# Stabilization of a Trajectory for Nonlinear Systems using the Time-varying Pole Placement Technique

Yasuhiko Mutoh and Shuhei Naitoh
*Department of Engineering and Applied Sciences, Sophia University, 7-1, Kioicho, Chiyoda-ku, Tokyo, Japan*

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Abstract: The author proposed the simple design procedure of pole placement controller for linear time-varying systems. The feedback gain can be obtained directly from the plant parameters without transforming the system into any standard form. This design method will be applied to the problem of stabilization of some desired trajectory of nonlinear systems.

## 1 INTRODUCTION

In general, to design a controller for nonlinear systems, we approximate the system around some equilibrium point by a linear time-invariant system, and then, linear control design methods are applied. But, if we need to stabilize some particular trajectory, in practice, we approximate the nonlinear system around multiple points for designing the controller. In such a case, gain scheduling method or some other similar scheme will be necessary. Nonlinear controllers, of course, are one of other choices. The most simple idea is to approximate the nonlinear system around some trajectory using a linear time-varying system. However, since, the design method for linear time-varying systems is not necessarily simple (Nguyen(1987)) (Valsek(1995)) (Valsek(1999)), the gain scheduling strategy may be the first choice for such a control design problem, in general. The author et. al. have proposed simple pole placement controller design method (Mutoh(2011))(Mutoh and Kimura (2011)). Such controller is obtained by finding a new output signal so that the relative degree from the input to this new output is equal to the system degree. We do not need to transform the system into any standard form for the controller design. In this paper, such a pole placement controller design procedure will be applied to the problem of the stabilization of some desired trajectory of nonlinear systems. Section 2 will present how to design the pole placement controller for linear time-varying systems. For the simplicity, we consider only single-input single-output systems. Then, Section 3 will show an example of stabilizing some desired trajectory of a nonlinear system.

## 2 POLE PLACEMENT FOR LINEAR TIME-VARYING SYSTEMS

Consider the following linear time-varying system.

$$\dot{x} = A(t)x + b(t)u \tag{1}$$

Here, $x(t) \in R^n$ is the state vector and $u(t) \in R^1$ is the input signal. $A(t) \in R^{n\times n}$ and $b(t) \in R^n$ are time varying coefficient matrices, which are smooth functions of $t$.

The controllability matrix, $U_c(t)$, of this system is

$$U_c(t) = [b_0(t),\ b_1(t),\ \cdots,\ b_{n-1}(t)] \tag{2}$$

where $b_i(t)$ is defined by the following recurrence equation.

$$\begin{cases} b_0(t) = b(t) \\ b_i(t) = A(t)b_{i-1}(t) - \dot{b}_{i-1}(t), \ \ i = 1,2,\cdots \end{cases} \tag{3}$$

The system (1) is controllable if and only if $U_c(t)$ is nonsingular for all $t$.

The problem is to find the state feedback

$$u(t) = k^T(t)x(t) \tag{4}$$

for the system (1) which makes the closed loop system equivalent to some time invariant linear system with arbitrarily stable poles.

For this purpose, consider the problem of finding a new output signal $y$ such that the relative degree from $u$ to $y$ is $n$. Here, $y$ has the following form.

$$y = c^T(t)x \tag{5}$$


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Then, the problem is to find a vector $c(t) \in R^n$ that satisfies this condition.

Let $c_i^T(t)$ be defined by

$$\begin{cases} c_0^T(t) = c^T(t) \\ c_i^T(t) = \dot{c}_{i-1}^T(t) + c_{i-1}^T(t)A(t), \quad i = 1, 2, \cdots \end{cases} \tag{6}$$

Then, we have the following lemma.

**Lemma 1.** *The relative degree from $u$ to $y$ is $n$ if and only if*

$$\begin{cases} c_0^T(t)b(t) = 0 \\ c_1^T(t)b(t) = 0 \\ \quad\vdots \\ c_{n-2}^T(t)b(t) = 0 \\ c_{n-1}^T(t)b(t) = \lambda(t), \quad \lambda(t) \neq 0 \end{cases} \tag{7}$$

**Proof:** By differentiating $y$ successively, we have the following equations from (7).

$$\begin{aligned} y &= c^T(t)x \\ &= c_0^T(t)x \\ \dot{y} &= \left(\dot{c_0}^T(t) + c_0^T(t)A(t)\right)x + c_0^T(t)b(t)u \\ &= c_1^T(t)x + c_0^T(t)b(t)u \\ &= c_1^T(t)x \\ \ddot{y} &= \left(\dot{c}_1^T(t) + c_1^T(t)A(t)\right)x + c_1^T(t)b(t)u \\ &= c_2^T(t)x + c_1^T(t)b(t)u \\ &= c_2^T(t)x \\ &\ \ \vdots \\ y^{(n-1)} &= c_{n-1}^T(t)x + c_{n-2}^T(t)b(t)u \\ &= c_{n-1}^T(t)x \\ y^{(n)} &= c_n^T(t)x + c_{n-1}^T(t)b(t)u \\ &= c_n^T(t)x + \lambda(t)u \end{aligned} \tag{8}$$

This implies that the relative degree from $u$ to $y$ becomes $n$ and vice versa. Note that $\lambda(t) = 1$ can be a simple choice. But, as shown later in examples, some particular choice for $\lambda(t)$ makes a design calculation simpler.

**Lemma 2.** *If the relative degree from $u$ to $y = c^T(t)x$ is $n$, we have the following relation.*

$$\begin{aligned} &[c_0^T(t)b(t),\ c_1^T(t)b(t),\ \cdots,\ c_{n-1}^T(t)b(t)] \\ &\quad = [c^T(t)b_0(t),\ c^T(t)b_1(t),\ \cdots,\ c^T(t)b_{n-1}(t)] \end{aligned} \tag{9}$$

**Proof:** From (3) and (6),

$$c_0^T(t)b(t) = c^T(t)b(t) = c^T(t)b_0(t) \tag{10}$$

Using (7), we have

$$\dot{c}_0^T(t)b(t) = -c_0^T(t)\dot{b}(t) \tag{11}$$

and then,

$$\begin{aligned} c_1^T(t)b(t) &= \dot{c}_0^T(t)b(t) + c_0^T(t)A(t)b(t) \\ &= -c_0^T(t)\dot{b}(t) + c_0^T(t)A(t)b(t) \\ &= c_0^T(t)b_1(t) \\ &= c^T(t)b_1(t) \end{aligned} \tag{12}$$

Similarly, using (7), the following relations are obtained.

$$\begin{aligned} \dot{c}_0^T(t)b_1(t) = -c_0^T(t)\dot{b}_1(t) \\ \dot{c}_1^T(t)b(t) = -c_1^T(t)\dot{b}(t) \end{aligned} \tag{13}$$

From which we have

$$\begin{aligned} c_2^T(t)b(t) &= \dot{c}_1^T(t)b(t) + c_1^T(t)A(t)b(t) \\ &= -c_1^T(t)\dot{b}(t) + c_1^T(t)A(t)b(t) \\ &= c_1^T(t)b_1(t) \\ &= \dot{c}_0^T(t)b_1(t) + c_0^T(t)A(t)b_1(t) \\ &= -c_0^T(t)\dot{b}_1(t) + c_0^T(t)A(t)b_1(t) \\ &= c_0^T(t)b_2(t) \\ &= c^T(t)b_2(t) \end{aligned} \tag{14}$$

By continuing this operation, (9) is obtained.

From the above, (9) can be written as

$$\begin{aligned} &[c_0^T(t)b(t), c_1^T(t)b(t), \cdots,\ c_{n-1}^T(t)b(t)] \\ &\quad = c^T(t)U_c(t) \\ &\quad = [0,\ 0,\ \cdots,\ \lambda(t)] \end{aligned} \tag{15}$$

then, we have the following Theorem.

**Theorem 1.** *If the system (1) is controllable, a time varying vector, $c^T(t)$, such that the relative degree from $u$ to a new output $y = c^T(t)x$ is $n$, can be obtained by the following equation.*

$$c^T(t) = [0,\ 0,\ \cdots,\ \lambda(t)]U_c^{-1}(t) \tag{16}$$

Let the stable characteristic polynomial of the desired linear time invariant system be

$$q(p) = p^n + \alpha_{n-1}p^{n-1} + \cdots + \alpha_0 \tag{17}$$

where $p$ is a differential operator. By multiplying $i$-th equation of (8) $\alpha_i$ ($\alpha_n = 1$) and summing them up, we have

$$q(p)y = d^T(t)x + \lambda(t)u \tag{18}$$

where $d^T(t)$ is defined by

$$d^T(t) = [\alpha_0, \alpha_1, \cdots, \alpha_{n-1}, 1] \begin{bmatrix} c_0^T(t) \\ c_1^T(t) \\ \vdots \\ c_{n-1}^T(t) \\ c_n^T(t) \end{bmatrix} \tag{19}$$

Then, by the state feedback

$$u = -\frac{1}{\lambda(t)}d^T(t)x \tag{20}$$

a new output $y$ satisfies the following closed loop system equation.

$$q(p)y = 0 \tag{21}$$

This implies that by applying the state feedback (20) to the system (1), the closed loop state equation becomes

$$\dot{x} = (A(t) - b(t)d^T(t))x. \tag{22}$$

Then, from (8), using the state transformation matrix

$$T(t) = \begin{bmatrix} c_0^T(t) \\ c_1^T(t) \\ \vdots \\ c_{n-1}^T(t) \end{bmatrix} \tag{23}$$

the following state transformation can be defined.

$$\xi = T(t)x \tag{24}$$

where

$$\xi = \begin{bmatrix} y(t) \\ \dot{y}(t) \\ \vdots \\ y^{(n-1)}(t) \end{bmatrix} \tag{25}$$

Then, (21) implies that the closed loop system (22) is equivalent to some linear time-invariant system that has desired stable eigenvalues, i.e.,

$$\begin{aligned} \dot{\xi} &= \{T(t)(A(t) - b(t)d^T(t))T^{-1}(t) - T(t)\dot{T}^{-1}(t)\}\xi \\ &= \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ \vdots & & & 1 \\ -\alpha_0 & \cdots & \cdots & -\alpha_{n-1} \end{bmatrix} \xi \\ &= A^*\xi \end{aligned} \tag{26}$$

where, the characteristic polynomial of $A^*$ is $q(s)$ in (17).

As for the nonsingularity of $T(t)$, we have the following Theorem.

**Theorem 2.** *If the system (1) is controllable, $T(t)$ defined by (23) is nonsingular.*

**Proof:** From (3) and (6),

$$\begin{aligned} c_{k+1}^T(t)b_j(t) &= \dot{c}_k^T(t)b_j(t) + c_k^T(t)A(t)b_j(t) \\ &= -c_k^T(t)\dot{b}_j(t) + c_k^T(t)A(t)b_j(t) \\ &= c_k^T(t)b_{j+1}(t) \end{aligned} \tag{27}$$

Then, from (7),

$$\begin{cases} c_0^T(t)b_0(t) = 0 \\ \quad \vdots \\ c_0^T(t)b_{n-2}(t) = 0 \\ c_0^T(t)b_{n-1}(t) = \lambda(t) \end{cases} \tag{28}$$

hence,

$$\begin{aligned} T(t)U_c(t) &= \begin{bmatrix} c_0^T(t) \\ \vdots \\ c_{n-1}^T(t) \end{bmatrix} \begin{bmatrix} b_0(t) & \cdots & b_{n-1}(t) \end{bmatrix} \\ &= \begin{bmatrix} 0 & \cdots & \lambda(t) \\ \vdots & \iddots & \vdots \\ \lambda(t) & \cdots & * \end{bmatrix} \end{aligned} \tag{29}$$

which implies that if the system (1) is controllable, $T(t)$ is nonsingular.

The transformation matrix $T(t)$ of a linear time-varying system is called a Lyapunov transformation if both of $T(t)$ and $T^{-1}(t)$ are continuous and bounded function of $t$ (Chen(1999)). It is known that if $T(t)$ is a Lyapunov transformation matrix, $\dot{x} = A(t)x$ is uniformly exponentially stable if and only if $\dot{x} = \{T(t)A(t)T^{-1}(t) - T(t)\dot{T}^{-1}(t)\}x$ is uniformly exponentially stable (Rugh(1987)). Hence, $T(t)$ should be a Lyapunov transformation matrix for the pole placement closed loop system to be uniformly exponentially stable.

In summary, when the linear time-varying system (1) is given, the design procedure of the pole placement is as follows.

**[Pole Placement Design Procedure].**

**[STEP 1].** Calculate $U_c(t)$ according to (2) and (3), and check the controllability of the system.

**[STEP 2].** Calculate $c^T(t)$ using (16).

**[STEP 3].** Calculate $c_0^T(t)$, $c_1^T(t)$, $\cdots$, $c_{n-1}^T(t)$ from $A(t)$ and $c^T(t)$ using (6).

**[STEP 4].** Determine the desired stable characteristic polynomial $q(s)$ in (17) and $d^T(t)$ by (19). Then the pole placement state feedback is obtained as $u = -\frac{1}{\lambda(t)}d^T(t)x$.

## 3 STABILIZATION OF A TRAJECTORY FOR NONLINEAR SYSTEMS

In this section, we consider the stabilization problem of some particular trajectory of a nonlinear system. For this purpose, we approximate the nonlinear system by using a linear time-varying system around some desired trajectory. And then, the simplified pole placement controller design procedure will be applied to stabilize this trajectory.

Consider the following nonlinear system.

$$\dot{x}(t) = f(x(t), u(t)) \tag{30}$$

Here, $x(t) \in R^n$ and $u(t) \in R^1$ are the state vector and the input signal. Let $x^*(t)$ and $u^*(t)$ be some desired trajectory and desired input signal, that is,

$$\dot{x}^*(t) = f(x^*(t), u^*(t)), \qquad x^*(0) = x_0^* \tag{31}$$

where the initial state, $x_0^*$ is on the desired trajectory. Let $\Delta x(t)$ and $\Delta u(t)$ be defined by

$$\begin{aligned} \Delta x(t) &= x(t) - x^*(t) \\ \Delta u(t) &= u(t) - u^*(t) \end{aligned} \tag{32}$$

Then, (30) can be approximated by the following linear time-varying system arround around $(x^*(t), u^*(t))$.

$$\Delta\dot{x}(t) = A(t)\dot{x}(t) + b(t)\Delta u(t) \tag{33}$$

where

$$\begin{aligned} A(t) &= \frac{\partial}{\partial x} f(x^*(t), u^*(t)) \\ b(t) &= \frac{\partial}{\partial u} f(x^*(t), u^*(t)) \end{aligned} \tag{34}$$

Here, $A(t)$ and $b(t)$ may be known as their explicit form of functions or known as numerical data. In any case, the pole placement controller design procedure can be applied to stabilize $\Delta x(t)$.

**[EXAMPLE 1].**

Consider the following Van Der Pol equation as an example of a nonlinear system.

$$\ddot{y} + (1 - y^2)\dot{y} + y = 0 \tag{35}$$

This equation can be presented by the following state equation.

$$\begin{cases} \dot{x_1} = x_2 \\ \dot{x_2} = -x_1 - (1 - x_1^2)x_2 + f(u) \end{cases} \tag{36}$$

Here, we put $u$ as a control input, and it is supposed that $u$ affects the system through the nonlinear function

$$f(u) = 2\left(\frac{2}{1 + exp(-0.5u)} - 1\right). \tag{37}$$

This function is shown in Fig.1.

This system has an unstable limit cycle shown in Fig.2 when the $u(t) = 0$. In fact, the trajectory moves away from the limit cycle as shown in Fig.3 if the inieial condition is slightly away from the limit cicle ($x_1(0) = 1.8$, $x_2(0) = 0$). As well, Fig.4 shows the trajectory when a disturbance $0.5\sin(10t)$ is added to the input signal with its initial condition on the limit cicle.

The linear time-varying system that approximate this nonlinear system around the unstable limit cycle $x_1^*(t)$, $x_2^*(t)$ is written as follows.

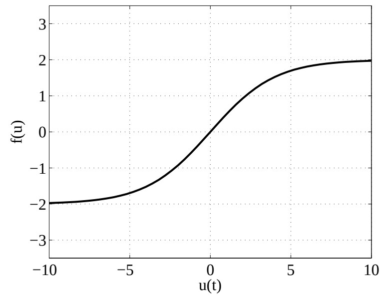


Figure 1: Nonlinear Input Function.

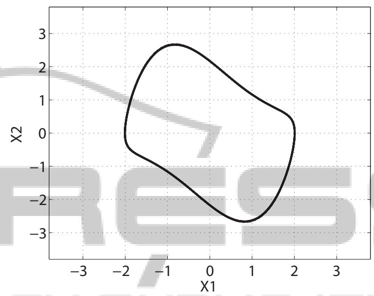


Figure 2: Limit Cycle of the System.

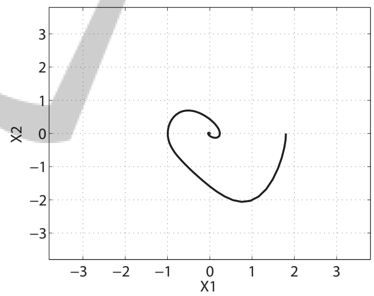


Figure 3: State Response with the Initial Condition near the Limit Cycle.

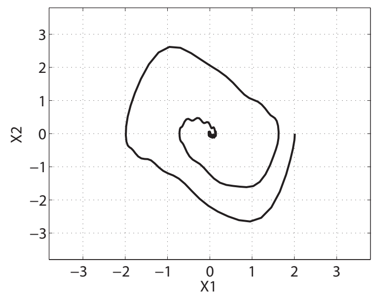


Figure 4: Disturbed State Response with the Initial Condition on the Limit Cycle.

$$\frac{d}{dt}\begin{bmatrix} \Delta x_1(t) \\ \Delta x_2(t) \end{bmatrix} = A(t)\begin{bmatrix} \Delta x_1(t) \\ \Delta x_2(t) \end{bmatrix} + b(t)\Delta u \tag{38}$$

where $\Delta x_1(t) = x_1(t) - x_1^*(t)$, $\Delta x_2(t) = x_2(t) - x_2^*(t)$,

$\Delta u(t) = u(t) - 0$, and, from (34), $A(t)$ and $b(t)$ are defined by the following equations.

$$\begin{aligned} A(t) &= \begin{bmatrix} 0 & 1 \\ -1 + 2x_1^*(t)x_2^*(t) & -1 + x_1^{*2}(t) \end{bmatrix} \\ b(t) &= \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} \end{aligned} \tag{39}$$

From the above, the controllability matrix $U_c(t)$ becomes

$$U_c(t) = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & x_1^{*2}(t) - 1 \end{bmatrix} \tag{40}$$

which implies that the system is controllable. Then, from (16), $c_0^T(t)$, $c_1^T(t)$ and $c_2^T(t)$ can be ontained as follows with $\lambda(t) = 1/2$.

$$\begin{aligned} c_0^T(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \\ c_1^T(t) &= \begin{bmatrix} 0 & 1 \end{bmatrix} \\ c_2^T(t) &= \begin{bmatrix} 2x_1^*(t)x_2^*(t) - 1 & x_1^{*2}(t) - 1 \end{bmatrix} \end{aligned} \tag{41}$$

Let the desired closed loop stable characteristic polynomial be defined as

$$q(p) = p^2 + 5q + 6. \tag{42}$$

From this and (19), (20), the pole placement state feedback is obtained as follows.

$$\begin{aligned} \Delta u(t) &= -2(5 + 2x_1^*(t)x_2^*(t))\Delta x_1(t) \\ &\quad -2(4 + x_1^{*2}(t))\Delta x_2(t) \end{aligned} \tag{43}$$

The state response of the closed loop system using the above state feedback with the same initial condition of Fig.3 is shown in Fig.5. The state feedback input is shown in Fig.6. Fig.7 and 8 show the state response and feedback input of the same closed loop system as the above with an input disturbance $0.5\sin(10t)$.

**[EXAMPLE 2].**

Consider the same system (36) and (37). Here, we define the desired trajectory by the circle

$$\begin{aligned} x_1^*(t) &= \sin t \\ x_2^*(t) &= \cos t. \end{aligned} \tag{44}$$

The desired input for this trajectory is

$$f(u^*(t)) = (1 - \sin^2 t)\cos t \tag{45}$$

This desired state response is shown in Fig.9. Fig.11 shows a disturbed state response when a disturbance signal described in Fig.10 is added to the desired input (45). The linear time-varying system that approximate this nonlinear system around the desired circle trajectory is written as follows.

$$\frac{d}{dt}\begin{bmatrix} \Delta x_1(t) \\ \Delta x_2(t) \end{bmatrix} = A(t)\begin{bmatrix} \Delta x_1(t) \\ \Delta x_2(t) \end{bmatrix} + b(t)\Delta u \tag{46}$$

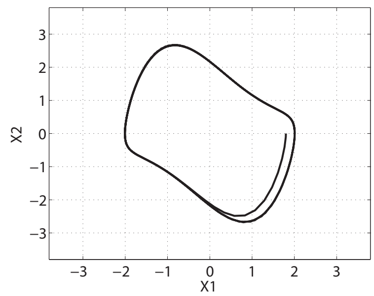


Figure 5: State Response of the Closed Loop with the Initial Condition near the Limit Cycle.

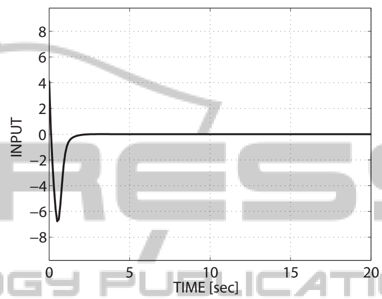


Figure 6: Feedback Input.

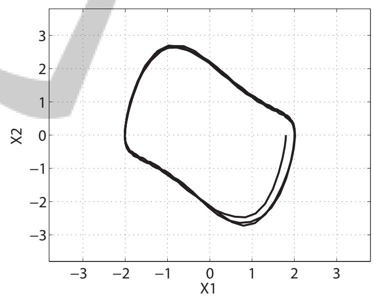


Figure 7: Disturbed State Response of the Closed Loop with the Initial Condition near the Limit Cycle.

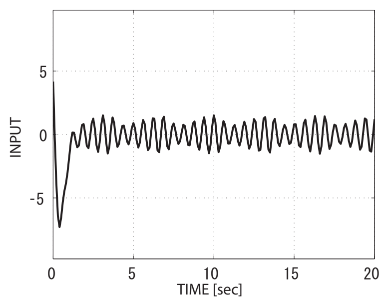


Figure 8: Feedback Input.

$\Delta x_1(t) = x_1(t) - x_1^*(t)$, $\Delta x_2(t) = x_2(t) - x_2^*(t)$, $\Delta u(t) = u(t) - u^*(t)$ where , $x_1^*(t)$, $x_2^*(t)$ and $u^*(t)$ are defined in (44) and (45). From (34), $A(t)$ and $b(t)$ become as follows.

$$\begin{aligned} A(t) &= \begin{bmatrix} 0 & 1 \\ -1+2x_1^*(t)x_2^*(t) & -1+x_1^{*2}(t)) \end{bmatrix} \\ b(t) &= \begin{bmatrix} 0 \\ \nu(t) \end{bmatrix} \end{aligned} \tag{47}$$

where

$$\nu(t) = \frac{2exp(-0.5u^*(t))}{(1+exp(-0.5u^*(t))^2} \tag{48}$$

From the above, the controllability matrix $U_c(t)$ becomes

$$U_c(t) = \nu(t)\begin{bmatrix} 0 & 1 \\ 1 & x_1^{*2}(t)-1 \end{bmatrix} \tag{49}$$

which implies that the system is controllable. Then, if we define $\lambda(t) = \nu(t)$, $c_0^T(t)$, $c_1^T(t)$ and $c_2^T(t)$ can be chosen as the same functions as in (41) with $x^*(t)$ in (44). Using the same desired closed loop characteristic polynomial defined in (42), the pole placement state feedback is obtained as follows.

$$\begin{aligned} \Delta u(t) &= -\frac{1}{\nu(t)}(5+2x_1^*(t)x_2^*(t))\Delta x_1(t) \\ &\quad -\frac{1}{\nu(t)}(4+x_1^{*2}(t))\Delta x_2(t) \end{aligned} \tag{50}$$

Fig.12 shows the state responce of the closed loop system with the same disturbance. Its initial condition is on the desired trajectory. The state feedback input is shown in Fig.13.

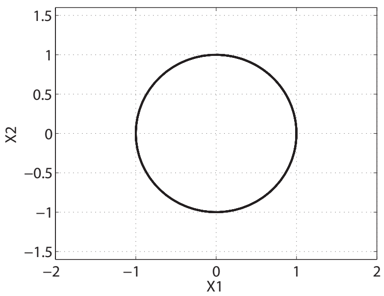


Figure 9: Desired Trajectory.

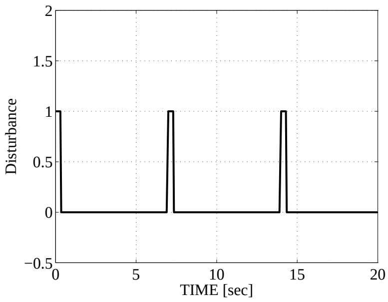


Figure 10: Disturbance Input Signal.

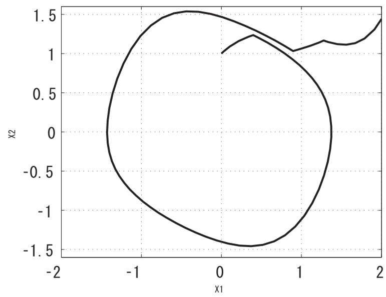


Figure 11: Disturbed State Response with the Initial Condition on the Desired Trajectory.

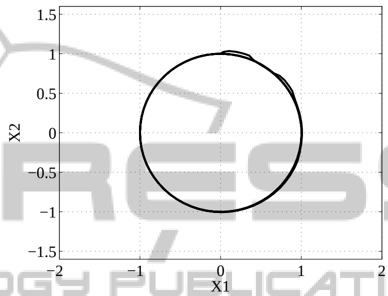


Figure 12: State Response of Stabilization of Limit Cycle.

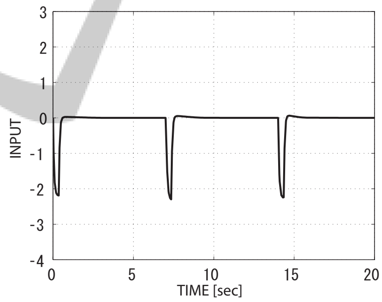


Figure 13: State Response of Stabilization of Limit Cycle.

## 4 CONCLUSIONS

This paper concerned the problem of stabilization of some desired trajectory of nonlinear systems. Nonlinear system can be approximated using a linear time-varying system around this trajectory. The author already proposed the simple design procedure of the pole placement controller for linear time-varying system. The paper showed that this design method can be applied to the trajectory stabilization control of nonlinear systems.

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