Finite Belief Fusion Model for Hidden Source Behavior Change Detection

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A person's beliefs and attitudes may change multiple times as they gain additional information/perceptions from various external sources, which in turn, may affect their subsequent behavior. Such influential sources, however, are often invisible to the public due to a variety of reasons – private communications, what one randomly reads or hears, and implicit social hierarchies, to name a few. Many efforts have focused on detecting distribution variations. However, the underlying reason for the variation has yet to be fully studied. In this paper, we present a novel approach and algorithm to detect such hidden sources, as well as capture and characterize the patterns of their impact with regards to the belief-changing trend. We formalize this problem as a *finite belief fusion model* and solve it via an optimization method. Finally, we compare our work with general mixture models, e.g. Gaussian Mixture Model. We present promising preliminary results obtained from proof-of-concept experiments conducted on both synthetic data and a real-world scenario.

1 INTRODUCTION

Abstract:

A person's beliefs and attitudes are key elements for inferring the meaning of opinions held by individuals and groups. These elements/perceptions, however, are not stable and may change over time through the processes of social interaction and firsthand experiences (Hill and Kriesi, 2001). Studies on social influence theories have shown that social influence may have qualitatively different effects, and that it may produce different kinds of change. One simple case is how likely an individual will adopt the attitudes and beliefs of other sources and by how much. For example, in the context of socialization of children, a child who has a strong bond with his family is inclined to take parental attitudes and actions with full trust. In contrast, people only selectively accept the arguments and views supported by online news sources, e.g. consumer review sites. Opinions adopted with different reliabilities will differ in terms of their qualitative characteristics, and affect a person's subsequent behavior. Moreover, the patterns of belief-changing behavior can be treated as an indicator of different types of social influence processes. For instance, in the process of selfidentification, the logic of how an individual actually believes in the opinions does not depend on observability of the influencing sources. It depends, nevertheless, on his identity activated at that given moment (Kelman, 1961). Thus, if we know the role that social influence plays in behavior change at each time period and the impact it has on a person's initial beliefs, we will be able to provide more insights and explanations on the observed opinion trend, and further, make predictions about other likely behavioral consequences.

However, the characteristics of opinion sources that affect people's beliefs and attitudes are rarely open to the public. Likewise, it is impossible to track how people view and adopt the opinions held by each of the sources they have interacted with. Such information can be concealed subconsciously when the influence is subtle or the reliability is not quantifiable, whereas sometimes people will intentionally conceal this information. For example, terrorists tend to protect criminal organizations by hiding their connections with the group. Therefore, it becomes critical to develop a flexible model that can support the social influence theory 1) of belief/opinion change; 2) detect and characterize the hidden influential sources; and, 3) discover the

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DOI: 10.5220/0004130100170024 In Proceedings of the International Conference on Knowledge Discovery and Information Retrieval (KDIR-2012), pages 17-24 ISBN: 978-989-8665-29-7 patterns/trend of the source's impact on the observed opinion change.

Bayesian approaches have been widely used to represent belief and opinions (Garg et al., 2004, Santos Jr. et al., 2011a). Among those, Bayesian Networks (BNs) (Pearl, 1988) are a popular probabilistic model due to its sound theoretical foundations in probability theory combined with efficient reasoning. For example, (Garg et al., 2004) introduces a BN based divergence minimization framework to integrate opinions from different sources in order to solve the problem of standard opinion pooling. However, people's belief, structured as a knowledge-based system, is necessarily associated with some degree of incompleteness, which turns out to be problematical to BNs, as they require a completely specified conditional probability table (CPT). BNs also require that information be topologically ordered which further restricts their general applicability to real-world situations. In this work, we build our model based on Bayesian Knowledge Bases (BKBs) (Santos and Santos, 1999), as it has been extensively used to model complex intent-driven scenarios (Santos et al., 2011a; Santos et al., 2011b).

At each time period, the formation of individual belief can be viewed as a process of aggregating opinion/information from different sources. The goal is to arrive at a single probability distribution that represents the integrated knowledge base. Santos et al. (2011c) proposed an algorithm to encode and fuse a set of belief networks from different sources into one unified BKB. Due to the nature of BKBs and the mathematical foundations of fusion, we derive a new modelling approach called a Finite Belief Fusion Model (FFM) to capture the characteristics of opinion-changing behavior. We can then show how to detect underlying hidden sources of change together with the corresponding influential factors through a non-linear optimization problem.

2 BELIEF FUSING MODEL

2.1 Related Work

Anomaly detection has been applied to detect the presence of any observations or patterns that are different from the normal behavior of the data (Das et al., 2008). Works based on Bayesian Networks include detecting anomalies in network intrusion detection (García-Teodoro et al., 2009) and disease outbreak detection (Wong et al., 2003). The typical

approach of BN-based anomaly detection is to compute the likelihood of each record in the dataset and report records with unusually low likelihoods as potential anomalies. Different from these approaches whose main goals are to achieve early detection and identify anomalous change in terms of a probability distribution (Das et al., 2008), we focus on detecting the reasons behind the behavior change. Moreover, many statistics-based anomaly detection methods only focus on detecting events whose patterns are anomalous enough to be distinguishable from normal data. Furthermore, they overlook the situation when certain external opinion sources that have subtle influences at present, may cause a butterfly effect later, as triggered by other events. We show that our work overcomes the above limitations by being able to detect less substantial influencing sources.

There are some other techniques that attempt to handle changing belief networks. Methods based on learning Dynamic Bayesian Networks (DBNs) (Dean and Kanazawa 1989) have provided mechanisms for identifying conditional dependencies in time-series data, such as for reconstructing transcriptional regulatory networks from gene expression data (Robinson and Hartemink, 2010) and speech recognition using HMM (Gale and Young, 2008). Nevertheless, most DBN implementations assume for the sake of efficiency that the Markov property holds for the domain they represent, which restricts knowledge engineering by requiring that the probability distribution of variables at time t depends solely on the single snapshot at time t - 1. Thus, for real world cases when the future outcomes are highly dependent on the hidden factors whose prior information is unidentified, we need another model that can easily express such abstract temporal relationships.

For each of the opinion sources, we would expect the probability of generating a series of responses follows a particular type of pattern. Similarly, the reliability of an opinion is also likely to vary across sources. This results in a natural expectation that we need a model that is capable of mixing belief networks from different sources together. Hill and Kriesi (2001) apply a Finite Mixture Model to support their theory of opinionchanging behavior, where the attitude of each of the group is represented by a distribution and the mixed distribution is described by a weighted aggregation of *n* different distributions. However, the Expectation-maximization (EM) based mixture decomposition methods show propensity to identify

local optima (McLaughlan and Peel, 2000), which makes it also sensitive to initial guesses. In addition, the separation of parameter estimation and component identification increases the probability of converging to boundary values when the number of model components exceeds the true one (Figueiredo and Jain, 2002). These considerations led us to develop a variant mixture model that is suitable for our problem of detecting hidden belief sources by taking advantage of time-varying information, as well as loosening the requirement of a predefined number of sources.

2.2 BKB

In this work, we assume that both of the initial beliefs and hidden influencing sources at each time period are represented by BKBs. BKBs are a rulebased probabilistic model that represents possible world states and their (causal) relationships using a directed graph. BKBs subsume BNs by specifying dependence at the instantiation level (versus BNs that specify only at the random variable level); by allowing for cycles between variables; and, by loosening the requirements for specifying complete probability distribution. BKBs collect the conditional probability rules (CPR) in an "if-then" style. Each instantiation of a random variable is represented by an I-node and the rule specifying the conditional probability of an I-node is encoded in an S-node with a certain weight/probability. Fig. 1 presents an example BKB fragment, with square blocks and circles representing I-nodes and S-nodes, respectively. Multiple fragments can be combined into a single BKB using the Bayesian fusion algorithm (Santos et al., 2011c). The idea behind this algorithm is to take the union of all input fragments by incorporating source nodes, indicating the source and reliability of the fragments. Reasoning algorithms are used in BKBs to make predictions and provide explanations (Santos and Santos, 1999).

2.3 Building a Model

Our goal is to detect hidden opinion sources and the corresponding impact patterns that result in behavior change over time. However, without a sound theoretical foundation, the methods developed will simply be ad hoc. Social influence theories show how the way people adopt beliefs and attitudes from other sources varies across conditions/situations. Sometimes, a person will not accept these external ideas in total but only adopts the pieces that fit into his own situation (Kelman, 1961). Therefore, we develop a model that is specifically tailored to take into account of all these points.



Figure 1: Sample BKB fragment from an intent framework.

2.3.1 Finite Belief Fusion Model (FFM)

We develop a finite belief fusion model to represent a person's actual belief distribution. Formally, a finite belief fusion model is defined as

$$\begin{cases} \boldsymbol{p}'(\boldsymbol{x}) = \boldsymbol{fuse}(\boldsymbol{w}_0, \boldsymbol{p}, \boldsymbol{\bar{w}}, \boldsymbol{\bar{h}}) \\ \sum_{i=0}^{n} \boldsymbol{w}_i = 1 \end{cases}$$

where p and w_0 denote the initial belief distribution and reliability, respectively. initial $\vec{w} =$ $[w_1, w_2, \dots, w_n]$ represents how a person views and trusts the opinion sources $\overline{h} = [h_1, h_2, ..., h_n]$, where both \vec{w} and \vec{h} are implicit to the observer. Instead of simply adding up the weighted input distributions linearly like general mixture model, the new belief distribution p' is generated through the BKB fusion algorithm. An important property of the fusion algorithm is the capability to support transparency in analysis. In other word, all perspectives are preserved in the fused BKB without loss of information. Since the fused belief is still a valid BKB, for each of the random variable v in p', let v_{na} be the parent variables of v. We have

$$\sum_{x} p'(v = x | v_{pa} = y) = \sum_{x} \frac{p'(v = x, v_{pa} = y)}{p'(v_{pa} = y)}$$
$$= \frac{\sum_{x} p'(v = x, v_{pa} = y)}{p'(v_{pa} = y)} \le 1$$

where

$$\begin{cases} p'(v = x, v_{pa} = y) = w_0 p(v = x, v_{pa} = y) + \\ \sum_{i=1}^{n} w_i h(v = x, v_{pa} = y) \\ p'(v_{pa} = y) = w_0 p(v_{pa} = y) + \sum_{i=1}^{n} w_i h(v_{pa} = y) \end{cases}$$

Social influence theory suggests that people's beliefs are partially affected by the external sources. In this work, we consider a simplified situation when each of the hidden sources only affects one part of the initial belief p, such that the conditional probability distribution of a particular variable v will not be changed by more than one source. This could happen when people prefer to take the attitude from the source whose belief/opinion is most convincing in a particular field of knowledge. Then, the above model can be simplified as:

$$\begin{cases} p'(v = x, v_{pa} = y) = wh(v = x, v_{pa} = y) + \\ (1 - w)p(v = x, v_{pa} = y) \end{cases}$$

$$p'(v_{pa} = y) = wh(v_{pa} = y) + (1 - w)p(v_{pa} = y) \\ \sum_{x} p'(v = x, v_{pa} = y) \le p'(v_{pa} = y) \end{cases}$$
(1)

where h is the only influencing source that affects v.

2.3.2 Detection Algorithm

Now, we generalize the problem by considering a series of beliefs: given belief trend $p_0, p_1, p_2, ..., p_t$ generated over t time periods, the goal is to learn the probability distribution for each of the potential hidden BKBs h_i (i = 1:n), as well as its time varying impact $w_{ij}(j = 1:t)$. Considering that the causal relationship in human belief systems is less likely to change, we assume that all belief networks share the same (causal) structure, but vary on probability distribution. Note, $w_{ij} = 0$ if source h_i has no impact at time j.

Let v_k be the variable influenced by source h_i and let θ and φ denote two states representing $\{v_{pa}^k = y\}$ and $\{v_k = x, v_{pa}^k = y\}$, we rewrite (1) as:

$$\begin{cases} p_j(\varphi) = w_{ij}h_i(\varphi) + (1 - w_{ij})p_0(\varphi) \\ p_j(\theta) = w_{ij}h_i(\theta) + (1 - w_{ij})p_0(\theta) \\ \sum_{\varphi} p_j(\varphi) \le p_j(\theta) \end{cases}$$
(2)

where w_{ij} , $h_i(\theta)$ and $h_i(\varphi)$ are *unknown* parameters needed to be learned from the given belief trend.

Let $\vec{w}_i = [w_{i1}, w_{i2}, ..., w_{it}]$ be the impact series of source h_i , we learn \vec{w}_i , $h_i(\theta)$ and $h_i(\varphi)$ via the following constrained optimization problem

$$[h_{i}^{*}(\theta), h_{i}^{*}(\varphi), \bar{w}_{i}^{*}] = \underset{\theta, \varphi, \bar{w}_{i}}{\operatorname{argmin}} \sum_{j=1}^{t} (\sum_{\theta} e^{f_{j}(\theta, w_{ij})} + \sum_{\varphi} e^{f_{j}(\varphi, w_{ij})})$$

s.t.
$$\forall \varphi, \sum_{\theta} p_j(\theta) - p_j(\varphi) \le 0, \ j = 1:t$$

where

$$f_j(x, y) = [p_j(x) - yh(x) - (1 - y)p_0(x)]^2$$

We apply Sequential Quadratic Programming (SQP) algorithm (Nocedal and Wright, 2006) to do the optimization, as the linear algebra routines it uses are more efficient in both memory usage and speed than the active-set routines.

So far, we have addressed the problem of characterizing the hidden source h_i and its impact pattern with respect to variable v_k . We apply the algorithm to all variables (k = 1:m) and get m impact trends.

Considering that some hidden sources may affect a fragment of initial belief that contains more than one variable, it is reasonable to believe that the variables that generate similar impact trends are affected by the same hidden source and should be represented in one distribution. We treat the weight at each time step as a feature and apply clustering algorithms (Xu and Wunsch, 2005) such as K-means to detect similar trends. The optimal number of hidden sources is achieved when the sum of interclass variance is less than a threshold.

3 EXPERIMENTS

In what follows, we present results of experiments that were carried out on both simulated data and a real world scenario. We studied the performance characteristics of our algorithm in simulation studies that vary by several orders of magnitude in the number of variables, number of hidden sources and number of time steps.

3.1 Simulated Data Set

To evaluate the effectiveness of our method, we simulate a person's actual belief trend from his initial belief and some hidden external sources, where the external sources are unknown to the detection model. We start with a small dataset, in which both of the initial belief and hidden sources are represented by a simulated five variables BKB (same structure, different distribution). In this experiment, we select only one hidden source. Then for every time period, we sample 1000 records from initial belief and hidden source respectively. The testing data is generated by mixing samples from two different distributions together with a randomly assigned hidden weight ranging from 0 to 1.



Figure 2: Comparison of detected weight for FFM (blue line) and GMM (green line) methods with different number of time steps. The red line with diamond mark is the real weight trend.

In order to examine how the amount of timevarying data affects our detection performance, we choose three different numbers of time steps: 10, 50 and 100. The conditional probability parameters of the belief network at each time step are learned from the testing data using smoothed maximum likelihood estimation (Das et al., 2008). To compare with the state-of-art mixture models, we run the same testing data on Gaussian Mixture models (GMMs), one of the most statistically mature methods for mixture model clustering. The weight of each component is learned through mixture decomposition. Figure 2 plots the impact trend detected using FFM and GMM respectively in terms of number of time steps, from which we can see that the hidden impact pattern we captured is pretty consistent with the true trend. Also, when the hidden impact values are very small, our detection results are still accurate. This fact enables us to detect less substantial influencing sources. The mean and standard deviation of the detection errors (difference between true and learned weight) can be found in Table 1a. In contrast to GMM, our method shows a higher accuracy with a smaller variance. Additionally, we see that the average accuracy of FFM increases with the number time steps, which indicates that our method is capable of improving detection performance by leveraging time-varying knowledge. Moreover, we compare the distribution of a hidden source learned during the detection process with the true one. Chan and Darviche (2002) proposed a distance measure between two probability distributions, where the distance is defined as:

$$D(P, P') = \ln \max_{w} \frac{P'(w)}{P(w)} - \ln \min_{w} \frac{P'(w)}{P(w)}$$

We apply this metric in our evaluation due to its ability to bound belief changes comparable to KLdivergence. The results provided in Table 1b suggest that the distribution of the hidden source we learned is closer to the real distribution than GMM.

Table 1a: Mean and Std of the detection errors.

Detection Error	10 steps	50 steps	100 steps
FFM (Mean)	0.0385	0.0281	0.0253
GMM (Mean)	0.3123	0.2219	0.2361
FFM (std)	0.0263	0.0160	0.0166
GMM (std)	0.1248	0.1386	0.1361

Table 1b: Distance measure between the true and learned probability distribution using different algorithm.

Distance	10 steps	50 steps	100 steps
FFM	0.5618	0.5137	0.4742
GMM	2.3145	1.9723	2.1687

To evaluate the scalability of our technique, we also simulate data from a 30 variables network with 100,000 mixture records generated at each time step.

We ran our experiment on nine different hidden sources and present the results in Figure 3. Apparently, our method scales well to large network.

Next, we conduct a more detailed analysis of performance by looking at detection results on each run. The largest error comes from the sixth trial. We examine the hidden sources involved in this trial and find that the distribution of the hidden source is very similar to the initial belief. Thus, it becomes more difficult to accurately detect the hidden impact, as the varied belief at each time step is insensitive to the value of impact.

Finally, we examine the ability of our method to detect multiple hidden sources. We choose n hidden



sources (n = 1:9), where each of them affects one fragment of the initial network with a certain weight. We follow the same procedure as the second experiment except that the mixture records are generated from n different hidden sources. Figure 4 depicts the average detection error with respect to the number of hidden sources. As we can see, the error grows with the number of sources. This is due to the increased degree of freedom brought about by multiple fragments fusion as it enlarges the potential solution space. Nevertheless, the largest error is still less than 0.1.



Figure 4: Average detection error in terms of the number of hidden sources.

3.2 H1N1

In this subsection, we apply our method to identify the impact patterns behind the events that happened during the H1N1 pandemic in Mexico. Santos et al. (2011a) conducted a Cross-Border Epidemic Spread project to study why and under what circumstances would people be driven to cross the border both legally and illegally with respect to epidemic spread. In order to understand such human behavior as well as the intent, they employed the intent framework

represented by BKBs to model people's reaction to the various events that took place during the pandemic in 2009. The whole intent system is constructed through the fusion of cultural BKB fragments that are created based on sources such as demographic information and news articles. When a major event occurs, the intent system will update its probability distribution adaptively to reflect an individual/group's belief change caused by the event. Therefore, the characteristics of these events and their impact patterns are key to analysing people's reactions. We apply our method on a series of intent systems modelled in the paper (Santos et al., 2011a) to detect the implicit events without any foreknowledge. Figure 5 displays our detection results, where two potential/unknown events, represented by blue and red dotted lines are successfully detected.

To figure out what these two events could be, we plot the probability of "people believe disease is contagious" over time in Figure 5. As we can see, the probability achieves its peak on May-1-09 and starts to decline on Jul-15-09, which shows a strong correlation with the impact pattern from the second event. This finding indicates that the breakout of the second event causes a temporary increase on people's belief regarding the contagious nature of the disease. In comparison, the event that happened on Apr-24-09 had no direct impact on such belief change. In fact, according to the timeline of H1N1, we find that two events: "WHO sends experts to Mexico" on Apr-24-09 and "Government published an announcement to advise people staying at home" on May-1-09 match perfectly with our detection result. The learned distribution of the WHO event suggests an increase in the probability of "believe healthcare is effective" by 0.225. However, there is

direct causal relationship between no the effectiveness of healthcare and the contagiousness of H1N1, so the impact from the "WHO" will not be reflected by how people think of the disease. This explains why people did not change their beliefs about the contagiousness of H1N1 until they received the government's announcement, even though the WHO was already sending in experts since April. Moreover, we compare the impact patterns we detected with the true trend. As shown in Figure 5, our results are very close to the modelled scenario (solid lines), which helps point towards the effectiveness of our approach



Figure 5: Detection results on event impact. Two events represented by blue and red lines are detected. The black solid line indicates the probability of "people believe disease is contagious".

CONCLUSIONS 4

In this paper, we presented a new approach to detect hidden sources of influence, as well as capture and characterize the patterns of their impact with regards to belief-changing trends. We formalize the problem as a finite belief fusion model and solve it via an optimization method. We demonstrate that FFM outperforms the classic Gaussian Mixture Models in both small and large synthetic datasets. In addition, we applied our method to identify implicit events that happened during the H1N1 pandemic in Mexico. Also, the detection results generated by FFM were consistent with the modelled scenario.

In future work, we will expand our approach by allowing multiple sources to affect the same part of the belief network. This happens when there is no convincing source for a particular fragment and the final knowledge/belief system is formed by integrating all possible explanations.

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