Functional Semantics for Non-prenex QBF

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Abstract: Quantified Boolean Formulae (or QBF) are suitable to represent finite two-player games. Current techniques to solve QBF are for prenex QBF and knowledge representation is rarely in this form. We propose in this article a functional semantics for non-prenex QBF. The proposed formalism is symmetrical for validity and non-validity and allows to give different interpretations to the quantifiers. With our formalism, the solution of a non-prenex QBF is consistent with the specification, directly readable by the designer of the QBF and the locality of the knolewge is preserved.

1 INTRODUCTION

Quantified Boolean Formula (or QBF) is a generalization of satisfiability in which propositional symbols may be universally and existentially quantified. Many important problems in Artificial Intelligence may be specified in QBF. The satisfiability problem (SAT) of propositional logic is nothing more than the validity problem for QBF constituted of a propositional formula embedded in existential quantifiers associated with their propositional symbols. Hence, most of the more recent decision procedures for the validity problem of QBF are based on the (propositional version of the) search-based algorithm of Davis, Logemann and Loveland (DLL) for SAT which is a direct consequence of the semantics of the existential quantifier.

The semantics of QBF is usually presented either in its decision form (Stockmeyer, 1977) either in a functional form but only for prenex QBF essentially thanks to Skolem functions (Kleine Büning et al., 2007; Benedetti, 2005a) [1] which may be expressed by policies (Coste-Marquis et al., 2006) or strategies (Bordeaux and Monfroy, 2002).

The QBF semantics presented in its decision form is very suitable for theoretical problems but does not allow to extract solutions of the QBF. QBF are also suitable to represent finite two-player games. The validity of a QBF ensures to the *existential player* that there exists a strategy to win whatever plays the *universal player*. But in this case the decision semantics is no more sufficient to help the existential player.

The QBF semantics presented in its functional form is restricted to prenex QBF[2] but this restriction is a major drawback: knowledge representation is rarely in prenex form. There exists a prenexing process which preserves validity but this prenexing process has many drawbacks:

- It is a non-deterministic process and the chosen prenexing strategy impacts the time complexity of the obtained QBF (Egly et al., 2003) (even the impact may be reduce by so-called dependency schemes (Lonsing and Biere, 2010)).
- The elimination of biconditionals leads to an exponential growth of the size of the formula (and of its search space, see (Da Mota et al., 2009) for a discussion about the translation of Plaisted-Greenbaum (Plaisted and Greenbaum, 1986) for QBF).
- The loss of locality of the quantified propositional symbols introduces an increase of the size of the search space (anyway many systems add *ab initio* miniscoping associated to quantifier trees in order to recover the lost scopes of the quantifiers (Benedetti, 2005b; Giunchiglia et al., 2006)).
- The choice of a total order induced by the partial order defined by the quantifiers introduces new dependencies between existentially quantified propositional symbols and universally quantified propositional symbols which does not exist from the QBF designer point of view and are, hence, interpreted in the solution with difficulty.

[1] Boolean functions associated to the existentially quantified propositional symbols which depend on the universally quantified propositional symbols which precede them

[2] i.e. nested quantifiers are forbidden

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- Parts of a solution may have no meaning at all for the QBF designer: for example in a two-player game over a space containing n moves, the height of the tree representing a winning strategy is necessarily n even in the subtrees where the victory conditions are fulfilled before the game space is completely filled[3].

These different drawbacks lead us to propose new procedures for QBF with a richer expressivity. But we face then to two issues:

- the lack of a functional semantics for non prenex QBF and
- the lack of techniques to verify the results of those new procedures.

In this article, we focus on the first issue. In order to explicit our motivations, we need some preliminaries (Section 2) which present some basic elements about propositional logic (§ 2), QBF syntax (§ 2) and decision and functional semantics for QBF (§ 2). We first present in Section 3 an example which give a more technical presentation of our issue (§ 3.1) then our proposition about a non-prenex QBF semantics (§ 3.2).

2 PRELIMINARIES

Propositional Logic. Boolean values are denoted **t** (for true) and **f** (for false) and the set of Boolean values is denoted **BOOL**. The set of propositional symbols is denoted $\mathcal{PS}$. Symbols $\top$ and $\bot$ stand for Boolean constants. Symbol $\wedge$ stands for conjunction, $\vee$ for disjunction, $\neg$ for negation, $\rightarrow$ for implication, $\leftrightarrow$ for biconditional. The set of binary operators $\{\wedge,\vee,\rightarrow,\leftrightarrow\}$ is denoted O. The set of propositional formulae **PROP** is defined inductively as follows: propositional symbols or constants are elements of **PROP**, if F is an element of **PROP** then $\neg F$ is also an element of **PROP**, if F and G are elements of **PROP** and $\circ$ an element of O then $(F \circ G)$ is an element of **PROP**. A literal is a propositional symbol or the negation of a propositional symbol. A cube is a conjunction of literals. A clause is a disjunction of literals. A valuation v is a function from $\mathcal{PS}$ to **BOOL** and the set of valuations is denoted **VAL_PROP**.

The semantics of the propositional formulae uses the semantics of propositional constants and operators which is defined as usual: To each constant and operator (resp. $\top$, $\bot$, $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$) is associated its semantics as a Boolean function (resp. $i_\top, i_\bot :\rightarrow \mathbf{BOOL}$, $i_\neg : \mathbf{BOOL} \rightarrow \mathbf{BOOL}$, $i_\wedge, i_\vee, i_\rightarrow, i_\leftrightarrow : \mathbf{BOOL} \times \mathbf{BOOL} \rightarrow \mathbf{BOOL}$). The semantics of the propositional formulae is defined inductively for any valuation v as follows: $v^*(\bot) = i_\bot = \mathbf{f}$, $v^*(\top) = i_\top = \mathbf{t}$, $v^*(x) = v(x)$ if $x \in \mathcal{PS}$, $v^*((F \circ G)) = i_\circ(v^*(F), v^*(G))$ if $F, G \in \mathbf{PROP}$ and $\circ \in O$ and $v^*(\neg F) = i_\neg(v^*(F))$ if $F \in \mathbf{PROP}$. A propositional formula F is a tautology if for every valuation v, $v^*(F) = \mathbf{t}$. A Boolean function f of arity n (i.e. a function from $\mathbf{BOOL}^n$ to **BOOL**) is *associated to a propositional formula* $\mu(f)$ on the propositional symbols $\{x_1, \ldots, x_n\}$ such that $v^*(\mu(f)) = \mathbf{t}$ if and only if $f(v(x_1), \ldots, v(x_n)) = \mathbf{t}$ for every valuation v.

Syntax of Quantified Boolean Formulae. Symbol $\exists$ stands for existential quantifier, $\forall$ stands for universal quantifier and q stands for any quantifier. The set **QBF** of quantified Boolean formulae is defined inductively as follows: if F is an element of **PROP** then it is also an element of **QBF**, if F is an element of **QBF** and x is a propositional symbol then $(\exists x\, F)$ and $(\forall x\, F)$ are elements of **QBF**, if F is an element of **QBF** then $\neg F$ is an element of **QBF**, if F and G are elements of **QBF** and $\circ$ is an element of O then $(F \circ G)$ is an element of **QBF**. An occurrence of a propositional symbol x is free if it does not appeared into the scope of $\exists x$ or $\forall x$. A QBF is closed if it contains no free occurrence of propositional symbol. A substitution is a function from the set of propositional symbols to the set of formulae. We define a substitution of x by F in G, denoted $[x \leftarrow F](G)$, as the formula obtained from G by replacing all the occurrences of x by F except for the occurrences of x under the scope of a quantifier associated to x. A binder is a character string $q_1x_1 \ldots q_nx_n$ with $x_1, \ldots, x_n$ some separate propositional symbols and $q_1, \ldots, q_n$ some quantifiers. A QBF QM is under prenex form if Q is a binder and M is a Boolean formula. We define the function $\overline{(.)}$ which inverts the quantifiers and is such that $\overline{(\exists x)} = \forall x$ and $\overline{(\forall x)} = \exists x$; this function is extended classically to the binder.

Quantified Boolean Formula Semantics. The QBF semantics $[[.]] : \mathbf{VAL\_PROP} \rightarrow \mathbf{BOOL}$ presented uses the semantics of the Boolean operators (and constants) of propositional logic and is defined inductively by:

[3] In fact, moves after victory are any and rules are not necessarily respected otherwise victory may be invalidated after the party is over.

$[[\bot]](v) = i_\bot$
$[[\top]](v) = i_\top$
$[[x]](v) = v(x)$ if $x \in \mathcal{PS}$
$[[(F \circ G)]](v) = i_\circ([[F]](v), [[G]](v))$
if $F, G \in \mathbf{QBF}$ and $\circ \in O$
$[[\neg F]](v) = i_\neg([[F]](v))$ if $F \in \mathbf{QBF}$
$[[(\exists x\, F)]](v) =$
$i_\vee([[F]](v[x := \mathbf{t}]), [[F]](v[x := \mathbf{f}]))$
if $F \in \mathbf{QBF}$
$[[(\forall x\, F)]](v) =$
$i_\wedge([[F]](v[x := \mathbf{t}]), [[F]](v[x := \mathbf{f}]))$
if $F \in \mathbf{QBF}$

A closed QBF F is valid if $[[F]](v) = \mathbf{t}$ for every valuation v. For example the QBF $\exists a \exists b \forall c((a \vee b) \leftrightarrow c)$ is not valid while the QBF $\forall c \exists a \exists b((a \vee b) \leftrightarrow c)$ is. This example shows that the order of quantifiers is crucial to decide the validity of a QBF.

As in the propositional case, an equivalence relation denoted $\equiv$ is defined for the QBF by $F \equiv G$ if $[[F]](v) = [[G]](v)$ for every valuation v. In connection with the above example, $\exists x \exists y F \equiv \exists y \exists x F$, $\forall x \forall y F \equiv \forall y \forall x F$, $\neg \exists x F \equiv \forall x \neg F$ and $\neg \forall x F \equiv \exists x \neg F$ but $\exists a \exists b \forall c((a \vee b) \leftrightarrow c) \not\equiv \forall c \exists a \exists b((a \vee b) \leftrightarrow c)$.

The decision semantics of QBF is extended to a functional semantics of prenex QBF thanks to the notion of *functional valuation*: a partial function sk of the set of propositional symbols to the set of the Boolean functions is a functional valuation for a prenex QBF if for every existentially quantified propositional symbol x there exists a unique pair $(x \mapsto \hat{x}) \in sk$ such that the Boolean function $\hat{x}$ has for arity the number of universally quantified propositional symbols which precede x in the binder[4]. The set of functional valuations is denoted **VAL_FONC**. The decision semantics of QBF is extended to a functional semantics $[[.]] : \mathbf{VAL\_PROP} \times \mathbf{VAL\_FONC} \to \mathbf{BOOL}$ for prenex QBF:

$[[F]](v, sk) = v^*(F)$ if $F \in \mathbf{PROP}$
$[[(\exists x\, F)]](v, sk) = [[F]](v[x := \hat{x}], sk)$
if $(x \mapsto \hat{x}) \in sk$ and $F \in \mathbf{QBF}$
$[[(\exists x\, F)]](v, sk) =$
$i_\vee([[F]](v[x := \mathbf{t}], sk), [[F]](v[x := \mathbf{f}], sk))$
if $(x \mapsto \hat{x}) \notin sk$ and $F \in \mathbf{QBF}$
$[[(\forall x\, F)]](v, sk)$
$= i_\wedge([[F]](v[x := \mathbf{t}], sk(\mathbf{t})),$
$[[F]](v[x := \mathbf{f}], sk(\mathbf{f})))$
if $F \in \mathbf{QBF}$.

A functional valuation sk is a QBF model for a prenex closed QBF F if $[[F]](v, sk) = \mathbf{t}$, for any valuation v. A closed prenex QBF is valid if and only if it admits (at least) a QBF model.

We recall that the decision problem of the satisfiability of a Boolean formula (SAT) is NP-complete while the decision problem of the validity of a QBF is PSPACE-complete (Stockmeyer, 1977).

[4] A functional valuation is a set of Skolem functions.

3 NON-PRENEX QBF SEMANTICS

The introduction shows that a functional semantics for non-prenex QBF is useful for QBF designer in order to preserve the expected meaning of quantifiers and locality of knowledge. In what follows we give a more technical presentation of our issue on an example (§ 3.1), then our proposition about a non-prenex QBF semantics (§ 3.2).

3.1 Motivations

Let $\rho = (\forall t\ (t \leftrightarrow ((\exists x\ \phi(x,t)) \vee \neg(\exists y\ \psi(y,t)))))$ be a QBF with $\phi(x,t)$ and $\psi(y,t)$ also two QBF.

If we linearize this QBF (minimizing the dependencies), we obtain the following prenex QBF:

$$\begin{array}{rcl} \rho & \equiv & \forall t((t \to ((\exists x\ \phi(x,t)) \vee \neg(\exists y\ \psi(y,t)))) \\ & & \wedge(((\exists x'\ \phi(x',t)) \vee \neg(\exists y'\ \psi(y',t))) \to t)) \\ & \equiv & \forall t \exists x \exists y' \forall x' \forall y((t \to (\phi(x,t) \vee \neg\psi(y,t))) \\ & & \wedge((\phi(x',t) \vee \neg\psi(y',t)) \to t)) \end{array}$$

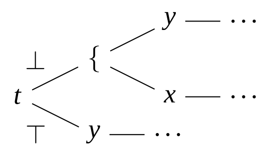
The designer of the QBF who chooses to model its problem by $\neg(\exists y\ \psi(y,t))$ and not by $(\forall y\ \neg\psi(y,t))$ waits for the existentially quantified propositional symbols x and y Skolem functions with parameter t. But the propositional symbol y is now universally quantified and is one of the parameters of the model associated to $\psi(y,t)$. Moreover, a new existentially quantified propositional symbol y' has appeared which has no meaning for the designer.

We have developped a QCSP search-based solver which is validity oriented: validity check $QF \equiv \top$ is treated unchanged but non validity check $QF \equiv \bot$ is replaced by the equivalent check $\overline{Q}\neg F \equiv \top$. If we look at the execution of a constraint-based validity-oriented solver, for the check $(\rho \equiv \top)$ we obtain the following propagations:

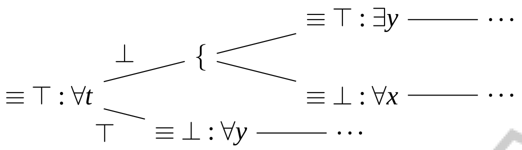
- If t is substituted by $\top$, then necessarily $((\exists x\ \phi(x,\top)) \vee \neg(\exists y\ \psi(y,\top))) \equiv \top$ and then either $(\exists x\ \phi(x,\top)) \equiv \top$ or $(\exists y\ \psi(y,\top)) \equiv \bot$.
- If t is substituted by $\bot$, then necessarily $((\exists x\ \phi(x,\bot)) \vee \neg(\exists y\ \psi(y,\bot))) \equiv \bot$ and then $(\exists x\ \phi(x,\bot)) \equiv \bot$ and $(\exists y\ \psi(y,\bot)) \equiv \top$.

For a validity-oriented solver, $(\exists y\ \psi(y,\top)) \equiv \bot$ is treated as $(\forall y\ \neg\psi(y,\top)) \equiv \top$ and $(\exists x\ \phi(x,\bot)) \equiv \bot$ is treated as $(\forall x\ \neg\phi(x,\bot)) \equiv \top$.

A model for a non-prenex QBF will have the following shape (if $(\exists y\ \psi(y,\top)) \equiv \bot$):



while a semantic certificate[5] for a validity-oriented solver will have the shape:



In the model, the binders are positively interpreted, i.e. respecting the modeling, while in the certificate, the binders are either positively interpreted ($\equiv \top$), as for example for the QBF ρ itself and $(\exists y\, \psi(y, \perp))$, or negatively ($\equiv \perp$), as for example for the QBF $(\exists y\, \psi(y, \top))$ and $(\exists x\, \phi(x, \perp))$ depending on whether the QBF is valid or not.

3.2 Functional Semantics for Non-prenex QBF

To be able to define models for non-prenex QBF, we need a new definition of QBF allowing easier access to nested binders.

Definition 1 *(QBF). Let $\mathcal{D}$ be a set of* definition symbols *such that $\mathcal{D} \cap \mathcal{PS} = \emptyset$. A* quantified Boolean formula *(or QBF) is a set of triplets*

$$def := Q\Sigma$$

($def \in \mathcal{D}$, Q a binder, Σ a propositional formula defined on $\mathcal{D} \cup \mathcal{PS}$ and $Q\Sigma$ a fragment*) associated to a partial order over the definition symbols with a least element, the definition symbol root, such that*

- *every definition symbol appears only once in the left-hand side;*
- *the definition symbol root only appears once in the right-hand side;*
- *any other definition symbol than the root appears only once in the right-hand side;*
- *every definition symbol in the left-hand side is smaller than the definition symbol which appears in the right-hand side;*
- *any binder except the one associated with the root, is empty.*

The function $(\dagger)$ is such that, for every triplet $def := Q\Sigma$ of a QBF, $def^{\dagger} = Q\Sigma$.

[5]A certificate is any piece of information that provides self-supporting evidence of the correctness of an execution.

The four first items define a tree structure while the last item allows to express easily the semantics.

If someone substitutes the definition symbols (except the root) of a QBF in their right-hand sides of the definitions, a "classical" QBF (i.e. in the meaning of the preliminary section) is obtained. In what follows, we makes no distinction between the QBF, its root and its "classical" definition.

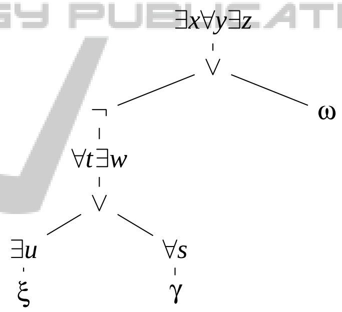
Example 1. We define a QBF with root ψ :

$$\begin{array}{lll} \psi & := & \exists x \forall y \exists z(\neg\psi_0 \vee \omega) \\ \psi_0 & := & \forall t \exists w(\psi_{0.0} \vee \psi_{0.1}) \\ \psi_{0.0} & := & \exists u \xi \\ \psi_{0.1} & := & \forall s \gamma \end{array}$$

with $\omega = (x \wedge y \wedge z)$, $\xi = ((u \vee y) \wedge \neg t)$ and $\gamma = ((s \vee w) \wedge t)$. This QBF is none other, by substitution, than the "classical" non-prenex QBF:

$$\psi = \exists x \forall y \exists z(\neg \forall t \exists w(\exists u \xi \vee \forall s \gamma) \vee \omega)$$

whose representation as a tree is:



To define the semantics of our definition of QBF, we associate first to the propositional symbols of the binder Boolean functions; such a function is either positive if the binder is considered unchanged or negative if the binder is considered in its reversed polarity: universal quantifiers are existantially interpreted and reciprocally.

Definition 2 *(Local Valuation). A* positive local valuation *(resp.* negative local valuation*) of a QBF, whose root definition has a binder Q, is a partial function vl from the set of propositional symbols to the set of Boolean functions such that for every existentially (resp. universally) quantified propositional symbol x of the binder Q, there exists a unique pair $(x \mapsto \hat{x}) \in vl$ such that the Boolean function $\hat{x}$ has for arity the number of universally (resp. existentially) quantified propositional symbols which precede x in the binder Q. The set of local valuations is denoted* **VAL_LOC**.

Example 2 *(example 1 continued)*

- *The partial function $vl_u = \{(u \mapsto (\mapsto \mathbf{t}))\}$ is a positive local valuation for the QBF of root $\psi_{0.0}$,*

- *the partial function* $vl_s = \{(s \mapsto (\mapsto \mathbf{f}))\}$ *is a negative local valuation for the QBF of root* $\psi_{0.1}$,
- *the partial function* $\emptyset$ *is a positive local valuation for the QBF of root* $\psi_{0.1}$,
- *the partial function* $vl_w = \{(w \mapsto \{(\mathbf{t} \mapsto \mathbf{t}), (\mathbf{f} \mapsto \mathbf{t})\})\}$ *is a positive local valuation for the QBF of root* ψ_0,
- *the partial function* $vl_y = \{(y \mapsto \{(\mathbf{t} \mapsto \mathbf{f}), (\mathbf{f} \mapsto \mathbf{f})\})\}$ *is a negative local valuation for the QBF of root* ψ,
- *finally, the partial function* $\{(x \mapsto (\mapsto \mathbf{f})), (z \mapsto \{(\mathbf{t} \mapsto \mathbf{f}), (\mathbf{f} \mapsto \mathbf{t})\})\}$ *is a positive local valuation for the QBF of root* ψ.

A definition symbol def of a triplet $def := Q\Sigma$, if it is interpreted positively, has for semantics a *definition valuation* which may contain

- not only a positive local valuation (i.e. a function on the combinations over the values for the universally quantified propositional symbols of the binder Q) to give a semantics to the existentially quantified propositional symbols of the binder Q,
- but also a semantics to every definition symbol of the right-hand side thanks to a function which associates to every combination over the values for the universally quantified propositional symbols a definition valuation.

Definition 3 *(Definition Valuation and Definition Boolean Function).* *A* definition valuation *of a QBF is a partial function vd from the set of definition symbols to the set of* (p, vl, fbd) *consting of a polarity* $p \in \{+,-\}$*, a local valuation vl and a definition Boolean function fbd under the following contraint: for every* $(d \mapsto (p, vl, fbd)) \in vd$ *if the polarity p is* $+$ *then the local valuation vl and the definition Boolean function fbd are positive for the QBF with root definition symbol d otherwise the polarity is* $-$ *and the local valuation and the definition Boolean function are negative. A* definition association *is a quadruplet* $(d \mapsto (p, vl, fbd))$*. The set of QBF valuations is denoted* **VAL_DEF**.

A positive definition Boolean function *(resp.* negative*) of a QBF, whose root definition has a binder Q, is a partial Boolean function fbd from the set of definition valuations such that fbd has for arity the number of universally (resp. existentially) quantified propositional symbols of the binder Q. The set of definition Boolean functions is denoted* **FBD**.

Example 3 *(example 2 continued)*

- *The quadruplet* $ad_{0.0} = (\psi_{0.0} \mapsto (+, vl_u, \emptyset))$ *is a definition association for the QBF of root* $\psi_{0.0}$ *(*vl_u *is a positive local valuation);*
- *the quadruplet* $ad^n_{0.1} = (\psi_{0.1} \mapsto (-, vl_s, \emptyset))$ *is a definition association for the QBF of root* $\psi_{0.1}$ *(*vl_s *is negative local valuation);*
- *the quadruplet* $ad^p_{0.0} = (\psi_{0.1} \mapsto (+, \emptyset, \emptyset))$ *is a definition association for the QBF of root* $\psi_{0.1}$.

Thus, the function $vd_0 = \{ad_{0.0}, ad^n_{0.1}\}$ *is a definition valuation.*

The partial function

$$fbd_{\psi_0} = \{(\mathbf{t} \mapsto vd_0), (\mathbf{f} \mapsto \{ad_{0.0}\})\}$$

is a positive definition Boolean function for the QBF of root ψ_0 *and the quadruplet* $ad_0 = (\psi_0 \mapsto (+, vl_w, fbd_{\psi_0}))$ *is its definition association.*

Finally, the partial function

$$fbd_{\psi} = \{((\mathbf{t},\mathbf{t}) \mapsto \emptyset), ((\mathbf{t},\mathbf{f}) \mapsto \{ad_0\}), \\ ((\mathbf{f},\mathbf{t}) \mapsto \{ad_0\}), ((\mathbf{f},\mathbf{f}) \mapsto \emptyset)\}$$

is a negative definition Boolean function for the QBF of root ψ *and the quadruplet* $(\psi \mapsto (+, \emptyset, fbd_{\psi}))$ *is its definition association.*

Definition 4 *(QBF Valuation).* *A* QBF valuation *is a pair consisting of a propositional valuation and a definition valuation. The set of QBF valuations is denoted* **VAL_QBF**.

Definition 5. The semantics of a QBF F is definied by a function

$$[[.]] : \mathbf{VAL\_QBF} \to \mathbf{BOOL}$$

associated with two auxiliary functions

$$[[.]]^+, [[.]]^- : \\ \mathbf{VAL\_PROP} \times \mathbf{VAL\_LOC} \times \mathbf{FBD} \to \mathbf{BOOL}$$

all inductively defined as follows.

for the propositional symbols:

($\mathcal{PS}$) $[[x]](v, vd) = v(x)$ if $x \in \mathcal{PS}$;

for the propositional logical connectors:

($\bot$) $[[\bot]](v, vd) = \mathbf{f}$;

($\top$) $[[\top]](v, vd) = \mathbf{t}$;

($\circ$) $[[(G \circ H)]](v, vd) = i_\circ([[G]](v, vd), [[H]](v, vd))$ if G, H are extended propositional formulas and $\circ \in \{\wedge, \vee, \rightarrow, \leftrightarrow, \oplus\}$;

($\neg$) $[[\neg G]](v, vd) = i_\neg([[G]](v, vd))$ if G is an extended propositional formula.

for the definition symbols:

($\mathcal{D}_p$) $[[d]](v, vd) = [[d^\dagger]]^+(v, vl, fbd)$ if $d \in \mathcal{D}$, and $vd(d) = (+, vl, fbd)$;

($\mathcal{D}_n$) $[[d]](v, vd) = i_\neg([[d^\dagger]]^-(v, vl, fbd))$ if $d \in \mathcal{D}$, and $vd(d) = (-, vl, fbd)$;

for the quantifiers:

$(\exists_1)$ $[[(\exists x\, G)]]^+(v, vl, fbd) =$
$[[G]]^+(v[x := \hat{x}], vl, fbd)$
if G is an extended propositional formula, $x \in \mathcal{PS}$ and $(x \mapsto (\mapsto \hat{x})) \in vl$;

$(\exists_2)$ $[[(\exists x\, G)]]^+(v, vl, fbd) =$
$i_\vee([[G]]^+(v[x := \mathbf{t}], vl, fbd),$
$[[G]]^+(v[x := \mathbf{f}], vl, fbd))$
if G is an extended propositional formula, $x \in \mathcal{PS}$ and there is no pair $(x \mapsto \hat{x}) \in vl$ ($\hat{x}$ a Boolean function) ;

$(\exists_3)$ $[[(\exists x\, G)]]^-(v, vl, fbd) =$
$i_\wedge([[G]]^-(v[x := \mathbf{t}], vl(\mathbf{t}), fbd(\mathbf{t})),$
$[[G]]^-(v[x := \mathbf{f}], vl(\mathbf{f}), fbd(\mathbf{f})))$
if G is an extended propositional formula and $x \in \mathcal{PS}$.

$(\forall_1)$ $[[(\forall x\, G)]]^-(v, vl, fbd) =$
$[[G]]^-(v[x := \hat{x}], vl, fbd)$
if G is an extended propositional formula, $x \in \mathcal{PS}$ and $(x \mapsto (\mapsto \hat{x})) \in vl$;

$(\forall_2)$ $[[(\forall x\, G)]]^-(v, vl, fbd) =$
$i_\vee([[G]]^-(v[x := \mathbf{t}], vl, fbd),$
$[[G]]^-(v[x := \mathbf{f}], vl, fbd))$
if G is an extended propositional formula, $x \in \mathcal{PS}$ and there is no pair $(x \mapsto \hat{x}) \in vl$ ($\hat{x}$ Boolean function).

$(\forall_3)$ $[[(\forall x\, G)]]^+(v, vl, fbd) =$
$i_\wedge([[G]]^+(v[x := \mathbf{t}], vl(\mathbf{t}), fbd(\mathbf{t})),$
$[[G]]^+(v[x := \mathbf{f}], vl(\mathbf{f}), fbd(\mathbf{f})))$
if G is an extended propositional formula and $x \in \mathcal{PS}$.

$(\mathcal{P}_p)$ $[[G]]^+(v, vl, \{(\mapsto vd)\}) = [[G]](v, vd)$
if G is an extended propositional formula.

$(\mathcal{P}_n)$ $[[G]]^-(v, vl, \{(\mapsto vd)\}) = i_\neg([[G]](v, vd))$
if G is an extended propositional formula.

If we restrict Definition 5 to the rules $(\top)$, $(\bot)$, $(\circ)$, $(\neg)$ and $(\mathcal{PS})$ (neglecting the arguments for the local valuation and the definition Boolean function and replacing $[[.]]^+$ by $[[.]]$), the propositional semantics is obtained. If we add the rules $(\exists_2)$ and $(\forall_3)$ the semantics of decision procedure for non-prenex QBF is then obtained. The rule $(\mathcal{D}_p)$ interprets the definition symbol and begins the interpretation of the binder, of the fragment associated to the symbol, positively; the rule $(\mathcal{P}_p)$ ends the interpretation of the binder and evaluates the extended propositional formula of the fragment. The rule $(\mathcal{D}_n)$ interprets the definition symbol and begins the interpretation of the binder, of the fragment associated to the symbol, negatively ; the rule $(\mathcal{P}_n)$ ends the interpretation of the binder and evaluates the extended propositional formula of the fragment ; both combinated for a fragment $Q\Sigma$ the rules apply the equivalence $Q\Sigma \equiv \neg\overline{Q}\neg\Sigma$.

Example 4 *(example 3 continued)***.** *We show in this example how the semantics is applied on the fragment* $\exists x \forall y \exists z(\neg\psi_0 \vee \omega)$ *which is negatively interpreted*

$$\exists x \forall y \exists z(\neg\psi_0 \vee \omega) \equiv \neg\forall x \exists y \forall z \neg(\neg\psi_0 \vee \omega) \qquad (1)$$

by:

1. *the application of the rule* $(\mathcal{D}_n)$ *which introduces the first negation, the one before the binder;*
2. *the elimination of the quantifiers with a reversed interpretation according to the rules* $(\exists_3)$ *and* $(\forall_2)$*,*
3. *the application of the rule* $(\mathcal{P}_n)$ *which introduces the seconde negation, the one after the binder.*

Let us compute $[[\psi]](\emptyset, vd)$ *with* $vd = \{(\psi \mapsto (-, \emptyset, fbd_\psi))\}$*. Let* $\sigma = (\neg\psi_0 \vee \omega)$*.*

$$\begin{array}{ll} & [[\psi]](\emptyset, \{(\psi \mapsto (-, \emptyset, fbd_\psi))\}) \\ \stackrel{\mathcal{D}_n}{=} & i_\neg([[\psi^\dagger]]^-(\emptyset, \emptyset, fbd_\psi)) \\ \stackrel{\dagger}{=} & i_\neg([[\exists x \forall y \exists z \sigma]]^-(\emptyset, \emptyset, fbd_\psi)) \\ \stackrel{\exists_3}{=} & i_\neg(i_\wedge(\ [[\forall y \exists z \sigma]]^-([x := \mathbf{t}], \emptyset, fbd_\psi(\mathbf{t})), \\ & \qquad [[\forall y \exists z \sigma]]^-([x := \mathbf{f}], \emptyset, fbd_\psi(\mathbf{f})))) \end{array}$$

The existential quantifier is negatively interpreted.

Let us denote $v^x = [x := \mathbf{t}]$ *and let us compute in details:*

$$\begin{array}{ll} & [[\forall y \exists z \sigma]]^-(v^z, \emptyset, fbd_\psi(\mathbf{t})) \\ \stackrel{\forall_2}{=} & i_\vee(\ [[\exists z \sigma]]^-(v^x[y := \mathbf{t}], \emptyset, fbd_\psi(\mathbf{t})), \\ & \qquad [[\exists z \sigma]]^-(v^x[y := \mathbf{f}], \emptyset, fbd_\psi(\mathbf{t}))) \end{array}$$

The universal quantifier is negatively interpreted and in its definitional version of its semantics (since the local valuation is empty).

Let us denote $v^y = v^x[y := \mathbf{t}]$ *and let us compute in details:*

$$\begin{array}{ll} & [[\exists z \sigma]]^-(v^y, \emptyset, fbd_\psi(\mathbf{t})) \\ \stackrel{\exists_3}{=} & i_\wedge(\ [[\sigma]]^-(v^y[z := \mathbf{t}], \emptyset, fbd_\psi(\mathbf{t}, \mathbf{t})), \\ & \qquad [[\sigma]]^-(v^y[z := \mathbf{f}], \emptyset, fbd_\psi(\mathbf{t}, \mathbf{f}))) \end{array}$$

Let us denote $v^z = v^y[z := \mathbf{f}]$*,* $fbd = fbd_\psi(\mathbf{t}, \mathbf{f})$ *and let us compute in details:*

$$\begin{array}{ll} & [[\sigma]]^-(v^y[z := \mathbf{f}], \emptyset, fbd) \\ \stackrel{\mathcal{P}_n}{=} & i_\neg([[\sigma]](v^z, fbd)) \\ \stackrel{\circ}{=} & i_\neg(i_\vee([[\neg\psi_0]](v^z, fbd), [[\omega]](v^z, fbd))) \\ \stackrel{\neg}{=} & i_\neg(i_\vee(i_\neg([[\psi_0]](v^z, fbd)), [[\omega]](v^z, fbd))) \end{array}$$

By the rule $(\mathcal{P}_n)$*, the negative interpretation of the end of the binder introduces the seconde negation of (1).*

Since the formula ω is propositional formula, let us denote $b_\omega = [[\omega]](v^z, fbd) = (v^z)^*(\omega)$.

$$\begin{aligned}&[[\sigma]]^-(v^y[z := \mathbf{f}], \emptyset, fbd)\\ &\quad = i_\neg(i_\vee(i_\neg([[\psi_0]](v^z, fbd)), b_\omega))\end{aligned}$$

If we add the rules $(\exists_1)$ to the rules $(\top)$, $(\bot)$, $(\circ)$, $(\neg)$, $(\mathcal{PS})$, $(\exists_2)$ and $(\forall_3)$ (ignoring the argument for the definition Boolean function) then the functional semantics of prenex QBF is obtained.

Example 5 *(example 4 continued). This example shows the positive interpretation of a binder according to rules $(\mathcal{D}_p)$, $(\forall_3)$, $(\exists_1)$ and $(\mathcal{P}_p)$ and the interpretation of the existential quantifier in its functional semantics and not decision one.*

We recall that $\psi_0 := \forall t \exists w(\psi_{0.0} \vee \psi_{0.1})$, $vl_w = \{(w \mapsto \{(\mathbf{t} \mapsto \mathbf{t}), (\mathbf{f} \mapsto \mathbf{t})\})\}$, $fbd_{\psi_0} = \{(\mathbf{t} \mapsto vd_0), (\mathbf{f} \mapsto \{ad_{0.0}\})\}$ and $vd_0 = \{(\psi_{0.0} \mapsto (+, vl_u, \emptyset)), (\psi_{0.1} \mapsto (-, vl_s, \emptyset))\}$.

Let us denote $\sigma_0 = (\psi_{0.0} \vee \psi_{0.1})$ and let us compute in details:

$$\begin{aligned}&[[\psi_0]](v^z, (\psi_0 \mapsto (+, vl_w, fbd_{\psi_0})))\\ &\overset{(\mathcal{D}_p)}{=} [[\psi_0^\dagger]]^+(v^z, vl_w, fbd_{\psi_0})\\ &\overset{\dagger}{=} [[\forall t \exists w \sigma_0]]^+(v^z, vl_w, fbd_{\psi_0})\\ &\overset{(\forall_3)}{=} i_\wedge([[\exists w \sigma_0]]^+(v^z[t := \mathbf{t}], vl_w(\mathbf{t}), fbd_{\psi_0}(\mathbf{t})),\\ &\qquad [[\exists w \sigma_0]]^+(v^z[t := \mathbf{f}], vl_w(\mathbf{f}), fbd_{\psi_0}(\mathbf{f})))\end{aligned}$$

We have $vl_w(\mathbf{t}) = \{(w \mapsto (\mapsto \mathbf{t}))\}$ and let us denote $v^t = v^z[t := \mathbf{t}]$. Let us compute in details:

$$\begin{aligned}&[[\exists w \sigma_0]]^+(v^t, vl_w(\mathbf{t}), fbd_{\psi_0}(\mathbf{t}))\\ &= [[\exists w \sigma_0]]^+(v^t, \{(w \mapsto (\mapsto \mathbf{t}))\}, fbd_{\psi_0}(\mathbf{t}))\\ &\overset{(\exists_1)}{=} [[\sigma_0]]^+(v^t[w := \mathbf{t}], vl_w(\mathbf{t}), fbd_{\psi_0}(\mathbf{t}))\end{aligned}$$

The existential quantifier is not interpreted with its decision semantics but the value of the propositional symbol is obtained from the local valuation.

We have $fbd_{\psi_0}(\mathbf{t}) = \{(\mapsto vd_0)\}$ and let us denote $v^w = v^t[w := \mathbf{t}]$.

$$\begin{aligned}&[[(\psi_{0.0} \vee \psi_{0.1})]]^+(v^w, vl_w(\mathbf{t}), \{(\mapsto vd_0)\})\\ &\overset{(\circ)}{=} i_\vee(\quad [[\psi_{0.0}]]^+(v^w, vl_w(\mathbf{t}), \{(\mapsto vd_0)\}),\\ &\qquad\quad [[\psi_{0.1}]]^+(v^w, vl_w(\mathbf{t}), \{(\mapsto vd_0)\}))\\ &\overset{(\mathcal{P}_p)}{=} i_\vee([[\psi_{0.0}]](v^w, vd_0), [[\psi_{0.1}]](v^w, vd_0))\end{aligned}$$

The propositional formula of a fragment may contain many definition symbols and the associated binders are not necessarily interpreted in the same manner.

Example 6 *(example 5 continued). This example shows the access to two definition associations and the application of the rules $(\mathcal{D}_p)$ and $(\mathcal{D}_n)$ to interpret the definition symbol and its associated binder.*

We recall that

$$vd_0 = \{(\psi_{0.0} \mapsto (+, vl_u, \emptyset)), (\psi_{0.1} \mapsto (-, vl_s, \emptyset))\}$$

Hence $vd_0(\psi_{0.0}) = (+, vl_u, \emptyset)$ and $vd_0(\psi_{0.1}) = (-, vl_s, \emptyset)$. Thus

$$\begin{aligned}&i_\vee([[\psi_{0.0}]](v^w, vd_0), [[\psi_{0.1}]](v^w, vd_0))\\ &\overset{(\mathcal{D}_p)\&(\mathcal{D}_n)}{=} \quad i_\vee([[\psi_{0.0}^\dagger]]^+(v^w, vl_u, \emptyset),\\ &\qquad\qquad [[\psi_{0.1}^\dagger]]^-(v^w, vl_s, \emptyset))\end{aligned}$$

We give without proof (but the arguments are clear from above) soundness and completeness theorems w.r.t. the decision semantics of the "classical" QBF.

Theorem 1 *(Soundness). Let σ be a QBF, v a propositional valuation and vd definition valuation. If $[[\sigma]](v, vd) = \mathbf{t}$ then $v^*(\sigma) = \mathbf{t}$.*

Theorem 2 *(Completeness). Let σ be a QBF and v a propositional valuation. If $v^*(\sigma) = \mathbf{t}$ then there exists (at least) a definition valuation vd such that $[[\sigma]](v, vd) = \mathbf{t}$.*

4 CONCLUSIONS

We have proposed in this article a functional semantics for non-prenex quantified Boolean formulas. The proposed formalism is symmetrical w.r.t. the validity or the non-validity and allows to associate different interpretations to quantifiers. In particular, it allows to follow the choice of the designer of the QBF w.r.t. the quantifiers. In case of QBF solvers based on a quantified search algorithm, the extraction of the solution is easy since our functional semantics follows the inductive structure of the non-prenex QBF. Since the prenexing process is not applied, dependencies between propositional symbols are kept and the computed QBF valuation may be directly interpreted by the designer as a solution of its problem. Moreover, a QBF valuation representing a winning strategy for a finite two-player game develops no definition valuation for the propositional symbols beyond the fulfilled victory conditions.

Our formalism is also enough flexible to allow to define the notion of certificate for search procedure for non-prenex QBF. The way the certificate certifies the soundness of the result is not developped due to lack of space but is independent of the specifications of the solver.

This work is implemented in Prolog[6] and is being implemented for a quantified search algorithm based

[6]This work is accessible at the URL: http://www.info.univ-angers.fr/pub/stephan/Research/Download.html

on the Gecode system[7] and will be available soon. In our implementation with the Gecode system, if the input format of QBF is "classical", the internal structure follows our definition of QBF.

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[7]The solver is in fact a QCSP solver with a constraint approach for QBF solving.