Nonlinear Models of BPSK Costas Loop

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Abstract: Rigorous nonlinear analysis of the *physical model* of Costas loop is very difficult task, so for analysis, simplified *mathematical models* and numerical simulation are widely used. In the work it is shown that the use of simplified mathematical models, and the application of non rigorous methods of analysis may lead to wrong conclusions concerning the operability of Costas loop.

1 INTRODUCTION

The Costas loop is a classical phase-locked loop (PLL) based circuit for carrier recovery (Costas, 1956; Costas, 1962). Nowadays various modifications of Costas loop circuit are used in many communication devices, e.g. Global Positioning Systems (GPS) (Kaplan and Hegarty, 2006). In the paper the classical analog Costas loop (Costas, 1956; Gardner, 1966; Lindsey, 1972; Best, 2007), used for BPSK demodulation, is considered (similar analysis can also be done for QPSK Costas loop).

Costas loop is essentially a nonlinear control system and its *physical model* is described by a nonlinear non-autonomous discontinuous system of differential equations (*mathematical model in the signal space*). This system is a slow-fast system since there is considered simultaneously both very fast time scale of signals and slow time scale of phase difference between the signals.

Thus in practice, for the analysis of Costas loop it is widely used various *simplified mathematical models* and their numerical simulation (Costas, 1956; Gardner, 1966; Lindsey, 1972; Best, 2007; Best et al., 2014b).

In the work it is shown that 1) the use of simplified mathematical models, and 2) the application of non rigorous methods of analysis (e.g., a simulation) may lead to wrong conclusions concerning the operability of *physical model* of Costas loop.

To demonstrate this, below Costas loop operation will be considered in details.

2 BPSK COSTAS LOOP OPERATION

Consider Costas loop operation (see Fig. 1) after transient processes.

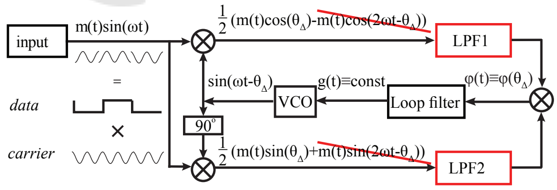

Figure 1: Costas loop is in lock (the case of nonequal frequencies of input carrier and free running VCO output): there is a constant phase difference θ_Δ after synchronization.

The input signal is BPSK signal, which is a product of the transferred data $m(t) = \pm 1$ and the harmonic carrier $\sin(\omega t)$ with a high frequency ω. Since the Costas loop is considered to be in the locked state, VCO (Voltage-Controlled Oscillator) signal is synchronized with the carrier.

Assumption 1. *The terms, whose frequency is about twice the carrier frequency, do not affect the synchronization of the loop.*

By Assumption 1 the outputs of low-pass filters LPF1 and LPF2 can be approximated in the following way. After the multiplication of VCO signal and the input signal by multiplier block ($\otimes$) on the upper branch one has

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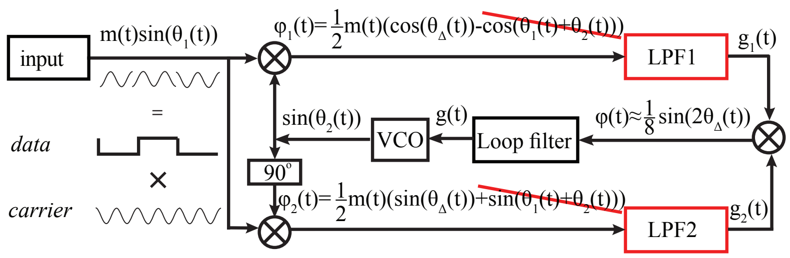

Figure 2: Costas loop is out of lock: there is time-varying phase difference $\theta_\Delta(t) = \theta_1(t) - \theta_2(t)$.

$$\varphi_1(t) = \frac{1}{2}\big(m(t)\cos(0) - m(t)\cos(2\omega t)\big) = \\ = \frac{1}{2}\big(m(t) - m(t)\cos(2\omega t)\big) \approx \frac{1}{2}m(t), \tag{1}$$

i.e the upper loop works as demodulator. On the lower branch the output signal of VCO, shifted by 90°, is multiplied by the input signal:

$$\begin{aligned} \varphi_2(t) &= m(t)\sin(\omega t)\cos(\omega t) = \\ &= \frac{1}{2}\big(m(t)\sin(0) + m(t)\sin(2\omega t)\big) = \\ &= \frac{1}{2}m(t)\sin(2\omega t) \approx 0. \end{aligned} \tag{2}$$

Since after a transient processes there is no phase difference, the control signal at the input of VCO, which is used for VCO frequency adjustment to the frequency of input carrier signal, has to be constant:

$$g(t) = const. \tag{3}$$

Consider Costas loop before synchronization (see Fig. 2) in the case when the phase of the input carrier $\theta_1(t)$ and the phase of VCO $\theta_2(t)$ are different:

$$\theta_\Delta(t) = \theta_1(t) - \theta_2(t) \neq 0. \tag{4}$$

In this case the signals $\varphi_{1,2}(t)$ on the upper and lower branches can be approximated as

$$\begin{aligned} &\varphi_1(t) = \frac{1}{2}m(t)\big(\cos(\theta_1(t) - \theta_2(t)) - \cos(\theta_1(t) + \theta_2(t))\big) \\ &\approx \frac{1}{2}m(t)\cos(\theta_\Delta(t)), \\ &\varphi_2(t) = \frac{1}{2}m(t)\big(\sin(\theta_1(t) - \theta_2(t)) + \sin(\theta_1(t) + \theta_2(t))\big) \\ &\approx \frac{1}{2}m(t)\sin(\theta_\Delta(t)). \end{aligned} \tag{5}$$

Assumption 2. *The data signal $m(t)$ does not affect the synchronization of the loop.*

Thus after multiplication of the outputs of LPF1 and LPF2 the transmitted data $m(t)$ is neglected in the signal $\varphi(t)$, i.e. one has $m(t)^2 = (\pm 1)^2 = 1$.

Approximations (6) depend on the phase difference of signals, i.e. two multiplier blocks ($\otimes$) on the upper and lower branches operate as phase detectors.

Caveat to Assumptions. Low-pass filters can not operate perfectly, especially, during the moments of changing $m(t)$, therefore the data pulse shapes are no longer ideal rectangular pulses after filtration due to distortion, created by the low-pass filters. This can lead to incorrect conclusions on the performance of the loop. One of such known examples is so-called false-lock: while for $m(t) \equiv const$ the loop acquires lock and proper synchronization of the carrier and VCO frequencies, for time-varying $m(t) \neq const$ the loop can acquire lock without proper synchronization of the frequencies (false lock) (Olson, 1975; Simon, 1978; Hedin et al., 1978). To avoid such undesirable situation one may try to choose loop parameters in such a way that the synchronization time is less than the time between changes in the data signal $m(t)$ or to modify the loop design (see, e.g., (Olson, 1975)).

Finally, by Assumption 2, one can use the following approximation for the input of loop filter

$$\varphi(t) \approx \varphi(\theta_\Delta(t)) = \frac{1}{8}\sin(2\theta_\Delta(t)). \tag{6}$$

Such an approximation, which depends on the phase difference of signals, is called a *phase detector characteristic of Costas loop*.

The relation between the input $\varphi(t)$ and the output $g(t)$ of linear loop filter (Thede, 2005) has the form

$$\frac{dx}{dt} = Ax + b\varphi(t), \; g(t) = c^*x + h\varphi(t), \tag{7}$$

where A is a constant matrix, the vector $x(t)$ is a filter state, b, c are constant vectors, h is constant, and $x(0)$ is initial state of filter. The control signal $g(t)$ is used to adjust VCO frequency to the frequency of input carrier signal

$$\dot{\theta}_2(t) = \omega_2(t) = \omega_2^{\text{free}} + Lg(t). \tag{8}$$

Here ω_2^{free} is free-running frequency of VCO and L is VCO gain.

If the frequency of input carrier is a constant:

$$\dot{\theta}_1(t) = \omega_1(t) \equiv \omega_1, \tag{9}$$

then equations (7)-(9) give the following mathematical model of Costas loop:

$$
\begin{aligned}
&\dot{x} = Ax + b\varphi(\theta_\Delta), \\
&\dot{\theta}_\Delta = \omega_\Delta - L(c^* x) - Lh\varphi(\theta_\Delta),
\end{aligned}
\tag{10}
$$

where $\theta_\Delta(t) = \theta_1(t) - \theta_2(t)$ and $\varphi(\theta_\Delta)$ corresponds to expression (7). Note that the initial frequencies difference (at $t = 0$) has the form

$$
\dot{\theta}_\Delta(0) = \omega_\Delta + Lc^* x(0) + Lh\varphi(\theta_\Delta(0)). \tag{11}
$$

Here to consider one-dimensional stability ranges defined only by $|\omega_\Delta|$ (see, e.g., capture and hold ranges) it is necessary to assume that $h = 0$ and $x(0) = 0$. For high-order filter, two different initial states $\tilde{x}(0)$ and $\tilde{\tilde{x}}(0)$ may lead to identical values of $\alpha_0(0, \tilde{x}(0)) = \alpha_0(0, \tilde{\tilde{x}}(0))$ but different functions $\alpha_0(t, \tilde{x}(0))$ and $\alpha_0(t, \tilde{\tilde{x}}(0))$ (to avoid this effect it is possible to require the observability of system (8)).

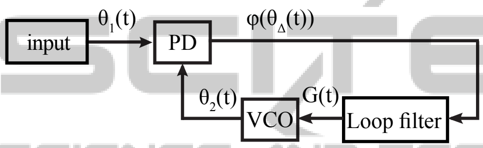

Figure 3: Classical (simplified) mathematical model of BPSK Costas loop in the signal's phase space.

System (11) considered in the classical works (Costas, 1962; Lindsey, 1972) and corresponds to the block-diagram shown in Fig. 3, where $\varphi(\theta_\Delta)$ is phase detector (PD) characteristic of Costas loop for sinusoidal signals.

Since mathematical model of Costas loop (11) is nonlinear, in practice, for its analysis numerical simulation and linearization are widely used (Gardner, 2005; Lindsey, 1972; Best, 2007). Note that the linearization without justification and the analysis of linearized models of nonlinear control systems may result in incorrect conclusions[1]. Also the application of standard numerical analysis of PLL-based circuits cannot guarantee to find undesired multiple steady-state solutions: see, e.g., examples of hidden oscillations[2].

In the next section a rigorous mathematical approach to derivation of nonlinear mathematical model of Costas loop will be discussed.

3 RIGOROUS MATHEMATICAL APPROACH TO DERIVATION OF NONLINEAR MATHEMATICAL MODEL OF COSTAS LOOP

The relation between the inputs $\varphi_{1,2}(t)$ and the outputs $g_{1,2}(t)$ of linear low-pass filters is as follows (Thede, 2005)

$$
\frac{dx_{1,2}}{dt} = A_{1,2}x_{1,2} + b_{1,2}\varphi_{1,2}(t),\ g_{1,2}(t) = c^*_{1,2}x_{1,2}. \tag{12}
$$

Here $A_{1,2}$ are constant matrices, the vectors $x_{1,2}(t)$ are filter states, $b_{1,2}, c_{1,2}$ are constant vectors, and the vectors $x_{1,2}(0)$ are initial states of filters. For the loop filter one can consider more general equation (8).

Taking into account (13), (8), and (9), one obtains the *mathematical model in the signal space* describing the *physical model* of BPSK Costas loop:

$$
\begin{aligned}
&\dot{x_1} = A_1 x_1 + b_1 m(t)\sin(\theta_1(t))\sin(\theta_2), \\
&\dot{x_2} = A_2 x_2 + b_2 m(t)\sin(\theta_1(t))\cos(\theta_2), \\
&\dot{x} = Ax + b(c_1^* x_1)(c_2^* x_2), \\
&\dot{\theta}_2 = \omega_2^{free} + L(c^* x) + Lh(c_1^* x_1)(c_2^* x_2).
\end{aligned}
\tag{13}
$$

Here $\theta_2(0)$ is the initial phase shift of VCO, and the vectors $x_{1,2}(0), x(0)$ are initial states of filters. Thus the initial VCO frequency (at $t = 0$) has the form

$$
\omega_2(0) = \omega_2^{free} + Lc^* x(0) + Lhc_1^* x_1(0) c_2^* x_2(0). \tag{14}
$$

The *mathematical model in the signal space* (14) is nonlinear nonautonomous discontinuous differential system, so in general case its analytical study is a difficult task even for the continuous case when $m(t) \equiv const$. Besides it is a slow-fast system, so its numerical study is rather complicated for the high-frequency signals. The problem is that it is necessary to consider simultaneously both very fast time scale of the signals $sin(\theta_{1,2}(t))$ and slow time scale of phase difference between the signals $\theta_\Delta(t)$, therefore one very small simulation time-step must be taken over a

[1] See, e.g., counterexamples to the filter hypothesis, hidden oscillations in counterexamples to Aizerman's and Kalman's conjectures on the absolute stability of nonlinear control systems (Bragin et al., 2011), and the Perron effects of the largest Lyapunov exponent sign reversal for a nonlinear system and its linearization (Kuznetsov and Leonov, 2005; Leonov and Kuznetsov, 2007).

[2] *An attractor is called a hidden attractor if its basin of attraction does not intersect with small neighborhoods of equilibria, otherwise it is called a self-excited attractor*. For example, hidden attractors are attractors in the systems with no-equilibria or with the only stable equilibrium (a special case of multistability and coexistence of attractors); they arise in the study of well-known fundamental problems such as 16th Hilbert problem, Aizerman & Kalman conjectures and in applied research of Chua circuits, drilling system, phase-locked loop based circuits, aircraft control systems and others (Kuznetsov et al., 2010; Bragin et al., 2011; Leonov et al., 2012; Kuznetsov et al., 2013; Andrievsky et al., 2013; Leonov and Kuznetsov, 2013; Leonov et al., 2014).

very long total simulation period (Goyal et al., 2006; Abramovitch, 2008a; Abramovitch, 2008b).

To overcome these problems, in place of using Assumption 1 one can apply averaging methods (Krylov and Bogolyubov, 1947; Mitropolsky and Bogolubov, 1961; Samoilenko and Petryshyn, 2004; Sanders et al., 2007) and consider *a simplified mathematical model in the signal's phase space*. However, this requires the consideration of constant data signal (Assumption 2) and constant frequency of input carrier (10):

$$\theta_1(t) = \omega_1 t + \theta_1(0).$$

In this case (14) is equivalent to

$$\begin{aligned}
\dot{x_1} &= A_1 x_1 + b_1 \sin(\omega_1 t + \theta_1(0)) \sin(\omega_1 t + \theta_1(0) + \theta_\Delta),\\
\dot{x_2} &= A_2 x_2 + b_2 \sin(\omega_1 t + \theta_1(0)) \cos(\omega_1 t + \theta_1(0) + \theta_\Delta),\\
\dot{x} &= Ax + b(c_1^* x_1)(c_2^* x_2),\\
\dot{\theta}_\Delta &= \omega_\Delta - L(c^* x) - Lh(c_1^* x_1)(c_2^* x_2),
\end{aligned} \tag{15}$$

Assuming that input carrier is a high-frequency signal (i.e. ω_1 is large), one can consider small parameter $\varepsilon = \frac{1}{\omega_1}$ and apply classical averring theory for the equations of low-pass filters. Thus one can obtain a *mathematical model of BPSK Costas loop in the signal's phase space* (see Fig. 4):

$$\begin{aligned}
\dot{x_1} &= A_1 x_1 + \frac{b_1}{2}\cos(\theta_\Delta),\\
\dot{x_2} &= A_2 x_2 + \frac{b_2}{2}\sin(\theta_\Delta),\\
\dot{x} &= Ax + b(c_1^* x_1)(c_2^* x_2),\\
\dot{\theta}_\Delta &= \omega_\Delta - L(c^* x) - Lh(c_1^* x_1)(c_2^* x_2).
\end{aligned} \tag{16}$$

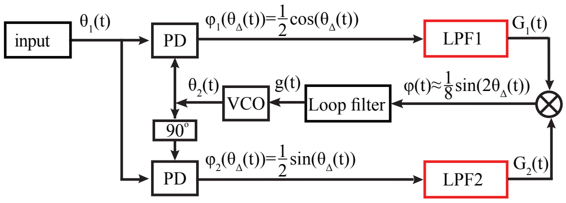

Figure 4: Mathematical model of BPSK Costas loop in the signal's phase space

Remark that here the initial frequencies difference (at $t = 0$)

$$\dot{\theta}_\Delta(0) = \omega_\Delta - Lc^* x(0) - Lhc_1^* x_1(0) c_2^* x_2(0) \tag{17}$$

is the same for system (17) and system (16) and it does not coincide with expression (18) for classical system (11). Here to consider one-dimensional stability ranges defined only by $|\omega_\Delta|$ (see, e.g., capture and hold ranges) it is necessary to assume that $h = 0$ and $x(0) = x_1(0) = x_2(0) = 0$.

In the general case one has to consider multidimensional stability domain $(\omega_\Delta, x(0), x_1(0), x_2(0))$.

4 COUNTEREXAMPLES TO THE ASSUMPTIONS

Note once more that various simplifications and the analysis of linearized models of control systems may result in incorrect conclusions[3]. At the same time the attempts to justify analytically the reliability of conclusions, based on engineering approaches, and rigorous study of nonlinear models are quite rare (see, e.g., (Abramovitch, 1990; Chang et al., 1993; Stensby, 1997; Shirahama et al., 1998; Watada et al., 1998; Hinz et al., 2000; Wu, 2002; Piqueira and Monteiro, 2003; Suarez and Quere, 2003; Margaris, 2004; Vendelin et al., 2005; Banerjee and Sarkar, 2006; Kudrewicz and Wasowicz, 2007; Wang et al., 2008; Bueno et al., 2010; Wiegand et al., 2010; Stensby, 2011; Suarez et al., 2012; Sarkar et al., 2014; Chicone and Heitzman, 2013; Yoshimura et al., 2013; Best et al., 2014a)). One of the reasons is that "*nonlinear analysis techniques are well beyond the scope of most undergraduate courses in communication theory*" (Tranter et al., 2010).

Further examples demonstrate that the use of Assumptions 1-2 requires further study and rigorous justification. The following examples demonstrate that for the same parameters the behaviors of considered models

- *physical model* (with data signal and low-pass filters) or its *mathematical model in the signal space* (Figs. 2,5 and system (14))
- *simplified mathematical model in the signal space* (without data signal and with low-pass filters)) (Figs. 2,5 with $m(t) = const$ and system (16));
- *simplified mathematical model the signal's phase space* (with low-pass filters and without data signal) (Fig. 4 and system (17));
- *classical mathematical model the signal's phase space* (without low-pass filters and data signal) (Fig. 3 and system (11))

may be very different from each other.

Simulation Parameters. Low pass filters transfer functions $H_{lpf}(s) = \frac{1}{s/\omega_3+1}$, $\omega_3 = 1.2566 * 10^6$ and corresponding equations (8) parameters are $A_{1,2} = -\omega_3$, $b_{1,2} = 1$, $c_{1,2} = 1$; Loop filter transfer function $H_{lf}(s) = \frac{\tau_2 s+1}{\tau_1 s}$, $\tau_2 = 3.9789 * 10^{-6}$, $\tau_1 = 2 * 10^{-5}$,

[3] see also counterexamples to the filter hypothesis, Aizerman's and Kalman's conjectures on the absolute stability of nonlinear control systems (Kuznetsov et al., 2011; Bragin et al., 2011; Leonov and Kuznetsov, 2013), and the Perron effects of the largest Lyapunov exponent sign inversions (Kuznetsov and Leonov, 2005; Leonov and Kuznetsov, 2007), etc.

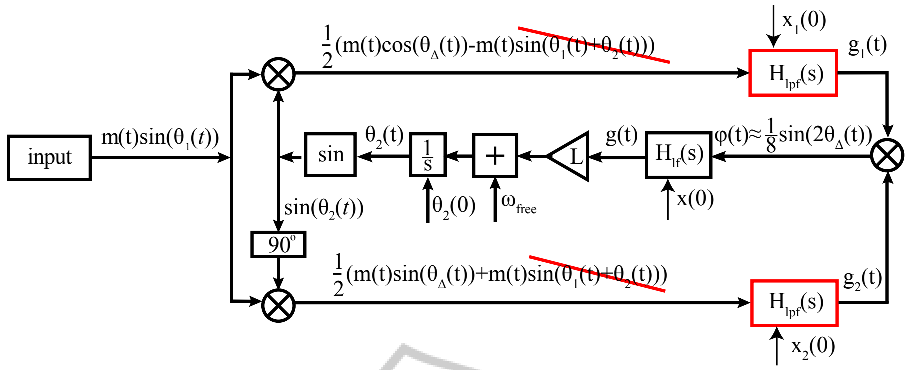

Figure 5: Block-diagram of Costas loop mathematical model in the signal space described by transfer functions and initial conditions.

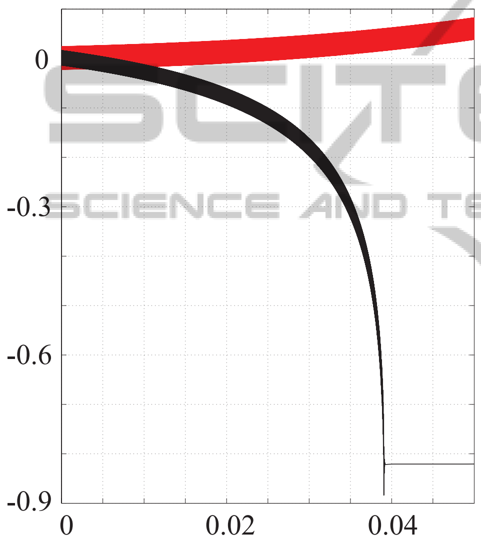

Figure 6: Loop filter output $g(t)$ for averaged model (17) (black) and physical model (red) in Fig. 2.

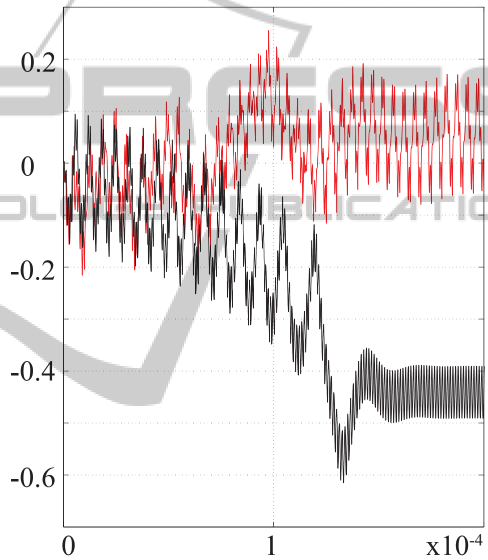

Figure 7: Loop filter output $g(t)$ for physical model (black) with periodic data signal, physical model (red) with constant data signal $m(t) \equiv 1$.

and corresponding differential equations (8) parameters are $A = 1, b = 0, c = \frac{1}{\tau_1}, h = \frac{\tau_2}{\tau_1}$; carrier frequency is $\omega_1 = 2 * \pi * 400000$ and initial carrier phase is zero: $\theta_1(0) = 0$.

Example 1. *In Fig. 6 is shown that Assumption 1 may not be valid: while simplified mathematical model in the signal's phase space* (17) *(see Fig. 4) acquires lock (black), physical model* (14) *(see Fig. 5) is out of lock (red).*

Here VCO free-running frequency: $\omega_2^{free} = 3.5500 * 10^6$*; initial states of filters are all zero* $\alpha_0(t) = x(0) = x_1(0) = x_20 = 0$.

Example 2. *In Fig. 7 is shown that Assumption 2 may not be valid: while simplified mathematical model in the signal space* (16) *(see physical model in Fig. 5 with constant data signal* $m(t) \equiv 1$*) acquire lock (black), physical model with periodic data signal* (14) *(see Fig. 5) is out of lock (red).*

Here VCO free-running frequency $\omega_2^{free} = 3.0753 * 10^6$*, initial states of filters are all zero* $\alpha_0(t) = x(0) = x_1(0) = x_2(0) = 0$*, data signal is periodic* $m(t) = sign\ \sin(100000 * 2 * \pi t)$.

Example 3. *In Fig. 8 is shown that low-pass filters may affect stability of models in the signal's phase space: while simplified mathematical model the signals phase space* (17) *(see Fig. 4) is out of lock (red), classical mathematical model the signals phase space* (11) *(see Fig. 3), where low-pass filters are not taken into account, acquires lock (black). Therefore the consideration of classical (simplified) model in signal's phase space (Fig. 3 and system* (11)*) may lead to wrong conclusion.*

Here VCO free-running frequency $\omega_2^{free} =$

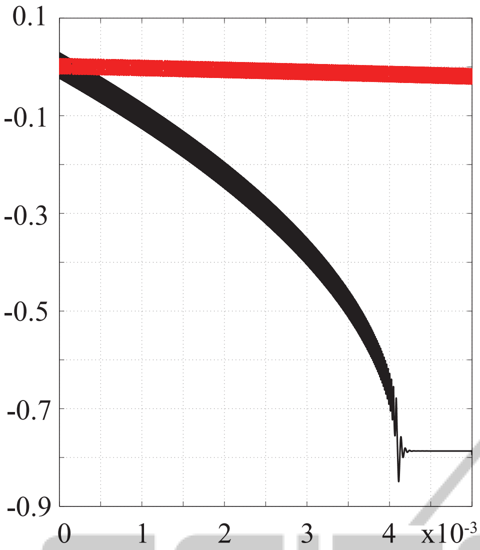

Figure 8: Loop filter output $g(t)$ for signal's phase space model (black curve) without low-pass filters, signal's phase space model (red curve) with low-pass filters.

$3.5133*10^6$, *initial states of filters are zero:* $\alpha_0(t) = x(0) = x_1(0) = x_2(0)$, *no data is being transmitted* $m(t) = 1$.

5 CONCLUSION

In the work it is shown that 1) the consideration of simplified mathematical models, constructed intuitively by engineers, and 2) the application of non rigorous methods of analysis can lead to wrong conclusions concerning the operability of Costas loop.

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