A Discrete-time Valuation of Callable Financial Securities with Regime Switches

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Keywords: Optimal Stopping, Game Option, Markov Chain, Regime Switching, Callable Securities, Stopping Boundaries.

Abstract: In this paper, we consider a model of valuing callable financial securities when the underlying asset price dynamic is modeled by a regime switching process. The callable securities enable both an issuer and an investor to exercise their rights to call. We show that such a model can be formulated as a coupled stochastic game for the optimal stopping problem with two stopping boundaries. We provide analytical results of optimal stopping rules of the issuer and the investor under general payoff functions defined on the underlying asset price, the state of the economy and the time. In particular, we derive specific stopping boundaries for the both players by specifying for the callable securities to be the callable American put option.

1 INTRODUCTION

The purpose of this paper is to develop a dynamic valuation framework for callable financial securities with general payoff function by explicitly incorporating the use of regime switches. Such examples of the callable financial security may include game options (Kifer, 2000), (Kyprianou, 2004), convertible bond (Yagi and Sawaki, 2005), (Yagi and Sawaki, 2007), callable put and call options (Black and Scholes, 1973), (Brennan and Schwartz, 1976), (Geske and Johnson, 1984). Most studies on these securities have focused on the pricing of the derivatives when the underlying asset price processes follow a Brownian motion defined on a single probability space. In other words the realizations of the price process come from the same source of the uncertainty over the planning horizon.

The Markov regime switching model make it possible to capture the structural changes of the underlying asset prices based on the macro-economic environment, fundamentals of the real economy and financial policies including international monetary cooperation. Such regime switching can be presented by the transition of the states of the economy, which follows a Markov chain. Recently, there is a growing interest in the regime switching model. (Naik, 1993), (Guo, 2001), (Elliott et al., 2005) address the European call option price formula. (Guo and Zhang, 2004) presents a valuation model for perpetual American put options. (Le and Wang, 2010) study the optimal stopping time for the finite time horizon, and derive the optimal stopping strategy and properties of the solution. They also derive the technique for computing the solution and show some numerical examples for the American put option.

In this paper we show that there exists a pair of optimal stopping rules for the issuer and of the investor and derive the value of the coupled game. Should the payoff functions be specified like options, some analytical properties of the optimal stopping rules and their values can be explored under the several assumptions. In particular, we are interested in the cases of callable American put option in which we may derive the optimal stopping boundaries of the both of the issuer and the investor, depending on the state of the economy. Numerical examples are also presented to illustrate these properties.

The organization of our paper is as follows: In section 2, we formulate a discrete time valuation model for a callable contingent claim whose payoff functions are in general form. And then we derive optimal policies and investigate their analytical properties by using contraction mappings. Section 3 discusses a case of the payoff functions to derive the specific stop and continue regions for callable put. In Sec-

Sato K. and Sawaki K..
A Discrete Time Valuation of Callable Financial Securities with Regime Switches.
DOI: 10.5220/0004201402250229
In *Proceedings of the 2nd International Conference on Operations Research and Enterprise Systems* (ICORES-2013), pages 225-229
ISBN: 978-989-8565-40-2

tion 4 we present numerical results for the American callable put option using binomial model. Finally, last section concludes the paper with further comments. It summarize results of this paper and raises further directions for future research.

2 A GENETIC MODEL OF CALLABLE-PUTABLE FINANCIAL COMMODITIES

In this section we formulate the valuation of callable securities as an optimal stopping problem in discrete time. Let $\mathcal{T}$ be the time index set $\{0,1,\cdots\}$. We consider a complete probability space $(\Omega,\mathcal{F},\mathcal{P})$, where $\mathcal{P}$ is a real-world probability. We suppose that the uncertainties of an asset price depend on its fluctuation and the economic states which are described by the probability space $(\Omega,\mathcal{F},\mathcal{P})$. Let $\{1,2,\cdots,N\}$ be the set of states of the economy and i or j denote one of these states. We denote $Z:=\{Z_t\}_{t\in\mathcal{T}}$ be the finite Markov chain with transition probability $P_{ij}=\Pr\{Z_{t+1}=j \mid Z_t=i\}$. A transition from i to j means a regime switch. Let r be the market interest rate of the bank account. We suppose that the price dynamics $B:=\{B_t\}_{t\in\mathcal{T}}$ of the bank account is given by

$$B_t=B_{t-1}e^r,\ B_0=1.$$

Let $S:=\{S_t\}_{t\in\mathcal{T}}$ be the asset price at time t. We suppose that $\{X_t^i\}$ be a sequence of i.i.d. random variable having mean μ_i with the probability distribution $F_i(\cdot)$ and its parameters depend on the state of the economy modeled by Z. Here, the sequence $\{X_t^i\}$ and $\{Z_t\}$ are assumed to be independent. Then, the asset price is defined as

$$S_{t+1}=S_tX_t^i. \tag{1}$$

The Esscher transform is well-known tool to determine an equivalent martingale measure for the valuation of options in an incomplete market ((Elliott et al., 2005) and (Ching et al., 2007)). (Ching et al., 2007) define the regime-swiching Esscher transform in discrete time and apply it to determine an equivalent martingale measure when the price dynamics is modeled by high-order Markov chain.

We define $Y_t^i=\log X_t^i$ and $Y:=\{Y_t\}_{t\in\mathcal{T}}$. Let $\mathcal{F}_t^Z$ and $\mathcal{F}_t^Y$ denote the σ-algebras generated by the values of Z and Y, respectively. We set $\mathcal{G}=\mathcal{F}_t^Z\vee\mathcal{F}_t^Y$ for $t\in\mathcal{T}$. We assume that θ_t be a $\mathcal{F}_T^Z$-measurable random variable for each $t=1,2,\cdots$. It is interpreted as the regime-switching Esscher parameter at time t conditional on $\mathcal{F}_T^Z$. Let $M_Y(t,\theta_t)$ denote the moment generating function of Y_t^i given $\mathcal{F}_T^Z$ under $\mathcal{P}$, that is, $M_Y(t,\theta_t):=E[e^{\theta_tY_t^i}\mid\mathcal{F}_T^Z]$. We define $\mathcal{P}^\theta$ as a equivalent martingale measure for $\mathcal{P}$ on $\mathcal{G}_T$ associated with $(\theta_1,\theta_2,\cdots,\theta_T)$.

The next proposition follows from (Ching et al., 2007).

Proposition 2.1. *The discounted price process* $\{S_t/B_t\}_{t\in\mathcal{T}}$ *is a* $(\mathcal{G},\mathcal{P}^\theta)$*-martingale if and only if* θ_t *satisfies*

$$\frac{M_Y(t+1,\theta_{t+1}+1)}{M_Y(t+1,\theta_{t+1})}=e^r. \tag{2}$$

If the dynamics Y is governed by the following Markov-modulated binomial model:

$$P(Y_t^i=y)=\begin{cases}p(Z_t), & \text{if } y=b(Z_t),\\ 1-p(Z_t), & \text{if } y=a(Z_t),\end{cases} \tag{3}$$

then the following proposition provides the Esscher transform of this process. For simplicity of notation, we write p_t, a_t and b_t instead of $p(Z_t)$, $a(Z_t)$ and $b(Z_t)$, respectively.

Proposition 2.2. *The Esscer transform of the Markov modulated binomial model with parameter* p_t *is again a binomial model with the parameter* $\frac{e^r-e^{a_t}}{e^{b_t}-e^{a_t}}$.

A callable contingent claim is a contract between an issuer I and an investor II addressing the asset with a maturity T. The issuer can choose a stopping time σ to call back the claim with the payoff function f_σ and the investor can also choose a stopping time τ to exercise his/her right with the payoff function g_τ at any time before the maturity. Should neither of them stop before the maturity, the payoff is h_T. The payoff always goes from the issuer to the investor. Here, we assume

$$0\le g_t\le h_t\le f_t,\ 0\le t<T$$

and

$$g_T=h_T. \tag{4}$$

The investor wishes to exercise the right to maximize the expected payoff. On the other hand, the issuer wants to call the contract to minimize the payment to the investor. Then, for any pair of the stopping times (σ,τ), define the payoff function by

$$R(\sigma,\tau)=f_\sigma 1_{\{\sigma<\tau\le T\}}+g_\tau 1_{\{\tau<\sigma\le T\}}+h_T 1_{\{\sigma\wedge\tau=T\}}. \tag{5}$$

When the initial asset price $S_0=s$, our stopping problem becomes the valuation of

$$v_0(s,i)=\min_{\sigma\in\mathcal{J}_{0,T}}\max_{\tau\in\mathcal{J}_{0,T}}E_{s,i}^\theta[\beta^{\sigma\wedge\tau}R(\sigma,\tau)], \tag{6}$$

where $\beta\equiv e^{-r}$, $0<\beta<1$ is the discount factor, $\mathcal{J}$ is the finite set of stopping times taking values in

$\{0,1,\cdots,T\}$, and $E^{\theta}[\cdot]$ is an expectation under $\mathcal{P}^{\theta}$. Since the asset price process follows a random walk, the payoff processes of g_t and f_t are both Markov types. We consider this optimal stopping problem as a Markov decision process. Let $v_n(s,i)$ be the price of the callable contingent claim when the asset price is s and the state is i. Here, the trading period moves backward in time indexed by $n = 0,1,2,\cdots,T$. It is easy to see that $v_n(s,i)$ satisfies

$$\begin{aligned} v_{n+1}(s,i) &\equiv (\mathcal{U}v_n)(s,i) \\ &\equiv \min\{f_{n+1}(s,i), \max(g_{n+1}(s,i), \mathcal{A}v_n)\} \end{aligned} \quad (7)$$

with the boundary conditions are $v_0(s,i) = h_0(s,i)$ for any s, i and $v_n(s,0) \equiv 0$ for any n and s. $\mathcal{A}$ is the operator defined by

$$(\mathcal{A}v_n)(s,i) \equiv \beta \sum_{j=1}^{N} P_{ij} \int_0^{\infty} v_n(sx,j)dF_i(x). \quad (8)$$

Remark 2.1. *The equation (7) can be reduced to the non-switching model when we set $P_{ii} = 1$ for all i, or $f_n(s,i) = f_n(s)$, $g_n(s,i) = g_n(s)$, $h_0(s,i) = h_0(s)$ and $\mu_i = \mu$ for all i, n and s.*

Let V be the set of all bounded measurable functions with the norm $\|v\| = \sup_{s\in(0,\infty)}|v(s,i)|$ for any i. For $u,v \in V$, we write $u \le v$ if $u(s,i) \le v(s,i)$ for all $s \in (0,\infty)$. A mapping $\mathcal{U}$ is called a contraction mapping if

$$\|\mathcal{U}u - \mathcal{U}v\| \le \beta\|u - v\|$$

for some $\beta < 1$ and for all $u,v \in V$.

Lemma 2.1. *The mapping $\mathcal{U}$ as defined by equation (7) is a contraction mapping.*

Corollary 2.1. *There exists a unique function $v \in V$ such that*

$$(\mathcal{U}v)(s,i) = v(s,i) \quad \text{for all } s,i. \quad (9)$$

Furthermore, for all $u \in V$,

$$(\mathcal{U}^T u)(s,i) \to v(s,i) \text{ as } T \to \infty,$$

where $v(s,i)$ is equal to the fixed point defined by equation (9), that is, $v(s,i)$ is a unique solution to

$$v(s,i) = \min\{f(s,i), \max(g(s,i), \mathcal{A}v)\}.$$

Since $\mathcal{U}$ is a contraction mapping from Corollary 2.1, the optimal value function v for the perpetual contingent claim can be obtained as the limit by successively applying an operator $\mathcal{U}$ to any initial value function v for a finite lived contingent claim.

To establish an optimal policy, we make some assumptions;

Assumption 2.1.

(i) $F_1(x) \ge F_2(x) \ge \cdots \ge F_N(x)$ for all x.

(ii) $f_n(s,i) \ge f_n(s,j)$, $g_n(s,i) \ge g_n(s,j)$ and $h_n(s,i) \ge h_n(s,j)$ for each n and s, and states i, j, $1 \le j < i \le N$.

(iii) $f_n(s,i)$, $g_n(s,i)$ and $h_n(s,i)$ are monotone in s for each i and n, and are non-decreasing in n for each s and i.

(iv) For each $k \le N$, $\sum_{j=k}^{N} P_{ij}$ is non-decreasing in i.

Assumption (i) means X_n^{i+1} first order stochastically dominates X_n^i for any i and n. That is, as the state i increases, the economy is going well. Thus, the state N represents that the most "Good" economy. Assumption (ii) implies that the payoff values increase as the economy is getting better. In addition, by Assumption (iii), the payoff values decreases as the maturity approaches. Assumption (iv) asserts that the probability of a transition into any block of states $\{k, k+1, \cdots\}$ is an increasing function of the present state.

Lemma 2.2. *Suppose Assumption 2.1 holds.*

(i) For each i, $(\mathcal{U}^n v)(s,i)$ is monotone in s for $v \in V$.

(ii) v satisfying $v = \mathcal{U}v$ is monotone in s.

(iii) Suppose $v_n(s,i)$ is monotone non-decreasing in s, then $v_n(s,i)$ is non-decreasing in i.

(iv) $v_n(s,i)$ is non-decreasing in n for each s and i.

(v) For each i, there exists a pair $(s_n^*(i), s_n^{**}(i))$, $s_n^{**}(i) < s_n^*(i)$, of the optimal boundaries such that

$$\begin{aligned} v_n(s,i) &\equiv (\mathcal{U}v_{n-1})(s) \\ &= \begin{cases} f_n(s,i), & \text{if } s_n^*(i) \le s, \\ \mathcal{A}v_{n-1}, & \text{if } s_n^{**}(i) < s < s_n^*(i), \\ g_n(s,i), & \text{if } s \le s_n^{**}(i), \end{cases} \end{aligned}$$

with $v_0(s,i) = h_0(s,i)$.

Corollary 2.2. *The relationship between g_n, f_n and $v_n(s,i)$ is given by*

$$g_n(s,i) \le v_n(s,i) \le f_n(s,i).$$

We define the stopping regions S^I for the issuer and S^{II} for the investor as

$$\begin{aligned} S_n^I(i) &= \{(s,n,i) \mid v_n(s,i) \ge f_n(s,i)\}, \quad (10) \\ S_n^{II}(i) &= \{(s,n,i) \mid v_n(s,i) \le g_n(s,i)\}. \quad (11) \end{aligned}$$

Moreover, the optimal exercise boundaries for the issuer and the investor are defined as

$$\begin{aligned} s_n^*(i) &= \inf\{s \in S_n^I(i)\}, \quad (12) \\ s_n^{**}(i) &= \inf\{s \in S_n^{II}(i)\}. \quad (13) \end{aligned}$$

3 A SIMPLE CALLABLE AMERICAN PUT OPTION WITH REGIME SWITCHING

Interesting results can be obtained for the special cases when the payoff functions are specified. In this section we consider callable American put option whose payoff functions are specified as a special case of callable contingent claim. If the issuer call back the claim in period n, the issuer must pay to the investor $g_n(s,i)+\delta_n^i$. Note that δ_n^i is the compensate for the contract cancellation, and varies depending on the state and the time period. If the investor exercises his/her right at any time before the maturity, the investor receives the amount $g_n(s,i)$. We discuss the optimal cancel and exercise policies both for the issuer and investor and show the analytical properties under some conditions.

We consider the case of a callable put option where $g_n(s,i)=\max\{K^i-s,0\}$ and $f_n(s,i)=g_n(s,i)+\delta_n^i$. The stopping regions for the issuer $S_n^I(i)$ and the investor $S_n^{II}(i)$ with respect to the callable put option are given by

$$\begin{cases} S_n^I(i)=\{s \mid v_n(s,i)\geq (K^i-s)^+ +\delta_n^i\}, & \text{for } n\geq 1,\\ S_n^I(i)=\phi, & \text{for } n=0,\\ S_n^{II}(i)=\{s \mid v_n(s,i)\leq (K^i-s)^+\}, & \text{for } n\geq 0.\end{cases}$$

For each i and n, we define the optimal exercise boundaries for the issuer $\tilde{s}_n^*(i)$ and the investor $\tilde{s}_n^{**}(i)$ as

$$\tilde{s}_n^*(i) = \inf\{s \mid v_n(s,i)=(K^i-s)^+ +\delta_n^i\}, \quad (14)$$
$$\tilde{s}_n^{**}(i) = \inf\{s \mid v_n(s,i)=(K^i-s)^+\}. \quad (15)$$

Assumption 3.1.

(i) $\beta\mu_N\leq 1$
(ii) $0\leq K^1\leq K^2\leq\cdots\leq K^N$.
(iii) $0\leq\delta_n^1\leq\delta_n^2\leq\cdots\leq\delta_n^N$ *for each n.*
(iv) $\delta_0^i=0$ *and* δ_n^i *is non-decreasing and concave in* $n>0$ *for each i.*
(v) $\beta\sum_{j=1}^N P_{ij}K^j-K^i$ *is non-decreasing in i.*

Assumption (i) means the expected rate of variability for the asset price is less than or equal to $\frac{1}{\beta}=e^r$. Assumption (ii) and (iii) imply that the strike price and the compensate increase as the economy is getting better. These assumptions consistent with the Assumption 2.1 (ii). Assumption (iv) shows that the compensate becomes smaller and smaller as the maturity approaches. Assumption (v) asserts that the difference between the discounted expected value of a strike price when the state transits to any state and the strike price of present state is an non-decrasing function of the present state.

Theorem 3.1. *Suppose that Assumption 3.1 (i)-(v) holds. The stopping regions for the issuer and investor can be obtained as follows;*

(i) The optimal stopping region for the issuer:

$$\begin{cases} S_n^I(i)=\{K^i\}, & \text{if } n_i^*\leq n\leq T,\\ S_n^I(i)=\phi, & \text{if } 0\leq n<n_i^*,\end{cases} \quad (16)$$

where $0\leq K^1\leq K^2\leq\cdots\leq K^N$, *and* $n_i^*\equiv\inf\{n \mid \delta_n^i\leq v_n^a(K^i,i)\}$ *which is non-decreasing in i. Here,* $v_n^a(s,i)=\max\{(K^i-s)^+,\mathcal{A}v_{n-1}(s,i)\}$.

(ii) The optimal stopping region for the investor:

$$\begin{cases} S_n^{II}(i)=[0,\tilde{s}_n^{**}(i)], & \text{if } n>0,\\ S_0^{II}(i)=\{K^i\}, & \text{if } n=0,\end{cases} \quad (17)$$

where $\tilde{s}_n^{**}(i)$ *is non-increasing in n and i. Moreover,* $\tilde{s}_n^{**}(i)\leq\tilde{s}_n^*(i)$ *for each i and n.*

4 NUMERICAL EXAMPLES

In this section we provide a numerical example for a callable American put option by using the binomial tree model. We assume that the transition probability matrix is given by

$$\boldsymbol{P}=\begin{pmatrix} p_1 & 1-p_1\\ 1-p_2 & p_2\end{pmatrix}. \quad (18)$$

For a fixed T, let us divide the interval $[0,T]$ into M subintervals such that $T=hM$. By Proposition 2.2, the probability of upward in the state i is given by

$$q_i=\frac{e^{rh}-d_i}{u_i-d_i}, \quad i=1,2, \quad (19)$$

where $u_i=e^{b_i}$, $d_i=e^{-b_i}$. Let $u_{i,j}$ and $d_{i,j}$ be the upward and downward rate when the state changes from i to j, respectively. The probability distribution function of X_t^i is described by

$$P(X_t^i=x)=\begin{cases} q_ip_i, & \text{if } x=u_{i,i},\\ q_i(1-p_i), & \text{if } x=u_{i,j},\\ (1-q_i)p_i, & \text{if } x=d_{i,i},\\ (1-q_i)(1-p_i), & \text{if } x=d_{i,j},\end{cases} \quad (20)$$

where $i=1,2$, $i\neq j$. It is easy to show that the process is a martingale. The asset price after n periods on tree can be obtained by

$$S_n=S_0u_0^{n_1}u_1^{n_2}d_0^{n_3}d_1^{n_4} \quad (21)$$

where $n_1+n_2+n_3+n_4=n$.

We set the parameters as $T=1$, $M=300$, $r=0.1$, $b_1=0.03$, $b_2=0.01$, $p_1=0.7$, $p_2=0.8$, $K^1=K^2=100$, $\delta_n^i=\delta^1=5$ and $\delta_n^2=\delta^2=6$ for all n. These parameters satisfy and Assumption 2.1 (i), (iv) and Assumption 3.1. The optimal exercise regions for the issuer and the investor is represented in Figure 1.

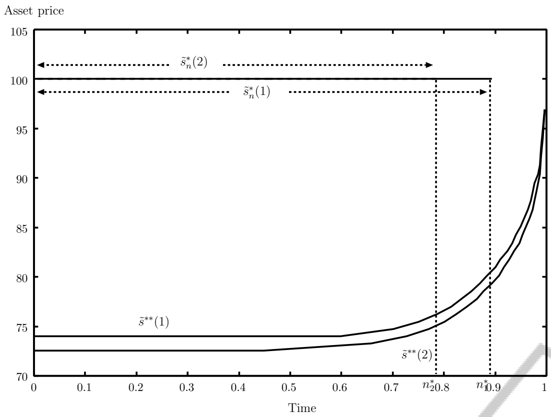

Figure 1: Optimal exercise boundaries for the callable American put.

5 CONCLUDING REMARKS

In this paper we consider the discrete time valuation model for callable contingent claims in which the asset price depends on a Markov environment process. The model explicitly incorporates the use of the regime switching. It is shown that such valuation model with the Markov regime switches can be formulated as a coupled optimal stopping problem of a two person game between the issuer and the investor. In particular, we show under some assumptions that there exists a simple optimal call policy for the issuer and optimal exercise policy for the investor which can be described by the control limit values. If the distributions of the state of the economy are stochastically ordered, then we investigate analytical properties of such optimal stopping rules for the issuer and the investor, respectively, possessing a monotone property.

ACKNOWLEDGEMENTS

This paper was supported in part by the Grant-in-Aid for Scientific Research (No. 20241037) of the Japan Society for the Promotion of Science in 2008-2012.

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