Simplified Computation of l_2-Sensitivity for 1-D and a Class of 2-D State-Space Digital Filters Considering 0 and ± 1 Elements

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Abstract: A simplified method of computing an improved l_2-sensitivity measure is developed for state-space digital filters by reducing the number of the Lyapunov equations, and it is expanded into a class of two-dimensional (2-D) state-space digital filters. First, a conventional improved l_2-sensitivity for state-space digital filters is reviewed and simplified to two novel forms so that the number of the Lyapunov equations is reduced. Next, the resulting mehod is expanded into a class of 2-D state-space digital filters. Finally, two numerical examples are presented to evaluate more precise (improved) l_2-sensitivity measures for 1-D and a class of 2-D state-space digital filters by employing the proposed methods.

1 INTRODUCTION

In the case when a state-space model is realized from a given transfer function and implemented with a finite binary representation, the truncation or rounding of the coefficients is required to meet the finite-word-length (FWL) constraints. As a result, unacceptable degradation of the characteristics of a recursive digital filter may be caused, and a stable recursive digital filter may be changed to an unstable one. This motivates the study of coefficient sensitivity analysis and its minimization problem. Several methods have been explored to evaluate the coefficient sensitivity of a state-space digital filter and to minimize the coefficient sensitivity (Thiele, 1984; Thiele, 1986; Iwatsuki et al., 1989; Li and Gevers, 1990; Li et al., 1992; Yan and Moore, 1992; Li and Gevers, 1992; Gevers and Li, 1993; Xiao, 1997; Hinamoto et al., 2005; Yamaki et al., 2006). The analysis and minimization problems of l_2-sensitivity have also been considered for two-dimensional (2-D) state-space digital filters (Kawamata et al., 1987; Hinamoto et al., 1990; Li, 1997; Li, 1998; Hinamoto et al., 2002; Hinamoto et al., 2006; Yamaki et al., 2007). Some of them evaluate the coefficient sensitivity by using a mixture of l_1/l_2 norms (Thiele, 1984; Thiele, 1986; Iwatsuki et al., 1989; Li and Gevers, 1990; Li et al., 1992; Kawamata et al., 1987; Hinamoto et al., 1990), while the others rely on the use of a pure l_2 norm (Yan and Moore, 1992; Li and Gevers, 1992; Gevers and Li, 1993; Xiao, 1997; Hinamoto et al., 2005; Li, 1997; Li, 1998; Hinamoto et al., 2002; Hinamoto et al., 2006; Yamaki et al., 2006; Yamaki et al., 2007). It is noted that the l_2-sensitivity measure is more natural and reasonable than the l_1/l_2 mixed sensitivity measure. In (Xiao, 1997), an improved l_2-sensitivity measure has been presented to evaluate l_2-sensitivity more precisely when the state-space model contains 0 and ± 1 coefficients. In (Hinamoto and Doi, 2012), simple l_2-sensitivity measures have been explored for evaluating the l_2-sensitivity of canonical forms in 1-D and 2-D separable-denominator state-space digital filters.

In this paper, a simplified method of computing an improved l_2-sensitivity measure for state-space digital filters is developed by reducing the number of the Lyapunov equations. The resulting method is expanded into a class of two-dimensional (2-D) state-space digital filters reported in (Hinamoto, 2001). This class of 2-D state-space digital filters can be viewed as a dual system of the Fornasini-Marchesini second local state-space model (Fornasini and Marchesini, 1978). First, a conventional improved l_2-sensitivity for state-space digital filters in

(Xiao, 1997) is reviewed and simplified to two novel forms so that the number of the Lyapunov equations is reduced. The novel contribution exists in two alternative formulations for the Lyapunov equations where two independent variables are replaced by a single independent variable. This enables one to reduce the amount of computations considerably. Furthermore, the possible coefficient values equal to 0 and ± 1 are considered as special cases. Then the simplified method of computing the improved l_2-sensitivity measure for 1-D filter is expanded into a class of 2-D state-space digital filters. The analysis is carried out more precisely than that in (Hinamoto et al., 2006) by taking into account 0 and ± 1 elements in the 2-D state-space digital filter. Finally, two numerical examples are presented to demonstrate the validity and effectiveness of simplified methods for computing more precise (improved) l_2-sensitivity measures in 1-D and a class of 2-D state-space digital filters.

2 REVIEW OF IMPROVED l_2-SENSITIVITY FOR STATE-SPACE DIGITAL FILTERS

Consider a stable, controllable and observable state-space digital filter $(\boldsymbol{A},\boldsymbol{b},\boldsymbol{c})_n$ described by

$$\begin{aligned} \boldsymbol{x}(k+1) &= \boldsymbol{A}\boldsymbol{x}(k)+\boldsymbol{b}u(k) \\ \boldsymbol{y}(k) &= \boldsymbol{c}\boldsymbol{x}(k) \end{aligned} \tag{1a}$$

where $\boldsymbol{x}(k)$ is an $n\times 1$ state-variable vector, $u(k)$ is a single input, $y(k)$ is a single output, and

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \tag{1b}$$

$$\boldsymbol{c} = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix}.$$

The transfer function of the filter in (1a) can be expressed as

$$H(z) = \boldsymbol{c}(z\boldsymbol{I}_n - \boldsymbol{A})^{-1}\boldsymbol{b}. \tag{2}$$

The l_2-sensitivity measure for the filter in (1a) is defined as (Yan and Moore, 1992)

$$\begin{aligned} S &= \sum_{k=1}^{n}\sum_{l=1}^{n}\frac{1}{2\pi j}\oint_{|z|=1}\left|\frac{\partial H(z)}{\partial a_{kl}}\right|^2\frac{dz}{z} \\ &+ \sum_{k=1}^{n}\frac{1}{2\pi j}\oint_{|z|=1}\left|\frac{\partial H(z)}{\partial b_k}\right|^2\frac{dz}{z} \\ &+ \sum_{l=1}^{n}\frac{1}{2\pi j}\oint_{|z|=1}\left|\frac{\partial H(z)}{\partial c_l}\right|^2\frac{dz}{z} \end{aligned} \tag{3a}$$

where

$$\begin{aligned} \frac{\partial H(z)}{\partial a_{kl}} &= \boldsymbol{G}(z)\boldsymbol{e}_k\boldsymbol{e}_l^T\boldsymbol{F}(z) \\ \frac{\partial H(z)}{\partial b_k} &= \boldsymbol{G}(z)\boldsymbol{e}_k, \qquad \frac{\partial H(z)}{\partial c_l} = \boldsymbol{e}_l^T\boldsymbol{F}(z) \end{aligned} \tag{3b}$$

with

$$\boldsymbol{F}(z) = (z\boldsymbol{I}_n - \boldsymbol{A})^{-1}\boldsymbol{b}, \qquad \boldsymbol{G}(z) = \boldsymbol{c}(z\boldsymbol{I}_n - \boldsymbol{A})^{-1}. \tag{3c}$$

It is noted that coefficients 0 and ± 1 can be realized precisely in the implementation of FWL digital systems. Therefore, system's l_2-sensitivity is not affected by these coefficients.

By taking this situation into account, the individual sensitivities for the elements of coefficient matrices $\boldsymbol{A}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ should be changed to (Xiao, 1997)

$$\begin{aligned} \frac{\partial H(z)}{\partial a_{kl}} &= \boldsymbol{G}(z)\boldsymbol{e}_k\boldsymbol{e}_l^T\boldsymbol{F}(z)\phi_{kl} \\ \frac{\partial H(z)}{\partial b_k} &= \boldsymbol{G}(z)\boldsymbol{e}_k\varphi_k, \qquad \frac{\partial H(z)}{\partial c_l} = \boldsymbol{e}_l^T\boldsymbol{F}(z)\psi_l \end{aligned} \tag{4a}$$

where

$$\begin{aligned} \phi_{kl} &= \begin{cases} 1 & \text{for} \quad a_{kl} \neq 0,\pm 1 \\ 0 & \text{for} \quad a_{kl} = 0,\pm 1 \end{cases} \\ \varphi_k &= \begin{cases} 1 & \text{for} \quad b_k \neq 0,\pm 1 \\ 0 & \text{for} \quad b_k = 0,\pm 1 \end{cases} \\ \psi_l &= \begin{cases} 1 & \text{for} \quad c_l \neq 0,\pm 1 \\ 0 & \text{for} \quad c_l = 0,\pm 1 \end{cases}. \end{aligned} \tag{4b}$$

Lemma: The improved l_2-sensitivity measure for a state-space model $(\boldsymbol{A},\boldsymbol{b},\boldsymbol{c})_n$ in (1a) is presented by (Xiao, 1997)

$$\begin{aligned} S_I &= \sum_{k=1}^{n}\sum_{l=1}^{n}\phi_{kl}\,[\,\boldsymbol{c} \quad \boldsymbol{0}\,]\,\boldsymbol{R}(k,l)\begin{bmatrix} \boldsymbol{c}^T \\ \boldsymbol{0} \end{bmatrix} \\ &+ \sum_{k=1}^{n}\varphi_k W_{kk} + \sum_{l=1}^{n}\psi_l K_{ll} \end{aligned} \tag{5a}$$

where $\boldsymbol{R}(k,l)$, K_{ll} ((l,l)th entry of $\boldsymbol{K}$) and W_{kk} ((k,k)th entry of $\boldsymbol{W}$) are obtained by solving the Lyapunov equations

$$\begin{aligned} \boldsymbol{R}(k,l) &= \begin{bmatrix} \boldsymbol{A} & \boldsymbol{e}_k\boldsymbol{e}_l^T \\ \boldsymbol{0} & \boldsymbol{A} \end{bmatrix}\boldsymbol{R}(k,l)\begin{bmatrix} \boldsymbol{A} & \boldsymbol{e}_k\boldsymbol{e}_l^T \\ \boldsymbol{0} & \boldsymbol{A} \end{bmatrix}^T + \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{b}\boldsymbol{b}^T \end{bmatrix} \\ \boldsymbol{K} &= \boldsymbol{A}\boldsymbol{K}\boldsymbol{A}^T + \boldsymbol{b}\boldsymbol{b}^T \\ \boldsymbol{W} &= \boldsymbol{A}^T\boldsymbol{W}\boldsymbol{A} + \boldsymbol{c}^T\boldsymbol{c} \end{aligned} \tag{5b}$$

for $k = 1,2,\cdots,n$ and $l = 1,2,\cdots,n$.

In the following two theorems, it is shown that the above improved l_2-sensitivity measure in (5a) can be modified to two novel forms so that the number of the Lyapunov equations is reduced.

Theorem 1: The improved l_2-sensitivity measure in (5a) is changed to the form

$$S_I' = \sum_{k=1}^{n}\sum_{l=1}^{n}\phi_{kl}\,[\,\boldsymbol{e}_l^T \quad \boldsymbol{0}\,]\,\boldsymbol{M}(k)\begin{bmatrix}\boldsymbol{e}_l\\ \boldsymbol{0}\end{bmatrix} + \sum_{k=1}^{n}\varphi_k W_{kk} + \sum_{l=1}^{n}\psi_l K_{ll} \tag{6a}$$

where $\boldsymbol{M}(k)$ is obtained by solving the Lyapunov equation

$$\boldsymbol{M}(k) = \begin{bmatrix}\boldsymbol{A} & \boldsymbol{bc}\\ \boldsymbol{0} & \boldsymbol{A}\end{bmatrix}\boldsymbol{M}(k)\begin{bmatrix}\boldsymbol{A} & \boldsymbol{bc}\\ \boldsymbol{0} & \boldsymbol{A}\end{bmatrix}^T + \begin{bmatrix}\boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{e}_k\boldsymbol{e}_k^T\end{bmatrix} \tag{6b}$$

for $k = 1, 2, \cdots, n$.

Proof: It is noted that

$$\boldsymbol{F}(z)\boldsymbol{G}(z) = (z\boldsymbol{I}_n - \boldsymbol{A})^{-1}\boldsymbol{bc}(z\boldsymbol{I}_n - \boldsymbol{A})^{-1} = [\,\boldsymbol{I}_n \quad \boldsymbol{0}\,]\left(z\boldsymbol{I}_{2n} - \begin{bmatrix}\boldsymbol{A} & \boldsymbol{bc}\\ \boldsymbol{0} & \boldsymbol{A}\end{bmatrix}\right)^{-1}\begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{I}_n\end{bmatrix}. \tag{7}$$

Defining $\Phi(z) = F(z)G(z)$, from (4a) and (7) it follows that

$$\begin{aligned}\frac{1}{2\pi j}\oint_{|z|=1}\left|\frac{\partial H(z)}{\partial a_{kl}}\right|^2\frac{dz}{z} &= \phi_{kl}\,\boldsymbol{e}_l^T\left[\frac{1}{2\pi j}\oint_{|z|=1}\Phi(z)\boldsymbol{e}_k\boldsymbol{e}_k^T\Phi^T(z^{-1})\frac{dz}{z}\right]\boldsymbol{e}_l\\ &= \phi_{kl}\,[\,\boldsymbol{e}_l^T \quad \boldsymbol{0}\,]\,\boldsymbol{M}(k)\begin{bmatrix}\boldsymbol{e}_l\\ \boldsymbol{0}\end{bmatrix}\end{aligned} \tag{8a}$$

where

$$\boldsymbol{M}(k) = \sum_{p=0}^{\infty}\begin{bmatrix}\boldsymbol{A} & \boldsymbol{bc}\\ \boldsymbol{0} & \boldsymbol{A}\end{bmatrix}^p\begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{e}_k\end{bmatrix}\begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{e}_k\end{bmatrix}^T\begin{bmatrix}\boldsymbol{A}^T & \boldsymbol{0}\\ (\boldsymbol{bc})^T & \boldsymbol{A}^T\end{bmatrix}^p \tag{8b}$$

which yields the Lyapunov equation in (6b). This completes the proof of Theorem 1.

Theorem 2: The improved l_2-sensitivity measure in (5a) can be written as

$$S_I'' = \sum_{k=1}^{n}\sum_{l=1}^{n}\phi_{kl}\,[\,\boldsymbol{0} \quad \boldsymbol{e}_k^T\,]\,\boldsymbol{N}(l)\begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{e}_k\end{bmatrix} + \sum_{k=1}^{n}\varphi_k W_{kk} + \sum_{l=1}^{n}\psi_l K_{ll} \tag{9a}$$

where $\boldsymbol{N}(l)$ is obtained by solving the Lyapunov equation

$$\boldsymbol{N}(l) = \begin{bmatrix}\boldsymbol{A} & \boldsymbol{bc}\\ \boldsymbol{0} & \boldsymbol{A}\end{bmatrix}^T\boldsymbol{N}(l)\begin{bmatrix}\boldsymbol{A} & \boldsymbol{bc}\\ \boldsymbol{0} & \boldsymbol{A}\end{bmatrix} + \begin{bmatrix}\boldsymbol{e}_l\boldsymbol{e}_l^T & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0}\end{bmatrix} \tag{9b}$$

for $l = 1, 2, \cdots, n$.

Proof: From (4a) and (7) it follows that

$$\begin{aligned}\frac{1}{2\pi j}\oint_{|z|=1}\left|\frac{\partial H(z)}{\partial a_{kl}}\right|^2\frac{dz}{z} &= \phi_{kl}\,\boldsymbol{e}_k^T\left[\frac{1}{2\pi j}\oint_{|z|=1}\Phi^T(z^{-1})\boldsymbol{e}_l\boldsymbol{e}_l^T\Phi(z)\frac{dz}{z}\right]\boldsymbol{e}_k\\ &= \phi_{kl}\,[\,\boldsymbol{0} \quad \boldsymbol{e}_k^T\,]\,\boldsymbol{N}(l)\begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{e}_k\end{bmatrix}\end{aligned} \tag{10a}$$

where

$$N(l) = \sum_{p=0}^{\infty}\begin{bmatrix}\boldsymbol{A}^T & \boldsymbol{0}\\ (\boldsymbol{bc})^T & \boldsymbol{A}^T\end{bmatrix}^p\begin{bmatrix}\boldsymbol{e}_l\\ \boldsymbol{0}\end{bmatrix}\begin{bmatrix}\boldsymbol{e}_l\\ \boldsymbol{0}\end{bmatrix}^T\begin{bmatrix}\boldsymbol{A} & \boldsymbol{bc}\\ \boldsymbol{0} & \boldsymbol{A}\end{bmatrix}^p \tag{10b}$$

which yields the Lyapunov equation in (9b). This completes the proof of Theorem 2.

It is noted that the novel contribution in this paper exists in two alternative formulations in (6b) and (9b) for the Lyapunov equations where two independent variables (k,l) in (5b) are replaced by a single independent variable either k or l. This makes it possible to reduce the amount of computations considerably.

3 MORE PRECISE l_2-SENSITIVITY FOR A CLASS OF 2-D STATE-SPACE DIGITAL FILTERS

Consider a stable, locally controllable and locally observable 2-D local state-space model $(\boldsymbol{A}_1, \boldsymbol{A}_2, \boldsymbol{b}, \boldsymbol{c}_1, \boldsymbol{c}_2, d)_n$ defined for a class of 2-D IIR digital filters (Hinamoto, 2001)

$$\begin{bmatrix}\boldsymbol{x}(i+1, j+1)\\ y(i,j)\end{bmatrix} = \begin{bmatrix}\boldsymbol{A}_1 & \boldsymbol{A}_2\\ \boldsymbol{c}_1 & \boldsymbol{c}_2\end{bmatrix}\begin{bmatrix}\boldsymbol{x}(i, j+1)\\ \boldsymbol{x}(i+1, j)\end{bmatrix} + \begin{bmatrix}\boldsymbol{b}\\ d\end{bmatrix}u(i,j) \tag{11a}$$

where $\boldsymbol{x}(i,j)$ is an $n \times 1$ local state vector, $u(i,j)$ is a single input, $y(i,j)$ is a single output, and

$$\boldsymbol{A}_i = \begin{bmatrix}a_{i11} & a_{i12} & \cdots & a_{i1n}\\ a_{i21} & a_{i22} & \cdots & a_{i2n}\\ \vdots & \vdots & \ddots & \vdots\\ a_{in1} & a_{in2} & \cdots & a_{inn}\end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix}b_1\\ b_2\\ \vdots\\ b_n\end{bmatrix}$$

$$\boldsymbol{c}_i = \begin{bmatrix}c_{i1} & c_{i2} & \cdots & c_{in}\end{bmatrix} \qquad \text{for } i = 1, 2. \tag{11b}$$

The transfer function of (11a) is given by (Hinamoto, 2001)

$$H(z_1,z_2) = (z_1^{-1}\boldsymbol{c}_1 + z_2^{-1}\boldsymbol{c}_2) \cdot \left(\boldsymbol{I}_n - z_1^{-1}\boldsymbol{A}_1 - z_2^{-1}\boldsymbol{A}_2\right)^{-1}\boldsymbol{b} + d. \tag{12}$$

A more precise (an improved) l_2-sensitivity measure for the local state-space model in (11a) can be defined as

$$\begin{aligned} M =& \sum_{k=1}^{n}\sum_{l=1}^{n}\frac{1}{(2\pi j)^2}\oint_{|z_1|=1}\oint_{|z_2|=1}\left|\frac{\partial H(z_1,z_2)}{\partial a_{1kl}}\right|^2\frac{dz_1dz_2}{z_1z_2} \\ &+ \sum_{k=1}^{n}\sum_{l=1}^{n}\frac{1}{(2\pi j)^2}\oint_{|z_1|=1}\oint_{|z_2|=1}\left|\frac{\partial H(z_1,z_2)}{\partial a_{2kl}}\right|^2\frac{dz_1dz_2}{z_1z_2} \\ &+ \sum_{k=1}^{n}\frac{1}{(2\pi j)^2}\oint_{|z_1|=1}\oint_{|z_2|=1}\left|\frac{\partial H(z_1,z_2)}{\partial b_k}\right|^2\frac{dz_1dz_2}{z_1z_2} \\ &+ \sum_{l=1}^{n}\frac{1}{(2\pi j)^2}\oint_{|z_1|=1}\oint_{|z_2|=1}\left|\frac{\partial H(z_1,z_2)}{\partial c_{1l}}\right|^2\frac{dz_1dz_2}{z_1z_2} \\ &+ \sum_{l=1}^{n}\frac{1}{(2\pi j)^2}\oint_{|z_1|=1}\oint_{|z_2|=1}\left|\frac{\partial H(z_1,z_2)}{\partial c_{2l}}\right|^2\frac{dz_1dz_2}{z_1z_2} \end{aligned} \tag{13a}$$

where

$$\begin{aligned} \frac{\partial H(z_1,z_2)}{\partial a_{ikl}} &= z_i^{-1}\boldsymbol{G}(z_1,z_2)\boldsymbol{e}_k\boldsymbol{e}_l^T\boldsymbol{F}(z_1,z_2)\phi_{ikl} \\ \frac{\partial H(z_1,z_2)}{\partial b_k} &= \boldsymbol{G}(z_1,z_2)\boldsymbol{e}_k\varphi_k \\ \frac{\partial H(z_1,z_2)}{\partial c_{il}} &= z_i^{-1}\boldsymbol{e}_l^T\boldsymbol{F}(z_1,z_2)\psi_{il} \qquad \text{for} \quad i=1,2 \end{aligned} \tag{13b}$$

with

$$\begin{aligned} \boldsymbol{F}(z_1,z_2) &= \left(\boldsymbol{I}_n - z_1^{-1}\boldsymbol{A}_1 - z_2^{-1}\boldsymbol{A}_2\right)^{-1}\boldsymbol{b} \\ \boldsymbol{G}(z_1,z_2) &= (z_1^{-1}\boldsymbol{c}_1 + z_2^{-1}\boldsymbol{c}_2) \cdot \left(\boldsymbol{I}_n - z_1^{-1}\boldsymbol{A}_1 - z_2^{-1}\boldsymbol{A}_2\right)^{-1} \\ \phi_{ikl} &= \begin{cases} 1 & \text{for} \quad a_{ikl} \neq 0, \pm 1 \\ 0 & \text{for} \quad a_{ikl} = 0, \pm 1 \end{cases} \\ \varphi_k &= \begin{cases} 1 & \text{for} \quad b_k \neq 0, \pm 1 \\ 0 & \text{for} \quad b_k = 0, \pm 1 \end{cases} \\ \psi_{il} &= \begin{cases} 1 & \text{for} \quad c_{il} \neq 0, \pm 1 \\ 0 & \text{for} \quad c_{il} = 0, \pm 1 \end{cases} \end{aligned} \tag{13c}$$

for $i = 1,2$.

It is noted that unlike those in (Hinamoto et al., 2006), the individual sensitivities in (13b) are taken into account 0 and ± 1 elements in the 2-D local state-space model of (11a) to evaluate the l_2-sensitivity more precisely.

By substituting (13b) into (13a), a more precise l_2-sensitivity measure for a 2-D state-space digital filter in (11a) is derived.

Theorem 3: The more precise l_2-sensitivity measure for the 2-D filter in (11a) can be computed from either of

$$\begin{aligned} M_I =& \sum_{k=1}^{n}\sum_{l=1}^{n}\phi_{1kl}\boldsymbol{e}_k^T\boldsymbol{M}_l\boldsymbol{e}_k + \sum_{k=1}^{n}\sum_{l=1}^{n}\phi_{2kl}\boldsymbol{e}_k^T\boldsymbol{M}_l\boldsymbol{e}_k \\ &+ \sum_{k=1}^{n}\varphi_k W_{kk} + \sum_{l=1}^{n}\psi_{1l}K_{ll} + \sum_{l=1}^{n}\psi_{2l}K_{ll} \end{aligned} \tag{14a}$$

$$\begin{aligned} M_I' =& \sum_{k=1}^{n}\sum_{l=1}^{n}\phi_{1kl}\boldsymbol{e}_l^T\boldsymbol{N}_k\boldsymbol{e}_l + \sum_{k=1}^{n}\sum_{l=1}^{n}\phi_{2kl}\boldsymbol{e}_l^T\boldsymbol{N}_k\boldsymbol{e}_l \\ &+ \sum_{k=1}^{n}\varphi_k W_{kk} + \sum_{l=1}^{n}\psi_{1l}K_{ll} + \sum_{l=1}^{n}\psi_{2l}K_{ll} \end{aligned} \tag{14b}$$

Proof: It follows from (13b) and (13c) that

$$\begin{aligned} &\frac{1}{(2\pi j)^2}\oint_{|z_1|=1}\oint_{|z_2|=1}\left|\frac{\partial H(z_1,z_2)}{\partial a_{ikl}}\right|^2\frac{dz_1dz_2}{z_1z_2} \\ &\qquad = \phi_{ikl}\boldsymbol{e}_k^T\boldsymbol{M}_l\boldsymbol{e}_k = \phi_{ikl}\boldsymbol{e}_l^T\boldsymbol{N}_k\boldsymbol{e}_l \\ &\frac{1}{(2\pi j)^2}\oint_{|z_1|=1}\oint_{|z_2|=1}\left|\frac{\partial H(z_1,z_2)}{\partial b_k}\right|^2\frac{dz_1dz_2}{z_1z_2} \\ &\qquad = \varphi_k\boldsymbol{e}_k^T\boldsymbol{W}\boldsymbol{e}_k \\ &\frac{1}{(2\pi j)^2}\oint_{|z_1|=1}\oint_{|z_2|=1}\left|\frac{\partial H(z_1,z_2)}{\partial c_{il}}\right|^2\frac{dz_1dz_2}{z_1z_2} \\ &\qquad = \psi_{il}\boldsymbol{e}_l^T\boldsymbol{K}\boldsymbol{e}_l \qquad\qquad \text{for} \quad i=1,2 \end{aligned} \tag{15a}$$

where

$$\begin{aligned} \boldsymbol{M}_l &= \frac{1}{(2\pi j)^2}\oint_{|z_1|=1}\oint_{|z_2|=1}[\boldsymbol{F}(z_1,z_2)\boldsymbol{G}(z_1,z_2)]^T\boldsymbol{e}_l \cdot \boldsymbol{e}_l^T\boldsymbol{F}(z_1^{-1},z_2^{-1})\boldsymbol{G}(z_1^{-1},z_2^{-1})\frac{dz_1dz_2}{z_1z_2} \\ \boldsymbol{N}_k &= \frac{1}{(2\pi j)^2}\oint_{|z_1|=1}\oint_{|z_2|=1}\boldsymbol{F}(z_1,z_2)\boldsymbol{G}(z_1,z_2)\boldsymbol{e}_k \cdot \boldsymbol{e}_k^T[\boldsymbol{F}(z_1^{-1},z_2^{-1})\boldsymbol{G}(z_1^{-1},z_2^{-1})]^T\frac{dz_1dz_2}{z_1z_2} \\ \boldsymbol{K} &= \frac{1}{(2\pi j)^2}\oint_{|z_1|=1}\oint_{|z_2|=1}\boldsymbol{F}(z_1,z_2)\boldsymbol{F}^T(z_1^{-1},z_2^{-1})\frac{dz_1dz_2}{z_1z_2} \\ \boldsymbol{W} &= \frac{1}{(2\pi j)^2}\oint_{|z_1|=1}\oint_{|z_2|=1}\boldsymbol{G}^T(z_1,z_2)\boldsymbol{G}(z_1^{-1},z_2^{-1})\frac{dz_1dz_2}{z_1z_2}. \end{aligned} \tag{15b}$$

Matrices $\boldsymbol{M}_l$, $\boldsymbol{N}_k$, $\boldsymbol{K}$ and $\boldsymbol{W}$ are the 2-D Gramians which can be derived from

$$\begin{aligned} M_l &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} H^T(i,j)e_l e_l^T H(i,j) \\ N_k &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} H(i,j)e_k e_k^T H^T(i,j) \\ K &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} f(i,j)f^T(i,j) \\ W &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} g^T(i,j)g(i,j) \end{aligned} \quad (16a)$$

where

$$\begin{aligned} f(i,j) &= A^{(i,j)}b \\ g(i,j) &= c_1 A^{(i-1,j)} + c_2 A^{(i,j-1)} \\ A^{(0,0)} &= I_n, \quad A^{(i,j)} = 0 \quad \text{for } i<0\,,\ j<0 \\ A^{(i,j)} &= A_1 A^{(i-1,j)} + A_2 A^{(i,j-1)} \\ &= A^{(i-1,j)}A_1 + A^{(i,j-1)}A_2 \quad \text{for } (i,j) > (0,0) \\ H(i,j) &= \sum\sum_{(0,0)\le(k,r)<(i,j)} f(k,r)g(i-k,j-r) \end{aligned} \quad (16b)$$

with the partial ordering for integer pairs (i,j) defined in (Roessor, 1975).

4 NUMERICAL EXAMPLES

Example 1: Let a state-space digital filter $(A,b,c)_3$ in (1a) be specified by (Xiao, 1997)

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.1732 & -1.0227 & 1.8155 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$c = \begin{bmatrix} 0.1174 & -0.3818 & 0.2984 \end{bmatrix}$$

From (6a) and (9a), the improved l_2-sensitivity was found to be

$$S_I' = 240.433072, \qquad S_I'' = 240.433072$$

which essentially coincide with $S_I = 240.43$ in (Xiao, 1997). The optimal state-space digital filter with the minimum l_2-sensitivity can be constructed as (Yan and Moore, 1992),(Xiao, 1997)

$$A^o = \begin{bmatrix} 0.6883 & -0.2234 & -0.0297 \\ -0.2234 & 0.5394 & -0.1394 \\ -0.0297 & -0.1394 & 0.5879 \end{bmatrix}$$

$$b^o = \begin{bmatrix} 0.5183 \\ 0.1718 \\ 0.0158 \end{bmatrix}$$

$$c^o = \begin{bmatrix} 0.5183 & 0.1718 & 0.0158 \end{bmatrix}$$

For this optimal realization, from (6a) and (9a), the improved l_2-sensitivity was found to be

$$S_I' = 2.458368, \qquad S_I'' = 2.458368$$

which essentially coincide with $S_I = 2.4579$ in (Xiao, 1997). In fact, $S_I = 2.458368$ was derived from (5a).

Example 2: Consider a local state-space model in (11a) specified by (Hinamoto et al., 2006)

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -0.0041 & 0.0801 & -0.4246 & 1.0446 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} -0.2261 & 1.6143 & 0.1005 & -0.0072 \\ -0.4059 & 1.6104 & -0.6062 & 0.2458 \\ -0.3096 & 1.0234 & -0.4532 & 0.3867 \\ -0.1447 & 0.4387 & -0.3102 & 0.5629 \end{bmatrix}$$

$$b = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T$$

$$c_1 = \begin{bmatrix} -0.0145 & 0.0123 & 0.0205 & 0.0476 \end{bmatrix}$$

$$c_2 = \begin{bmatrix} 0.01190 & 0.0235 & -0.0064 & 0.0209 \end{bmatrix}.$$

In (16a), the Gramians M_l, N_k, K, and W were computed over $(0,0) \le (i,j) \le (100,100)$. From (14a) and (14b), the more precise l_2-sensitivity was found to be

$$M_I = 1.426236 \times 10^5, \qquad M_I' = 1.426236 \times 10^5$$

which are smaller than the value of the l_2-sensitivity measure: 1.579936×10^5 reported in (Hinamoto et al., 2006) since coefficients 0 and ±1 were not taken into account in (Hinamoto et al., 2006).

The optimal 2-D state-space digital filter structure that minimizes the l_2-sensitivity subject to l_2-scaling constraints was constructed as (Hinamoto et al., 2006)

$$A_1^o = \begin{bmatrix} 0.31822 & 0.36329 & -0.21491 & -0.14635 \\ -0.01438 & 0.13734 & 0.56514 & -0.07384 \\ -0.08182 & -0.07710 & 0.16782 & 0.18698 \\ -0.01344 & 0.07553 & -0.03754 & 0.42122 \end{bmatrix}$$

$$A_2^o = \begin{bmatrix} 0.53774 & -0.04378 & 0.15039 & 0.21219 \\ 0.08849 & 0.38433 & 0.01571 & 0.00945 \\ -0.22483 & 0.36107 & 0.11917 & -0.07048 \\ -0.09396 & -0.04963 & 0.14221 & 0.45275 \end{bmatrix}$$

$$b^o = \begin{bmatrix} -0.38108 \\ -0.36371 \\ -0.77857 \\ 0.53703 \end{bmatrix}$$

$$c_1^o = \begin{bmatrix} -0.19351 & -0.07547 & 0.06028 & -0.01237 \end{bmatrix}$$

$$c_2^o = \begin{bmatrix} 0.15022 & 0.38727 & 0.40379 & 0.99327 \end{bmatrix}$$

For this optimal realization that does not contain 0 and ±1 elements, from (14a) and (14b) the more precise l_2-sensitivity was found to be

$$M_I = 372.778156, \qquad M_I' = 372.778156$$

which essentially coincide with the value of an l_2-sensitivity measure: 372.776303 in (Hinamoto et al., 2006).

5 CONCLUSIONS

This paper has developed a simplified method of computing an improved l_2-sensitivity measure for state-space digital filters by reducing the number of the Lyapunov equations. The simplified method has also been expanded into a class of 2-D state-space digital filters. First, a conventional improved l_2-sensitivity for state-space digital filters has been reviewed and its computation method has been simplified with two novel forms such that the number of the Lyapunov equations is reduced. Next, the resulting method has been applied to a class of 2-D state-space digital filters. This has been done more precisely by taking into account 0 and ± 1 elements in the filter. Finally, two numerical examples have been presented to explain the validity and effectiveness of simplified methods of computing more precise (improved) l_2-sensitivity measures for 1-D as well as a class of 2-D state-space digital filters.

The simplified method has also been investigated for computing a more precise l_2-sensitivity measure in 2-D state-space digital filters described by the Roessor model (Roessor, 1975) and the results will appear elsewhere.

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