cap (\mathcal{B} - tcons(S)) = \emptyset$, $atoms(\mathbb{S})$, $qatoms(\mathbb{S})$, $\mathcal{B}$, $tcons(S)$, $\mathcal{B} - tcons(S)$ are countable; there exist $\gamma_2 \leq \omega$ and a sequence $\delta_2 : \gamma_2 \longrightarrow \mathcal{B} - tcons(S)$ of $\mathcal{B} - tcons(S)$. Let $\varepsilon_1, \varepsilon_2 \in \mathcal{B}$. $\varepsilon_1 \triangleq \varepsilon_2$ iff there exists an equality chain $\varepsilon_1 \Xi \varepsilon_2$ of $\mathbb{S}$. Note that $\triangleq$ is a binary symmetric transitive relation on $\mathcal{B}$. $\varepsilon_1 \lhd \varepsilon_2$ iff there exists an increasing chain $\varepsilon_1 \Xi \varepsilon_2$ of $\mathbb{S}$. Note that $\lhd$ is a binary transitive relation on $\mathcal{B}$.

$$\begin{aligned} &0 \not\triangleq 1, 1 \not\triangleq 0, 0 \lhd 1, 1 \not\lhd 0, \\ &\text{for all } \varepsilon \in \mathcal{B}, \varepsilon \not\lhd 0, 1 \not\lhd \varepsilon, \varepsilon \not\lhd \varepsilon. \end{aligned} \tag{52}$$

The proof is straightforward; we have that there does not exist a contradiction of $\mathbb{S}$. Note that $\lhd$ is also irreflexive and a partial strict order on $\mathcal{B}$.

Let $tcons(S) \subseteq X \subseteq \mathcal{B}$. A partial valuation $\mathcal{V}$ is a mapping $\mathcal{V} : X \longrightarrow [0,1]$ such that $\mathcal{V}(0) = 0$, $\mathcal{V}(1) = 1$, for all $\bar{c} \in tcons(S) \cap \overline{C}_{\mathcal{L}}$, $\mathcal{V}(\bar{c}) = c$. We denote $dom(\mathcal{V}) = X$, $tcons(S) \subseteq dom(\mathcal{V}) \subseteq \mathcal{B}$. We define a partial valuation $\mathcal{V}_\alpha$ by recursion on $\alpha \leq \gamma_2$ in Table 8.

Table 8: $\mathcal{V}_\alpha$.

$$\mathcal{V}_0 = \{(0,0),(1,1)\} \cup \{(\bar{c}, c) \mid \bar{c} \in tcons(S) \cap \overline{C}_{\mathcal{L}}\};$$

$$\mathcal{V}_\alpha = \mathcal{V}_{\alpha-1} \cup \{(\delta_2(\alpha-1), \lambda_{\alpha-1})\} \quad (1 \leq \alpha \leq \gamma_2 \text{ is a successor ordinal}),$$

$$\mathbb{E}_{\alpha-1} = \{\mathcal{V}_{\alpha-1}(a) \mid a \triangleq \delta_2(\alpha-1), a \in dom(\mathcal{V}_{\alpha-1})\},$$

$$\mathbb{D}_{\alpha-1} = \{\mathcal{V}_{\alpha-1}(a) \mid a \lhd \delta_2(\alpha-1), a \in dom(\mathcal{V}_{\alpha-1})\},$$

$$\mathbb{U}_{\alpha-1} = \{\mathcal{V}_{\alpha-1}(a) \mid \delta_2(\alpha-1) \lhd a, a \in dom(\mathcal{V}_{\alpha-1})\},$$

$$\lambda_{\alpha-1} = \begin{cases} \dfrac{\bigvee \mathbb{D}_{\alpha-1} + \bigwedge \mathbb{U}_{\alpha-1}}{2} & \text{if } \mathbb{E}_{\alpha-1} = \emptyset, \\ \bigvee \mathbb{E}_{\alpha-1} & \text{else}; \end{cases}$$

$$\mathcal{V}_{\gamma_2} = \bigcup_{\alpha < \gamma_2} \mathcal{V}_\alpha \quad (\gamma_2 \text{ is a limit ordinal})$$

$$\text{For all } \alpha \leq \alpha' \leq \gamma_2,\ \mathcal{V}_\alpha \text{ is a partial valuation, } dom(\mathcal{V}_\alpha) = tcons(S) \cup \delta_2[\alpha],\ \mathcal{V}_\alpha \subseteq \mathcal{V}_{\alpha'}. \tag{53}$$

The proof is by induction on $\alpha \leq \gamma_2$.

$$\begin{aligned} &\text{For all } \alpha \leq \gamma_2, \text{ for all } a, b \in dom(\mathcal{V}_\alpha), \\ &\text{if } a \triangleq b, \text{ then } \mathcal{V}_\alpha(a) = \mathcal{V}_\alpha(b); \\ &\text{if } a \lhd b, \text{ then } \mathcal{V}_\alpha(a) < \mathcal{V}_\alpha(b). \end{aligned} \tag{54}$$

The proof is by induction on $\alpha \leq \gamma_2$ using the assumption that $tcons(S)$ is admissible with respect to suprema and infima.

We put $\mathcal{V} = \mathcal{V}_{\gamma_2}$, $dom(\mathcal{V}) \overset{(53)}{=\!=} tcons(S) \cup \delta_2[\gamma_2] = \mathcal{B}$.

$$\begin{aligned} &\text{For all } a, b \in \mathcal{B} = dom(\mathcal{V}), \\ &\text{if } a \triangleq b, \text{ then } \mathcal{V}(a) = \mathcal{V}(b); \\ &\text{if } a \lhd b, \text{ then } \mathcal{V}(a) < \mathcal{V}(b); \\ &\text{if } a = \forall x c, \text{ then } \mathcal{V}(a) = \textstyle\bigwedge_{u \in \mathcal{U}_{\mathfrak{A}}} \mathcal{V}(c(x/u)); \\ &\text{if } a = \exists x c, \text{ then } \mathcal{V}(a) = \textstyle\bigvee_{u \in \mathcal{U}_{\mathfrak{A}}} \mathcal{V}(c(x/u)). \end{aligned} \tag{55}$$

The proof. A straightforward consequence of (54).

We put

$$
\begin{aligned}
&\mathit{if}\ qatoms(S) = \emptyset, \\
&\quad f^{\mathfrak{A}}(u_1,\dots,u_\tau) = f(u_1,\dots,u_\tau), \\
&\qquad\qquad f \in Func_{\mathcal{L}\cup P}, u_i \in \mathcal{U}_{\mathfrak{A}}; \\
&\quad p^{\mathfrak{A}}(u_1,\dots,u_\tau) = \\
&\qquad \begin{cases} \mathcal{V}(p(u_1,\dots,u_\tau)) & \mathit{if}\ p(u_1,\dots,u_\tau) \in \mathcal{B}, \\ 0 & \mathit{else}, \end{cases} \\
&\qquad\qquad p \in Pred_{\mathcal{L}\cup P}, u_i \in \mathcal{U}_{\mathfrak{A}}; \\
&\quad \mathfrak{A} = \big(\mathcal{U}_{\mathfrak{A}}, \{f^{\mathfrak{A}} \,|\, f \in Func_{\mathcal{L}\cup P}\}, \\
&\qquad \{p^{\mathfrak{A}} \,|\, p \in Pred_{\mathcal{L}\cup P}\}\big), \\
&\qquad \mathit{an\ interpretation\ for}\ \mathcal{L}\cup P;
\end{aligned}
$$

$$
\begin{aligned}
&\mathit{if}\ qatoms(S) \neq \emptyset, \\
&\quad f^{\mathfrak{A}}(u_1,\dots,u_\tau) = f(u_1,\dots,u_\tau), \\
&\qquad\qquad f \in Func_{\mathcal{L}\cup\tilde{\mathbb{W}}\cup P}, u_i \in \mathcal{U}_{\mathfrak{A}}; \\
&\quad p^{\mathfrak{A}}(u_1,\dots,u_\tau) = \\
&\qquad \begin{cases} \mathcal{V}(p(u_1,\dots,u_\tau)) & \mathit{if}\ p(u_1,\dots,u_\tau) \in \mathcal{B}, \\ 0 & \mathit{else}, \end{cases} \\
&\qquad\qquad p \in Pred_{\mathcal{L}\cup\tilde{\mathbb{W}}\cup P}, u_i \in \mathcal{U}_{\mathfrak{A}};
\end{aligned}
$$

$$
\begin{aligned}
&\mathfrak{A} = \big(\mathcal{U}_{\mathfrak{A}}, \{f^{\mathfrak{A}} \,|\, f \in Func_{\mathcal{L}\cup\tilde{\mathbb{W}}\cup P}\}, \\
&\quad \{p^{\mathfrak{A}} \,|\, p \in Pred_{\mathcal{L}\cup\tilde{\mathbb{W}}\cup P}\}\big), \\
&\quad \mathit{an\ interpretation\ for}\ \mathcal{L}\cup\tilde{\mathbb{W}}\cup P.
\end{aligned}
$$

Hence, it is straightforward to prove that for all $a \in \mathcal{B}$ and $e \in \mathcal{S}_{\mathfrak{A}}$, $\|a\|_e^{\mathfrak{A}} = \mathcal{V}(a)$; for all $l \in \mathbb{S}$ and $e \in \mathcal{S}_{\mathfrak{A}}$, $\|l\|_e^{\mathfrak{A}} = 1$; for all $C \in S$ and $e \in \mathcal{S}_{\mathfrak{A}}$, $e|_{freevars(C)} \in Subst_{\mathcal{L}\cup\tilde{\mathbb{W}}\cup P}$, $dom(e|_{freevars(C)}) = freevars(C)$, $range(e|_{freevars(C)}) = \emptyset$, $C(e|_{freevars(C)}) \in clo^{\mathcal{BH}}(S)$, there exists $l^* \in C(e|_{freevars(C)})$ and $l^* \in \mathbb{S}$, $\|l^*\|_e^{\mathfrak{A}} = 1$; there exists $l^{**} \in C$ and $l^* = l^{**}(e|_{freevars(C)})$, $\|l^{**}\|_e^{\mathfrak{A}} = \|l^{**}(e|_{freevars(C)})\|_e^{\mathfrak{A}} = \|l^*\|_e^{\mathfrak{A}} = 1$; $\mathfrak{A} \models_e C$; $\mathfrak{A} \models S$, $\mathfrak{A}|_{\mathcal{L}\cup P} \models S$; S is satisfiable. $\square$

Consider $S = \{0 \prec a\} \cup \{a \prec \overline{\tfrac{1}{n}} \,|\, n \in \mathbb{N}\} \subseteq OrdCl_{\mathcal{L}}$, $a \in Pred_{\mathcal{L}} - Tcons_{\mathcal{L}}$, $ar_{\mathcal{L}}(a) = 0$. $tcons(S)$ is not admissible with respect to suprema and infima; for $\{0\}$ and $\{\overline{\tfrac{1}{n}} \,|\, n \in \mathbb{N}\}$, $\bigvee\{0\} = \bigwedge\{\tfrac{1}{n} \,|\, n \in \mathbb{N}\} = 0$, $0 \in \{0\}$, $0 \notin \{\tfrac{1}{n} \,|\, n \in \mathbb{N}\}$. S is unsatisfiable; both the cases $\|a\|^{\mathfrak{A}} = 0$ and $\|a\|^{\mathfrak{A}} > 0$ lead to $\mathfrak{A} \not\models S$ for every interpretation $\mathfrak{A}$ for $\mathcal{L}$. However, $\square \notin clo^{\mathcal{H}}(S) = S$. So, the condition on $tcons(S)$ being admissible with respect to suprema and infima, is necessary.

Corollary 4.5. *Let $\mathcal{L}$ contain a constant symbol. Let $n_0 \in \mathbb{N}$, $\phi \in Form_{\mathcal{L}}$, $T \subseteq Form_{\mathcal{L}}$, $tcons(T)$ be admissible with respect to suprema and infima. There exist $J_T^{\phi} \subseteq \{(i,j) \,|\, i \geq n_0\}$ and $S_T^{\phi} \subseteq SimOrdCl_{\mathcal{L}\cup\{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_T^{\phi}\}}$ such that $tcons(S_T^{\phi})$ is admissible with respect to suprema and infima; $T \models \phi$ if and only if $\square \in clo^{\mathcal{H}}(S_T^{\phi})$.*

Proof. A straightforward consequence of Corollary 3.5 and Theorem 4.4. $\square$

In Table 9, we show that $\phi = \forall x\, (q_1(x) \rightarrow \overline{0.3}) \rightarrow (\exists x\, q_1(x) \rightarrow \overline{0.5}) \in Form_{\mathcal{L}}$ is logically valid using the proposed translation to order clausal form and the order hyperresolution calculus.

5 CONCLUSIONS

In the paper, we have proposed a modification of the hyperresolution calculus from (Guller, 2012) which is suitable for automated deduction with explicit partial truth. The first-order Gödel logic is expanded by a countable set of intermediate truth constants $\bar{c}$, $c \in (0,1)$. We have modified translation of a formula to an equivalent satisfiable finite order clausal theory, consisting of order clauses. An order clause is a finite set of order literals of the form $\varepsilon_1 \diamond \varepsilon_2$ where $\diamond$ is a connective either $\eqcirc$ or $\prec$. $\eqcirc$ and $\prec$ are interpreted by the equality and standard strict linear order on $[0,1]$, respectively. We have investigated the so-called canonical standard completeness, where the semantics of the first-order Gödel logic is given by the standard G-algebra and truth constants are interpreted by themselves. The modified hyperresolution calculus is refutation sound and complete for a countable order clausal theory if the set of all truth constants occurring in the theory is admissible with respect to suprema and infima. This condition covers the case of finite order clausal theories.

Let $\phi \in Form_{\mathcal{L}}$; ϕ contains a finite number of truth constants. Then the problem that ϕ is unsatisfiable can be reduced to the deduction problem $\phi \models 0$ (after a constant number of steps). As an immediate consequence of Corollary 3.5 and Theorem 4.4, if $\phi \models 0$, then we can decide it after a finite number of steps. This straightforwardly implies that the set of unsatisfiable formulae of $\mathcal{L}$ (in the general first-order Gödel logic with intermediate truth constants) is recursively enumerable.

Table 9: An example: $\phi = \forall x\,(q_1(x) \rightarrow \overline{0.3}) \rightarrow (\exists x\, q_1(x) \rightarrow \overline{0.5})$.

$$\phi = \forall x\,(q_1(x) \rightarrow \overline{0.3}) \rightarrow (\exists x\, q_1(x) \rightarrow \overline{0.5})$$

$$\{\tilde{p}_0(x) \prec 1, \big(\underbrace{\forall x\,(q_1(x) \rightarrow \overline{0.3})}_{\tilde{p}_1(x)} \rightarrow \underbrace{(\exists x\, q_1(x) \rightarrow \overline{0.5})}_{\tilde{p}_2(x)}\big) \rightarrow \tilde{p}_0(x)\} \qquad (27)$$

$$\{\tilde{p}_0(x) \prec 1, \tilde{p}_2(x) \prec \tilde{p}_1(x) \vee \tilde{p}_2(x) \eqcirc 1 \vee \tilde{p}_0(x) \eqcirc 1, \tilde{p}_2(x) \prec \tilde{p}_0(x) \vee \tilde{p}_2(x) \eqcirc \tilde{p}_0(x), \tilde{p}_1(x) \rightarrow \forall x\,(\underbrace{q_1(x) \rightarrow \overline{0.3}}_{\tilde{p}_3(x)}), (\underbrace{\exists x\, q_1(x)}_{\tilde{p}_4(x)} \rightarrow \underbrace{\overline{0.5}}_{\tilde{p}_5(x)}) \rightarrow \tilde{p}_2(x)\} \qquad (33),(27)$$

$$\{\tilde{p}_0(x) \prec 1, \tilde{p}_2(x) \prec \tilde{p}_1(x) \vee \tilde{p}_2(x) \eqcirc 1 \vee \tilde{p}_0(x) \eqcirc 1, \tilde{p}_2(x) \prec \tilde{p}_0(x) \vee \tilde{p}_2(x) \eqcirc \tilde{p}_0(x), \tilde{p}_1(x) \prec \forall x\, \tilde{p}_3(x) \vee \tilde{p}_1(x) \eqcirc \forall x\, \tilde{p}_3(x), \tilde{p}_3(x) \rightarrow (\underbrace{q_1(x)}_{\tilde{p}_6(x)} \rightarrow \underbrace{\overline{0.3}}_{\tilde{p}_7(x)}),$$

$$\tilde{p}_5(x) \prec \tilde{p}_4(x) \vee \tilde{p}_5(x) \eqcirc 1 \vee \tilde{p}_2(x) \eqcirc 1, \tilde{p}_5(x) \prec \tilde{p}_2(x) \vee \tilde{p}_5(x) \eqcirc \tilde{p}_2(x), \tilde{p}_4(x) \rightarrow \exists x\, \underbrace{q_1(x)}_{\tilde{p}_8(x)}, \overline{0.5} \prec \tilde{p}_5(x) \vee \overline{0.5} \eqcirc \tilde{p}_5(x)\} \qquad (25),(37)$$

$$S^{\phi} = \left\{ \begin{array}{ll} \boxed{\tilde{p}_0(x) \prec 1} & [1] \\ \tilde{p}_2(x) \prec \tilde{p}_1(x) \vee \tilde{p}_2(x) \eqcirc 1 \vee \boxed{\tilde{p}_0(x) \eqcirc 1} & [2] \\ \boxed{\tilde{p}_2(x) \prec \tilde{p}_0(x)} \vee \tilde{p}_2(x) \eqcirc \tilde{p}_0(x) & [3] \\ \boxed{\tilde{p}_1(x) \prec \forall x\, \tilde{p}_3(x) \vee \tilde{p}_1(x) \eqcirc \forall x\, \tilde{p}_3(x)} & [4] \\ \boxed{\tilde{p}_3(x) \prec \tilde{p}_7(x) \vee \tilde{p}_3(x) \eqcirc \tilde{p}_7(x)} \vee \tilde{p}_6(x) \prec \tilde{p}_7(x) \vee \tilde{p}_6(x) \eqcirc \tilde{p}_7(x) & [5] \\ \boxed{q_1(x) \prec \tilde{p}_6(x) \vee q_1(x) \eqcirc \tilde{p}_6(x)} & [6] \\ \boxed{\tilde{p}_7(x) \prec \overline{0.3} \vee \tilde{p}_7(x) \eqcirc \overline{0.3}} & [7] \\ \tilde{p}_5(x) \prec \tilde{p}_4(x) \vee \tilde{p}_5(x) \eqcirc 1 \vee \boxed{\tilde{p}_2(x) \eqcirc 1} & [8] \\ \boxed{\boxed{\tilde{p}_5(x) \prec \tilde{p}_2(x)} \vee \tilde{p}_5(x) \eqcirc \tilde{p}_2(x)} & [9] \\ \boxed{\tilde{p}_4(x) \prec \exists x\, \tilde{p}_8(x) \vee \tilde{p}_4(x) \eqcirc \exists x\, \tilde{p}_8(x)} & [10] \\ \boxed{\tilde{p}_8(x) \prec q_1(x) \vee \tilde{p}_8(x) \eqcirc q_1(x)} & [11] \\ \boxed{\overline{0.5} \prec \tilde{p}_5(x) \vee \overline{0.5} \eqcirc \tilde{p}_5(x)} & [12] \end{array} \right\}$$

Rule (46) : [1][2] :

$$\tilde{p}_2(x) \prec \tilde{p}_1(x) \vee \boxed{\tilde{p}_2(x) \eqcirc 1} \qquad [13]$$

Rule (46) : [3][13] :

$$\boxed{\tilde{p}_2(x) \eqcirc \tilde{p}_0(x)} \vee \tilde{p}_2(x) \prec \tilde{p}_1(x) \qquad [14]$$

Rule (46) : [1][13][14] :

$$\boxed{\tilde{p}_2(x) \prec \tilde{p}_1(x)} \qquad [15]$$

Rule (46) : [8][15] :

$$\tilde{p}_5(x) \prec \tilde{p}_4(x) \vee \boxed{\tilde{p}_5(x) \eqcirc 1} \qquad [16]$$

Rule (46) : [9][16] :

$$\boxed{\tilde{p}_5(x) \eqcirc \tilde{p}_2(x)} \vee \tilde{p}_5(x) \prec \tilde{p}_4(x) \qquad [17]$$

Rule (46) : [15][16][17] :

$$\boxed{\tilde{p}_5(x) \prec \tilde{p}_4(x)} \qquad [18]$$

Rule (48) : $\forall x\, \tilde{p}_3(x)$:

$$\boxed{\forall x\, \tilde{p}_3(x) \prec \tilde{p}_3(x) \vee \forall x\, \tilde{p}_3(x) \eqcirc \tilde{p}_3(x)} \qquad [19]$$

repeatedly **Rule (46)** : [4][5][7][9][12][15][19] :

$$\boxed{\tilde{p}_6(x) \prec \tilde{p}_7(x) \vee \tilde{p}_6(x) \eqcirc \tilde{p}_7(x)} \qquad [20]$$

Rule (51) : $\exists x\, \tilde{p}_8(x), \overline{0.5}$:

$$\overline{0.5} \prec \tilde{p}_8(\tilde{w}_{(0,0)}) \vee \boxed{\exists x\, \tilde{p}_8(x) \prec \overline{0.5} \vee \exists x\, \tilde{p}_8(x) \eqcirc \overline{0.5}} \qquad [21]$$

repeatedly **Rule (46)** : [10][12][18][21] :

$$\boxed{\overline{0.5} \prec \tilde{p}_8(\tilde{w}_{(0,0)})} \qquad [22]$$

repeatedly **Rule (46)** : [6][7][11][20]; $\tilde{w}_{(0,0)}$: [22] :

$$\square \qquad [23]$$

REFERENCES

Biere, A., Heule, M. J., van Maaren, H., and Walsh, T. (2009). *Handbook of Satisfiability*, volume 185 of *Frontiers in Artificial Intelligence and Applications*. IOS Press, Amsterdam.

Davis, M., Logemann, G., and Loveland, D. (1962). A machine program for theorem-proving. *Commun. ACM*, 5(7):394–397.

Davis, M. and Putnam, H. (1960). A computing procedure for quantification theory. *J. ACM*, 7(3):201–215.

de la Tour, T. B. (1992). An optimality result for clause form translation. *J. Symb. Comput.*, 14(4):283–302.

Esteva, F., Gispert, J., Godo, L., and Noguera, C. (2007a). Adding truth-constants to logics of continuous t-norms: axiomatization and completeness results. *Fuzzy Sets and Systems*, 158(6):597–618.

Esteva, F., Godo, L., and Montagna, F. (2001). The ŁΠ and Ł$\Pi\frac{1}{2}$ logics: two complete fuzzy systems joining Łukasiewicz and Product logics. *Arch. Math. Log.*, 40(1):39–67.

Esteva, F., Godo, L., and Noguera, C. (2007b). On completeness results for the expansions with truth-constants of some predicate fuzzy logics. In Stepnicka, M., Novák, V., and Bodenhofer, U., editors, *New Dimensions in Fuzzy Logic and Related Technologies. Proceedings of the 5th EUSFLAT Conference, Ostrava, Czech Republic, September 11-14, 2007, Volume 2: Regular Sessions*, pages 21–26. Universitas Ostraviensis.

Esteva, F., Godo, L., and Noguera, C. (2009). First-order t-norm based fuzzy logics with truth-constants: distinguished semantics and completeness properties. *Ann. Pure Appl. Logic*, 161(2):185–202.

Esteva, F., Godo, L., and Noguera, C. (2010a). Expanding the propositional logic of a t-norm with truth-constants: completeness results for rational semantics. *Soft Comput.*, 14(3):273–284.

Esteva, F., Godo, L., and Noguera, C. (2010b). On expansions of WNM t-norm based logics with truth-constants. *Fuzzy Sets and Systems*, 161(3):347–368.

Gallier, J. H. (1985). *Logic for Computer Science: Foundations of Automatic Theorem Proving*. Harper & Row Publishers, Inc., New York, NY, USA.

Guller, D. (2010). A DPLL procedure for the propositional Gödel logic. In Filipe, J. and Kacprzyk, J., editors, *ICFC-ICNC 2010 - Proceedings of the International Conference on Fuzzy Computation and International Conference on Neural Computation, [parts of the International Joint Conference on Computational Intelligence IJCCI 2010], Valencia, Spain, October 24-26, 2010*, pages 31–42. SciTePress.

Guller, D. (2012). An order hyperresolution calculus for Gödel logic - General first-order case. In Rosa, A. C., Correia, A. D., Madani, K., Filipe, J., and Kacprzyk, J., editors, *IJCCI 2012 - Proceedings of the 4th International Joint Conference on Computational Intelligence, Barcelona, Spain, 5 - 7 October, 2012*, pages 329–342. SciTePress.

Hähnle, R. (1994). Short conjunctive normal forms in finitely valued logics. *J. Log. Comput.*, 4(6):905–927.

Hájek, P. (2001). *Metamathematics of Fuzzy Logic*. Trends in Logic. Springer.

Nonnengart, A., Rock, G., and Weidenbach, C. (1998). On generating small clause normal forms. In Kirchner, C. and Kirchner, H., editors, *Automated Deduction - CADE-15, 15th International Conference on Automated Deduction, Lindau, Germany, July 5-10, 1998, Proceedings*, volume 1421 of *Lecture Notes in Computer Science*, pages 397–411. Springer.

Novák, V., Perfilieva, I., and Močkoř, J. (1999). *Mathematical Principles of Fuzzy Logic*. The Springer International Series in Engineering and Computer Science. Springer US.

Pavelka, J. (1979). On fuzzy logic I, II, III. Semantical completeness of some many-valued propositional calculi. *Mathematical Logic Quarterly*, 25(2529):45–52, 119–134, 447–464.

Plaisted, D. A. and Greenbaum, S. (1986). A structure-preserving clause form translation. *J. Symb. Comput.*, 2(3):293–304.

Robinson, J. A. (1965a). Automatic deduction with hyper-resolution. *Internat. J. Comput. Math.*, 1(3):227–234.

Robinson, J. A. (1965b). A machine-oriented logic based on the resolution principle. *J. ACM*, 12(1):23–41.

Savický, P., Cignoli, R., Esteva, F., Godo, L., and Noguera, C. (2006). On Product logic with truth-constants. *J. Log. Comput.*, 16(2):205–225.