About Convergence for Finite-difference Equations of Incompressible Fluid with Boundary Conditions by Woods Formulas

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Abstract: In this paper, mathematical aspects of stability, convergence and numerical implementation of two-dimensional differential problem for incompressible fluid equations in “stream function, vorticity” variables defined on a symmetrical template of finite-difference grid studied by method of a priori estimates are considered. Approximate boundary conditions for the vorticity are chosen in the form of Woods formula. In case of a linear Stokes problem, it is shown that the numerical solution of the difference problem converges to the solution of the differential problem with second order accuracy and two algorithms of numerical implementation, for which the rates of convergence obtained, are considered. In the case of non-linear Navier-Stokes equations, estimates of the convergence of a solution of the difference problem to the solution of the differential problem, as well as estimation of the convergence of a considered iterative algorithm with the assumption that the condition is equivalent to the condition of uniqueness of nonlinear difference problem are obtained.

1 INTRODUCTION

Sufficient number of scientific publications is devoted to the problems of numerical solution of two-dimensional boundary value problems for incompressible fluid equations in “stream function, vorticity” variables. Descriptions of the most well-known computing technologies are used during the computational experiments to study various flows of incompressible fluid can be found in monographs (Chuhg, 2002), (Hirsch, 2002), (Kwak and Kiris, 2013). As is known, the main difficulties encountered in the numerical solution of the Navier-Stokes equations for an incompressible fluid, associated with the implementation of the boundary conditions for the vorticity. Generally, in practice, to find the values of the vorticity on the boundary, formulas approximating the conditions of adhesion and impermeability of the velocity components in the physical formulation of the problems considered are used (Danaev and Smagulov, 1991), (Vabishchevich, 1983), (Weinan and Liu, 1996). The most famous among them are Tom and Woods formulas (Tom and Aplt, 1964) having first and second order accuracy, respectively, for determining the vorticity on the boundary. Sufficient number of papers devoted to theoretical and practical aspects of using the Tom’s formula for the calculation of incompressible fluid flow (Li and Wang, 2003). In the paper (Voevodin, 1993) the absolute stability of the classical implicit difference schemes for two-dimensional Stokes equations is proven and stable direct and iterative methods for solving difference boundary value problems by the method of operator inequalities are proposed. In the paper (Voevodin and Yushkova, 1999), on the basis of the method of splitting into physical processes, the numerical method for solving initial-boundary value problems for the Navier-Stokes equations written in “stream function, vorticity” variables is proposed. To solve systems of implicit difference equations, a modification of “two-field” calculation of the stream function and vorticity values is used. The investigation of stability is conducted using the linear approximation of differential schemes. In the paper (Voevodin, 1998), using the method of a priori estimates, it is shown that the solution of the difference scheme converges to the solution of differential equations on a symmetrical grid pattern with the order $O(h^{3/2})$ in the case of choice of the boundary condition for the vorticity in form of Tom’s formula on the boundary, where $h = max(h_1, h_2)$, h_1, h_2 are steps of finite differential grid. Mathematical justification of implicit iterative methods for their numerical im-

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plementation is given. Review of existing literature shows that in the case of selecting Woods formula to calculate the values of the vorticity at the boundary theoretical studies are virtually absent. Issues of convergence of difference schemes have not been investigated. There are no estimates of the rate of convergence of iterative algorithms for numerical implementation of solutions of corresponding grid equations or existing studies cover only the case of linear difference schemes (Danaev and Amenova, 2013).

2 STATEMENT OF THE PROBLEM AND FINITE-DIFFERENTIAL EQUATIONS

In a domain $D=\{0\le x,y\le 1\}$ two-dimensional system of stationary Navier-Stokes equations for an incompressible fluid of the following form is considered (Rouch, 1980):

$$\left(\Omega\frac{\partial\Psi}{\partial y}\right)_x-\left(\Omega\frac{\partial\Psi}{\partial x}\right)_y=\nu\Delta\Omega+f(x,y), \quad (1)$$

$$\Delta\Psi=\Omega, \quad (x,y)\in D, \quad (2)$$

with boundary conditions

$$\Psi=\frac{\partial\Psi}{\partial n}=0, \quad (x,y)\in\partial D, \quad (3)$$

where $\vec{n}$ is the outward normal to the boundary of the domain, Δ is the two-dimensional Laplace operator, Ψ is the stream function, Ω is the vorticity, ν is a viscosity factor, and $f(x,y)$ is a given function.

For approximation of equations (1), (2) in the computational domain

$$D_h=\left\{(kh_1,mh_2),\ k\in\overline{1,N_1-1},\ m\in\overline{1,N_2-1}\right\},$$

where h_1 and h_2 are steps of the finite-differential grid in the directions of x and y, respectively, the differential scheme on the symmetrical template of the following form is considered:

$$L_h(\Omega)\Psi=\nu\Delta_h\Omega+f, \quad (4)$$

$$\Delta_h\Psi=\Omega, \quad (5)$$

where the differential operator L_h corresponds to the approximation of convective terms of equations (1) and is given in the form

$$L_h(\Omega)\Psi=\left(\Omega\Psi_{\overset{0}{y}}\right)_{\overset{0}{x}}-\left(\Omega\Psi_{\overset{0}{x}}\right)_{\overset{0}{y}}, \quad (6)$$

here and further $\Psi_{\overset{0}{x}}$, $\Psi_{\overset{0}{y}}$ means symmetrical difference derivatives in the directions of x and y, respectively.

On the border

$$\Psi_{0,m}=\Psi_{N_1,m}=0,\ m\in\overline{1,N_2-1},$$

$$\Psi_{k,0}=\Psi_{k,N_2}=0,\ k\in\overline{1,N_1-1} \quad (7)$$

for the vorticity, boundary conditions are selected in the form of Woods formula (Rouch, 1980), for example:

$$\Omega_{0,m}+\frac{1}{2}\Omega_{1,m}=\frac{3}{h_1}\Psi_{x,0,m}\ \ldots,\ m\in\overline{1,N_2-1}. \quad (8)$$

3 LINEAR DIFFERENTIAL STOKES PROBLEM

Study of the stability and convergence of iterative algorithms of numerical implementation of solving the Navier-Stokes grid equations for an incompressible fluid (4)-(8) is essentially based on the results which can be obtained for the case of a linear Stokes problem.

$$\Delta\Omega=f(x,y), \quad (9)$$

$$\Delta\Psi=\Omega, \quad (x,y)\in D, \quad (10)$$

with boundary conditions of the form (3). Here, for simplicity, we assume that $\nu=1$.

In this case relations (4),(5) can be presented in the following form:

$$\Delta_h\Omega_{k,m}=\frac{\Omega_{k+1,m}-2\Omega_{k,m}+\Omega_{k-1,m}}{h_1^2}+ +\frac{\Omega_{k,m+1}-2\Omega_{k,m}+\Omega_{k,m-1}}{h_2^2}=f_{k,m}, \quad (11)$$

$$\Delta_h\Psi_{k,m}=\frac{\Psi_{k+1,m}-2\Psi_{k,m}+\Psi_{k-1,m}}{h_1^2}+ +\frac{\Psi_{k,m+1}-2\Psi_{k,m}+\Psi_{k,m-1}}{h_2^2}=\Omega_{k,m}, \quad (12)$$

$$k\in\overline{1,N_1-1},\ m\in\overline{1,N_2-1}.$$

Hereinafter, the following well-known inequalities will be used (Samarski, 1989)

$$\delta_0||u||^2\le||\nabla_h u||^2,\ \delta_0||u||\le||\Delta_h u||,$$

$$||\Delta_h u||^2\le\frac{8}{h^2}||\nabla_h u||^2, \quad (13)$$

which hold for any grid function $u\in\overset{0}{\Omega}_h$, where $h=min(h_1,h_2)$, $\delta_0>0$ is minimal eigenvalue of the difference Laplace operator, $\overset{0}{\Omega}_h(D_h)$ is the space of grid functions with zero boundary values defined at the grid D_h.

Let us investigate the stability of a solution of difference problem (11),(12) with the boundary conditions of the form (7),(8). The relation (11) is multiplied by $\Psi_{k,m}h_1h_2$ summed over the internal nodes of grid D_h, next, using the formulas of summation by parts and the boundary conditions (7) we have the energy identity

$$\sum_{m=1}^{N_2-1}\left(\Omega_{0,m}\Psi_{x,0,m}-\Omega_{N_1,m}\Psi_{\bar{x},N_1,m}\right)h_2+$$

$$+\sum_{k=1}^{N_1-1}\left(\Omega_{k,0}\Psi_{y,k,0}-\Omega_{k,N_2}\Psi_{\bar{y},k,N_2}\right)h_1+$$

$$+\|\Delta_h\Psi\|^2=(f,\Psi).$$

Hereinafter, $\|f\|$ is the norm of the grid function in the space $L_{2,h}(D_h)$.

Hence, using the boundary conditions of the form (8), after simple transformations, we have

$$\|\Delta_h\Psi\|^2+\frac{h_1h_2}{4}\left(\sum_{m=1}^{N_2-1}\left(|\Omega_{0,m}|^2+|\Omega_{N_1,m}|^2\right)+\right.$$

$$\left.+\sum_{k=1}^{N_1-1}\left(|\Omega_{k,0}|^2+|\Omega_{k,N_2}|^2\right)\right)-$$

$$-\frac{h_1h_2}{12}\left(\sum_{m=1}^{N_2-1}(|\Omega_{1,m}|^2+|\Omega_{N_1-1,m}|^2)+\right. \tag{14}$$

$$\left.+\sum_{k=1}^{N_1-1}(|\Omega_{k,1}|^2+|\Omega_{k,N_2-1}|^2)\right)+$$

$$+\frac{h_1h_2}{12}\left(\sum_{m=1}^{N_2-1}\left((\Omega_{0,m}+\Omega_{1,m})^2+(\Omega_{N_1,m}+\Omega_{N_1-1,m})^2\right)+\right.$$

$$\left.+\sum_{k=1}^{N_1-1}\left((\Omega_{k,0}+\Omega_{k,1})^2+(\Omega_{k,N_2}+\Omega_{k,N_2-1})^2\right)\right)=(f,\Psi).$$

Therefore, we can write

$$\frac{11}{12}\|\Delta_h\Psi\|^2\le|(f,\Psi)|.$$

Hence, using the Cauchy-Bunyakovsky's inequality, we obtain the estimation

$$\|\Delta_h\Psi\|\le c_0\|f\|.$$

Here and below, we will designate the bounded positive constants non-dependent from the grid parameter h_1,h_2 by c_0.

3.1 On the Convergence of Linear Difference Stokes Problem

Assuming that the solution of differential problem (9),(10) with the boundary conditions (3) has a sufficient smoothness required for our analysis, we will study the order of convergence of the difference problem (7),(8),(11),(12) to the solution of the differential problem.

Let us designate discrepancy of differential equations (11),(12) as R_h and Q_h, respectively, i.e.

$$R_h=\Delta_h\Omega_h-f(x,y),$$

$$Q_h=\Delta_h\Psi_h-\Omega_h,\ \ (x,y)\in D_h,$$

where solutions of differential problem (3),(9),(10) in nodes of finite-differential grid are designated as Ψ_h, Ω_h.

Obviously, since chosen approximation formulas for derivatives are symmetrical,

$$R_h=O\left(h^2\right),\ Q_h=O\left(h^2\right),\ \ h=\max(h_1,h_2).$$

Let us introduce the following designations:

$$\Phi=\Psi_h-\Psi,\ Z=\Omega_h-\Omega.$$

Then for solution errors we have the following relations:

$$\Delta_hZ=R_h, \tag{15}$$

$$\Delta_h\Phi=Z+Q_h \tag{16}$$

with boundary conditions

$$\Phi(x,y)=0,\ (x,y)\in\partial D_h, \tag{17}$$

$$Z_{0,m}+\frac{1}{2}Z_{1,m}=\frac{3}{h_1}\Phi_{x,0,m}+r_{0,m},\ \ldots \tag{18}$$

$$m=\overline{1,N_2-1},$$

where $r_{0,m}=r_{N_1,m}=r_{k,0}=r_{k,N_2}=O\left(h^2\right)$.

To obtain an estimate of convergence, let us multiply the relation (15) by Φh_1h_2 and sum over the internal nodes of grid D_h. In this case, the main energy identity considering conditions (17) has the form

$$(Z,\Delta_h\Phi)+\sum_{m=1}^{N_2-1}\left(Z_{0,m}\Phi_{x,0,m}-Z_{N_1,m}\Phi_{\bar{x},N_1,m}\right)h_2+$$

$$+\sum_{k=1}^{N_1-1}\left(Z_{k,0}\Phi_{y,k,0}-Z_{k,N_2}\Phi_{\bar{y},k,N_2}\right)h_1=(R_h,\Phi).$$

Considering the relation (16) and the boundary conditions (18) we will get:

$$(\Delta_h\Phi-Q_h,\Delta_h\Phi)+\frac{h_1}{3}\sum_{m=1}^{N_2-1}\Big[Z_{0,m}\big(Z_{0,m}+\frac{1}{2}Z_{1,m}-$$

$$-r_{0,m}\big)+Z_{N_1,m}(Z_{N_1,m}+\frac{1}{2}Z_{N_1-1,m}-r_{N_1,m})\Big]h_2+$$

$$+\frac{h_2}{3}\sum_{k=1}^{N_1-1}\Big[Z_{k,0}\big(Z_{k,0}+\frac{1}{2}Z_{k,1}-r_{k,0}\big)+$$

$$+Z_{k,N_2}\big(Z_{k,N_2}+\frac{1}{2}Z_{k,N_2-1}-r_{k,N_2}\big)\Big]h_1=(R_h,\Phi).$$

Applying simple transformations, we have:

$$\frac{11}{12}\|\Delta_h\Phi\|^2\leq\frac{h_1}{6}\sum_{m=1}^{N_2-1}(|r_{0,m}|^2+|r_{N_1,m}|^2)h_2+$$

$$+\frac{h_2}{6}\sum_{k=1}^{N_1-1}(|r_{k,0}|^2+|r_{k,N_2}|^2)h_1+$$

$$+|(Q_h,\Delta_h\Phi)|+|(R_h,\Phi)|.$$

Using the "ε" inequality and relations (13), we will get the inequality which holds for any positive $\varepsilon_1,\varepsilon_2$:

$$(\frac{11}{12}-\varepsilon_1-\varepsilon_2)\|\Delta_h\Phi\|^2\leq c_0\big(\frac{1}{4\varepsilon_1}\|Q_h\|^2+\frac{1}{4\varepsilon_2}\|R_h\|^2\big)+$$

$$\frac{h_1h_2}{6}\Big(\sum_{m=1}^{N_2-1}(|r_{0,m}|^2+|r_{N_1,m}|^2)+\sum_{k=1}^{N_1-1}(|r_{k,0}|^2+|r_{k,N_2}|^2)\Big).$$

Choosing $\varepsilon_1,\varepsilon_2$ satisfying condition

$$\frac{11}{12}-\varepsilon_1-\varepsilon_2\geq\delta>0,$$

considering the order of smallness of values $R_h,\ Q_h,\ r_{0,m},\ r_{N_1,m},\ r_{k,0},\ r_{k,N_2}$, we finally have

$$\delta\|\Delta_h\Phi\|^2\leq c_0h^4,$$

that is

$$\|\Delta_h\Phi\|\leq c_0h^2,$$

which means that the solution of the difference scheme converges to the solutions of differential problem with the second order of accuracy.

3.2 Study of Convergence of Iterative Algorithm I

For the numerical solution of equations (7),(8),(11),(12) first we will consider iterative algorithm of the following form (Algorithm I)

$$\frac{\Omega_{k,m}^{n+1}-\Omega_{k,m}^n}{\tau}=\Delta_h\Omega_{k,m}^n-f_{k,m},\tag{19}$$

$$\Delta_h\Psi_{k,m}^{n+1}=\Omega_{k,m}^{n+1},\tag{20}$$

with boundary conditions

$$\Psi^{n+1}=0,\ \ (x,y)\in\partial D_h,\tag{21}$$

$$\Omega_{0,m}^{n+1}+\frac{1}{2}\Omega_{1,m}^{n+1}=\frac{3}{h_1}\Psi_{x,0,m}^{n+1},\ \dots\ m\in\overline{1,N_2-1}.\tag{22}$$

Hereinafter, for iterative algorithms, we assume that initial values assignment for stream function is expected.

Let us introduce designations

$$\Phi_{k,m}^n=\Psi_{k,m}^n-\Psi_{k,m},\ \ Z_{k,m}^n=\Omega_{k,m}^n-\Omega_{k,m},$$

where $\Psi_{k,m}^n,\ \Omega_{k,m}^n$ are solutions of differential problem (19)-(22), $\Psi_{k,m},\ \Omega_{k,m}$ are solutions of differential problem (7),(8),(11),(12).

Then for iteration errors we have the following relations:

$$\frac{Z_{k,m}^{n+1}-Z_{k,m}^n}{\tau}=\Delta_hZ_{k,m}^n,\tag{23}$$

$$\Delta_h\Phi_{k,m}^{n+1}=Z_{k,m}^{n+1},\tag{24}$$

with boundary conditions

$$\Phi^{n+1}=0,\ (x,y)\in\partial D_h,\tag{25}$$

$$Z_{0,m}^{n+1}+\frac{1}{2}Z_{1,m}^{n+1}=\frac{3}{h_1}\Phi_{x,0,m}^{n+1},\dots\ m=\overline{1,N_2-1}.\tag{26}$$

We multiply the relation (25) by $2\tau\Phi_{k,m}^{n+1}h_1h_2$ and sum over internal nodes of the grid D_h. Considering boundary conditions , we have

$$\|\nabla_h\Phi^{n+1}\|^2-\|\nabla_h\Phi^n\|^2+\|\nabla_h(\Phi^{n+1}-\Phi^n)\|^2+$$

$$2\tau\Big(\sum_{m=1}^{N_2-1}\big(Z_{0,m}^n\Phi_{x,0,m}^{n+1}-Z_{N_1,m}^n\Phi_{\bar{x},N_1,m}^{n+1}\big)h_2+$$

$$+\sum_{k=1}^{N_1-1}\big(Z_{k,0}^n\Phi_{y,k,0}^{n+1}-Z_{k,N_2}^n\Phi_{\bar{y},k,N_2}^{n+1}\big)h_1\Big)+$$

$$+2\tau(\Delta_h\Phi^n,\Delta_h\Phi^{n+1})=0,\tag{27}$$

where

$$\|\nabla_h\Phi\|^2=\Big(\sum_{k=1}^{N_1}\sum_{m=1}^{N_2-1}|\Phi_{\bar{x},k,m}|^2+\sum_{k=1}^{N_1-1}\sum_{m=1}^{N_2}|\Phi_{\bar{y},k,m}|^2\Big)h_1h_2,$$

$$\forall\Phi\in\overset{0}{\Omega}_h(D_h).$$

Let us rewrite (27) in the following form

$$\|\nabla_h\Phi^{n+1}\|^2-\|\nabla_h\Phi^n\|^2+\|\nabla_h(\Phi^{n+1}-\Phi^n)\|^2+$$

$$+2\tau\Big(\sum_{m=1}^{N_2-1}\big(Z_{0,m}^n\Phi_{x,0,m}^n-Z_{N_1,m}^n\Phi_{\bar{x},N_1,m}^n\big)h_2+$$

$$+\sum_{k=1}^{N_1-1}\big(Z_{k,0}^n\Phi_{y,k,0}^n-Z_{k,N_2}^n\Phi_{\bar{y},k,N_2}^n\big)h_1\Big)+$$

$$+2\tau\Big(\sum_{m=1}^{N_2-1}\big(Z_{0,m}^n(\Phi^{n+1}-\Phi^n)_{x,0,m}-$$

$$-Z_{N_1,m}^n(\Phi^{n+1}-\Phi^n)_{\bar{x},N_1,m}\big)h_2+$$

$$+2\tau\Big(\sum_{k=1}^{N_1-1}\big(Z_{k,0}^n(\Phi^{n+1}-\Phi^n)_{y,k,0}-$$

$$-Z_{k,N_2}^n(\Phi^{n+1}-\Phi^n)_{\bar{y},k,N_2}\big)h_1+$$

$$+2\tau(\Delta_h\Phi^n,\Delta_h\Phi^{n+1})=0.\tag{28}$$

Transforming the corresponding terms in the equation (28) as in the case of the relationship (14) and considering known inequalities (13) after simple transformations we will get

$$\|\nabla_h\Phi^{n+1}\|^2+(1-\frac{10\tau}{h^2})\|\nabla_h(\Phi^{n+1}-\Phi^n)\|^2-$$

$$-\|\nabla_h\Phi^n\|^2+\frac{5\tau}{6}\Big(\|\Delta_h\Phi^{n+1}\|^2+\|\Delta_h\Phi^n\|^2\Big)\leq$$

$$\leq\frac{2\tau}{h_1^2}\sum_{m=1}^{N_2-1}\Big(|(\Phi^{n+1}-\Phi^n)_{x,0,m}|^2+$$

$$+|(\Phi^{n+1}-\Phi^n)_{\bar{x},N_1,m}|^2\Big)h_1h_2+$$

$$+\frac{2\tau}{h_2^2}\sum_{k=1}^{N_1-1}\Big(|(\Phi^{n+1}-\Phi^n)_{y,k,0}|+$$

$$+|(\Phi^{n+1}-\Phi^n)_{\bar{y},k,N_2}|^2\Big)h_1h_2.$$

Therefore, under condition

$$1-\frac{10\tau}{h^2}\geq 0, \tag{29}$$

the following inequality holds

$$\|\nabla_h\Phi^{n+1}\|^2-\|\nabla_h\Phi^n\|^2+$$

$$+\frac{5\tau\delta_0}{6}\Big(\|\nabla_h\Phi^{n+1}\|^2+\|\nabla_h\Phi^n\|^2\Big)\leq 0$$

i.e.

$$\|\nabla_h\Phi^{n+1}\|\leq q\|\nabla_h\Phi^n\|$$

where

$$q=\sqrt{\frac{1-\tau\beta}{1+\tau\beta}}<1,\ \ \beta=\frac{5\delta_0}{6}.$$

Hence, we can conclude that when condition (29) holds iterations by Algorithm I converge at a geometric rate with denominator $q<1$. Thus, it is possible to ensure that value $q^n\leq\varepsilon$, where ε the number characterizing iteration accuracy if

$$n\geq n_0(\varepsilon)\approx O(\frac{1}{h^2})\ln\frac{1}{\varepsilon}.$$

3.3 Study of Convergence of Iterative Algorithm II

Further, let us consider iterative algorithm of the following form (Algorithm II)

$$\frac{\Omega_{k,m}^{n+1}-\Omega_{k,m}^n}{\tau}=\Delta_h\Omega_{k,m}^{n+1}-f_{k,m}, \tag{30}$$

$$\Delta_h\Psi_{k,m}^{n+1}=\Omega_{k,m}^{n+1}, \tag{31}$$

with boundary conditions

$$\Psi^{n+1}=0,\ (x,y)\in\partial D_h, \tag{32}$$

$$\Omega_{0,m}^{n+1}+\frac{1}{2}\Omega_{1,m}^{n+1}=\frac{3}{h_1}\Psi_{x,0,m}^n,\dots\ m\in\overline{1,N_2-1}. \tag{33}$$

For iteration errors we have relationships

$$\frac{Z_{k,m}^{n+1}-Z_{k,m}^n}{\tau}=\Delta_hZ_{k,m}^{n+1}, \tag{34}$$

$$\Delta_h\Phi_{k,m}^{n+1}=Z_{k,m}^{n+1}, \tag{35}$$

with boundary conditions

$$\Phi^{n+1}=0,\ (x,y)\in\partial D_h, \tag{36}$$

$$Z_{0,m}^{n+1}+\frac{1}{2}Z_{1,m}^{n+1}=\frac{3}{h_1}\Phi_{x,0,m}^n,\dots\ m\in\overline{1,N_2-1}. \tag{37}$$

In this case, considering boundary condition (36) the main energy identity has the form

$$\|\nabla_h\Phi^{n+1}\|^2-\|\nabla_h\Phi^n\|^2+\|\nabla_h(\Phi^{n+1}-\Phi^n)\|^2+$$

$$+2\tau\Big(\sum_{m=1}^{N_2-1}\big(Z_{0,m}^{n+1}\Phi_{x,0,m}^{n+1}-Z_{N_1,m}^{n+1}\Phi_{\bar{x},N_1,m}^{n+1}\big)h_2+$$

$$+\sum_{k=1}^{N_1-1}\big(Z_{k,0}^{n+1}\Phi_{y,k,0}^{n+1}-Z_{k,N_2}^{n+1}\Phi_{\bar{y},k,N_2}^{n+1}\big)h_1\Big)+$$

$$+2\tau\|\Delta_h\Phi^{n+1}\|^2=0.$$

Considering boundary conditions (37) and transforming it, we will get the inequality

$$\|\nabla_h\Phi^{n+1}\|^2-\|\nabla_h\Phi^n\|^2+(1-\frac{2\tau}{h^2})\|\nabla_h(\Phi^{n+1}-\Phi^n)\|^2+$$

$$+\frac{11\tau}{6}\|\Delta_h\Phi^{n+1}\|^2\leq 0.$$

Consequently, under the condition

$$1-\frac{2\tau}{h^2}\geq 0, \tag{38}$$

we have that

$$\|\nabla_h\Phi^{n+1}\|\leq\|\nabla_h\Phi^n\|,$$

where

$$q=\sqrt{\frac{1}{1+\frac{11\tau\delta_0}{6}}}<1.$$

i.e. we can conclude that when condition (38) holds, iterations by Algorithm II also converge at a geometric rate with ratio $q<1$ and for $n_0(\varepsilon)$ the following relation holds

$$n\geq n_0(\varepsilon)\approx O(\frac{1}{h^2})\ln\frac{1}{\varepsilon}.$$

4 STUDY OF BOUNDARY VALUE PROBLEM FOR NON-LINEAR NAVIER-STOKES EQUATIONS

Note that for the differential operator L_h the following relations are valid (Danaev and Amenova, 2013)

$$|(L_h(\omega)u,v)| \le c_0\|\omega\|\|\Delta_h u\|\|\Delta_h v\|, \qquad (39)$$

$$(L_h(\omega)u,u) = 0, \ \forall u,v \in \overset{0}{\Omega}_h(D_h), \qquad (40)$$

where c_0 is a uniformly bounded constant.

Due to the fact that the equation (40) holds, proceeding as in the case of the linear problem, we will have a priori estimate of the solution for the solution of the differential problem (4)-(8):

$$\nu\|\Delta_h\Psi\| \le c_0\|f\|.$$

4.1 Uniqueness Condition of Solutions Non-linear Differential Navier-stokes Equations

Let us show that under the condition $c_0\frac{\|f\|}{\nu^2} < 1$, the solution of the problem (4)-(8) is unique.

Assume that there are two solutions (Ψ_1,Ω_1) and (Ψ_2,Ω_2). Then for differences $\Phi = \Psi_1 - \Psi_2$ $Z = \Omega_1 - \Omega_2$, we have the differential problem:

$$L_h(\Omega_1)\Phi + L_h(Z)\Psi_2 = \nu\Delta_h Z,$$

$$\Delta_h\Phi = Z,$$

with the following boundary conditions:

$$\Phi = 0, \ \ Z_{0,m} + \frac{1}{2}Z_{1,m} = \frac{3}{h_1}\Phi_{x,0,m}, \dots \ m \in \overline{1,N_2-1}.$$

We have

$$\nu\|\Delta_h\Phi\|^2 \le |(L_h(Z)\Psi_2,\Phi)| \le c_0\|\Delta_h\Phi\|^2\|\Delta_h\Psi_2\|,$$

$$\left(\nu - c_0\|\Delta_h\Psi_2\|\right)\|\Delta_h\Phi\|^2 \le 0.$$

Hence, if

$$\nu - c_0\|\Delta_h\Psi_2\| > 0, \quad \frac{c_0\|f\|}{\nu^2} < 1, \qquad (41)$$

then it should be

$$\|\Delta_h\Phi\| = 0,$$

i.e. the solution is unique.

4.2 Study of Convergence of Non-linear Difference Equations

Assuming sufficient smoothness of solutions of the differential problem (1)-(3), we will study the convergence of the solution of grid equations (4)-(8).

Let us designate discrepancy of the differential scheme (4) for the equation of motion as R_h, and discrepancy of the differential relation (5) as Q_h:

$$R_h = L_h(\Omega_h)\Psi_h - \nu\Delta_h\Omega_h + f(x,y), \ (x,y) \in D_h,$$
$$Q_h = \Delta_h\Psi_h - \Omega_h.$$

where solutions of the differential problem (1)-(3) in nodes of finite-differential grid are designated as Ψ_h, Ω_h.

Obviously, because chosen approximation templates of derivatives are symmetrical

$$R_h = O\left(h^2\right), \ Q_h = O\left(h^2\right),$$

where $h = \max(h_1,h_2)$.

Let us introduce

$$\Phi = \Psi_h - \Psi, \ \ Z = \Omega_h - \Omega.$$

Then for solution error we have following relationships:

$$L_h(\Omega_h)\Phi + L_h(Z)\Psi = \nu\Delta_h Z + R_h, \qquad (42)$$

$$\Delta_h\Phi = Z + Q_h, \qquad (43)$$

with boundary conditions

$$\Phi(x,y) = 0, \ (x,y) \in \partial D_h, \qquad (44)$$

$$Z_{0,m} + \frac{1}{2}Z_{1,m} = \frac{3}{h_1}\Phi_{x,0,m} + r_{0,m}, \dots \ m \in \overline{1,N_2-1}, \qquad (45)$$

where $r_{0,m} = r_{N_1,m} = r_{k,0} = r_{k,N_2} = O\left(h^2\right)$.

In order to obtain estimation of convergence, we multiply the relation (42) by Φ and sum by internal nodes of grid D_h. Then we apply Green's difference formula, take into account the given condition (44), and as a result, we have the following main energy identity:

$$\nu\Big((Z,\Delta_h\Phi) + \sum_{m=1}^{N_2-1}(Z_{0,m}\Phi_{x,0,m} - Z_{N_1,m}\Phi_{\bar{x},N_1,m})\,h_2 +$$
$$+ \sum_{k=1}^{N_1-1}\left(Z_{k,0}\Phi_{y,k,0} - Z_{k,N_2}\Phi_{\bar{y},k,N_2}\right)h_1 +$$
$$+(R_h,\Phi) = (L_h(Z)\Psi,\Phi).$$

Considering the relation (43) and boundary conditions (45), we will get:

$$\nu(\Delta_h\Phi - Q_h,\Delta_h\Phi) + \frac{\nu h_1}{3}\sum_{m=1}^{N_2-1}\Big[Z_{0,m}\big(Z_{0,m} + \frac{1}{2}Z_{1,m} -$$

$$-r_{0,m}) + Z_{N_1,m}(Z_{N_1,m} + \frac{1}{2}Z_{N_1-1,m} - r_{N_1,m})\Big]h_2 +$$

$$+\frac{\nu h_2}{3}\sum_{k=1}^{N_1-1}\Big[Z_{k,0}\big(Z_{k,0} + \frac{1}{2}Z_{k,1} - r_{k,0}\big) + Z_{k,N_2}\big(Z_{k,N_2}$$

$$+\frac{1}{2}Z_{k,N_2-1} - r_{k,N_2}\big)\Big]h_1 + (R_h,\Phi) = (L_h(Z)\Psi,\Phi).$$

Hence, we will get inequality:

$$\frac{11}{12}\nu\|\Delta_h\Phi\|^2 \le \frac{\nu h_1 h_2}{6}\Big(\sum_{m=1}^{N_2-1}(|r_{0,m}|^2 + |r_{N_1,m}|^2) +$$

$$+\sum_{k=1}^{N_1-1}(|r_{k,0}|^2 + |r_{k,N_2}|^2)\Big) +$$

$$+\nu|(Q_h,\Delta_h\Phi)| + |(R_h,\Phi)| + |(L_h(Z)\Psi,\Phi)|.$$

Applying the "ε" -inequality and the inequality (39) for the operator L_h, we have

$$\Big(\frac{11}{12}\nu - \nu\varepsilon_1 - \varepsilon_2 - c_0\|\Delta_h\Psi\|(1+\varepsilon_3)\Big)\|\Delta_h\Phi\|^2 \le$$

$$\le \frac{\nu}{4\varepsilon_1}\|Q_h\|^2 + \frac{1}{4\varepsilon_2}\|R_h\|^2 + c_0\frac{\|\Delta_h\Psi\|}{4\varepsilon_3}\|Q_h\|^2 +$$

$$+\frac{h_1 h_2}{6}\Big(\sum_{m=1}^{N_2-1}(|r_{0,m}|^2 + |r_{N_1,m}|^2) + \sum_{k=1}^{N_1-1}(|r_{k,0}|^2 + |r_{k,N_2}|^2)\Big).$$

Choosing positive parameters $\varepsilon_1,\ \varepsilon_2,\ \varepsilon_3$ satisfying the inequality

$$\frac{11}{12}\nu - \nu\varepsilon_1 - \varepsilon_2 - c_0\|\Delta_h\Psi\|(1+\varepsilon_3) \ge \delta > 0, \quad (46)$$

we have

$$\delta\|\Delta_h\Phi\|^2 \le Mh^4,$$

where $0 < M < \infty$ uniformly bounded constant non-dependent from grid steps.

Therefore,

$$\|\Delta_h\Phi\| \le c_0 h^2.$$

i.e. under the condition (46) which is equivalent to the uniqueness condition (41), the solutions of differential equation (4)-(8) converge to the solution of the differential problem as in the case of the linear Stokes problem with the second order of accuracy.

4.3 Study of Convergence of Iterative Algorithm for Non-linear Problem

To find a numerical solution of the differential problem (4)-(8), let us consider the iterative algorithm of the following form:

$$\frac{\Omega^{n+1} - \Omega^n}{\tau} + L_h(\Omega^n)\Psi^n = \nu\Delta_h\Omega^n + f, \quad (47)$$

$$\Delta_h\Psi^{n+1} = \Omega^{n+1}, \quad (48)$$

with boundary conditions

$$\Psi^0(x,y) = \Psi_0(x,y) = 0,\ \ (x,y) \in D_h$$

$$\Psi^{n+1} = 0,\ \ (x,y) \in \partial D_h, \quad (49)$$

$$\Omega^{n+1}_{0,m} + \frac{1}{2}\Omega^{n+1}_{1,m} = \frac{3}{h_1}\Psi^{n+1}_{x,0,m}, \ldots\ m \in \overline{1,N_2-1}. \quad (50)$$

Let us show that the solution of differential scheme (47)-(51) converges to the solution of the differential problem (4)-(8) and we obtain estimation of the convergence rate.

For iteration errors, we have the relations:

$$\frac{Z^{n+1} - Z^n}{\tau} + L_h(\Omega)\Phi^n + L_h(Z^n)\Phi = \nu\Delta_h Z^n, \quad (51)$$

$$\Delta_h\Phi^{n+1} = Z^{n+1}, \quad (52)$$

$$\Psi^{n+1} = 0,\ \ (x,y) \in \partial D_h, \quad (53)$$

$$Z^{n+1}_{0,m} + \frac{1}{2}Z^{n+1}_{1,m} = \frac{3}{h_1}\Phi^{n+1}_{x,0,m}, \ldots\ m \in \overline{1,N_2-1}. \quad (54)$$

We multiply (51) by $2\tau\Phi^{n+1}$ and sum by nodes of the grid. Considering conditions (53) we have the following identity

$$\|\nabla_h\Phi^{n+1}\|^2 - \|\nabla_h\Phi^n\|^2 + \|\nabla_h(\Phi^{n+1} - \Phi^n)\|^2 +$$

$$+\tau\nu(\Delta_h\Phi^n, \Delta_h\Phi^{n+1}) +$$

$$+2\tau\nu\Big(\sum_{m=1}^{N_2-1}\big(Z^n_{0,m}\Phi^{n+1}_{x,0,m} - Z^n_{N_1,m}\Phi^{n+1}_{\bar{x},N_1,m}\big)h_2 +$$

$$+\sum_{k=1}^{N_1-1}\big(Z^n_{k,0}\Phi^{n+1}_{y,k,0} - Z^n_{k,N_2}\Phi^{n+1}_{\bar{y},k,N_2}\big)h_1\Big) =$$

$$+2\tau(L_h(Z^n)\Psi, \Phi^{n+1}).$$

Considering boundary conditions (54), applying simple transformations, and applying known inequalities (13) we will get

$$\|\nabla_h\Phi^{n+1}\|^2 - \|\nabla_h\Phi^n\|^2 +$$

$$+\tau\nu\|\Delta_h\Phi^{n+1}\|^2 + \frac{5\tau\nu}{6}\|\Delta_h\Phi^n\|^2 -$$

$$-(1 - \frac{10\tau\nu}{h^2})\Big(\sum_{m=1}^{N_2-1}\Big(\big|(\Phi^{n+1} - \Phi^n)_{x,0,m}\big|^2 +$$

$$+\big|(\Phi^{n+1} - \Phi^n)_{\bar{x},N_1,m}\big|^2\Big) + \sum_{k=1}^{N_1-1}\Big(\big|(\Phi^{n+1} - \Phi^n)_{y,k,0}\big|^2 +$$

$$+\big|(\Phi^{n+1} - \Phi^n)_{\bar{y},k,N_2}\big|^2\Big)h_1 h_2 +$$

$$+(1 - \frac{8\tau\nu}{h^2})\Big[\sum_{k=2}^{N_1-2}\sum_{m=1}^{N_2-1}\Big(|\Phi^{n+1}_{\bar{x},k,m} - \Phi^n_{\bar{x},k,m}|^2\Big) +$$

$$+\sum_{k=2}^{N_1-1}\sum_{m=1}^{N_2-2}\Big(|\Phi^{n+1}_{\bar{y},k,m} - \Phi^n_{\bar{y},k,m}|^2\Big)\Big]h_1 h_2 \le$$

$$\leq 2\tau c_0\|\Delta_h\Psi\|\|\Delta_h\Phi^n\|\|\Delta_h\Phi^{n+1}\|.$$

Therefore, under the condition

$$1-\frac{10\tau\nu}{h^2}\geq 0, \tag{55}$$

we have the following inequalities

$$\|\nabla_h\Phi^{n+1}\|^2-\|\nabla_h\Phi^n\|^2+$$
$$+\tau\nu\|\Delta_h\Phi^{n+1}\|^2+\frac{5\tau\nu}{6}\|\Delta_h\Phi^n\|^2\leq$$
$$\leq\tau c_0\|\Delta_h\Psi\|\left(\|\Delta_h\Phi^{n+1}\|^2+\|\Delta_h\Phi^n\|^2\right),$$
$$\|\nabla_h\Phi^{n+1}\|^2-\|\nabla_h\Phi^n\|^2+$$
$$+\tau\big(\nu-c_0\|\Delta_h\Psi\|\big)\|\Delta_h\Phi^{n+1}\|^2+$$
$$+\tau\Big(\frac{5}{6}\nu-c_0\|\Delta_h\Psi\|\Big)\|\Delta_h\Phi^n\|\leq 0,$$
$$\Big(1+\tau\delta_0\big(\nu-c_0\|\Delta_h\Psi\|\big)\|\nabla_h\Phi^{n+1}\|^2\leq$$
$$\leq\left(1-\tau\delta_0\Big(\frac{5}{6}\nu-c_0\|\Delta_h\Psi\|\Big)\right)\|\nabla_h\Phi^n\|^2.$$

Assume that

$$\frac{5\nu}{6}\nu-c_0\|\Delta_h\Psi\|\geq\delta>0, \tag{56}$$

then

$$\|\nabla_h\Phi^{n+1}\|\leq q\|\nabla_h\Phi^n\|$$

where

$$q=\sqrt{\frac{1-\tau\delta_0\delta}{1+\tau\delta_0\delta}}<1.$$

Therefore, when condition (55) at chosen parameters τ, h and the inequality (56) both hold iterations converge at a geometric rate with denominator less than one and for $n_0(\varepsilon)$ as in the case of the linear problem following relationship is valid

$$n\geq n_0(\varepsilon)\approx O(\frac{1}{h^2})\ln\frac{1}{\varepsilon}.$$

5 CONCLUSION

In the paper, the study of the differential scheme written on a symmetrical grid pattern and methods of their numerical implementation for an incompressible fluid for equations in case of the choice of boundary conditions for grid values of the vorticity at the boundary by Woods formula are conducted. It is shown that the order of accuracy of the differential scheme in case of the choice of the Woods formula is better in comparison with the case of using the Tom's formula. For the Stokes difference problem, two algorithms for the numerical implementation of the solution of the difference problem are considered and the influence of the boundary conditions on the iteration layers is studied. In the case of the non-linear Navier-Stokes problem for the considered iterative algorithm, it is shown that the assumption of certain conditions which are equivalent to the uniqueness condition of the differential problem, the convergence rate coincides with the convergence rate in the case of the linear Stokes problem.

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