Nonlinear Analysis of Costas Loop Circuit

N. V. Kuznetsov[1,2], G. A. Leonov[1], M. V. Yuldashev[1,2] and R. V. Yuldashev [1,2]
[1]*Dept. of Mathematical Information Technology, University of Jyväskylä,*
P.O. Box 35 (Agora), FIN-40014, Jyväskylä, Finland
[2]*Faculty of Mathematics and Mechanics, Saint-Petersburg State University,*
Universitetsky pr. 28, Saint-Petersburg, 198504, Russia

Keywords: Costas Loop, Phase-locked Loop, Phase Detector Characteristic, Nonlinear Analysis.

Abstract: Problems of rigorous mathematical analysis of Costas Loop are considered. The analytical method for phase detector characteristics computation is proposed and new classes of phase detector characteristics are computed for the first time. Effective methods for nonlinear analysis of Costas Loop are discussed.

1 INTRODUCTION

The Costas loop was invented in 1950s by American electrical engineer John P. Costas (Costas, 1956). It is one of a few schemes of carrier recovery loop and it is widely used in practice for binary phase-shift keying (BPSK) demodulation technique.

Various methods for analysis of PLL, and particularly Costas loop, are well developed by engineers (Gardner, 1966; Lindsey, 1972; Kroupa, 2003) but the problems of construction of adequate nonlinear models and nonlinear analysis of such models are still far from being resolved. As noted by D. Abramovitch in his keynote talk at American Control Conference (Abramovitch, 2002), the main tendency in a modern literature on analysis of stability and design of PLL is the use of simplified linearized models, application of the methods of linear analysis, a rule of thumb, and simulation. However it is known that the application of linearization methods and linear analysis for control systems can lead to untrue results (e.g. the counterexamples to conjectures on absolute stability and on harmonic linearization and to filter hypothesis (Leonov et al., 2010a; Leonov and Kuznetsov, 2011; Bragin et al., 2011)) and requires special justifications. Also simple numerical analysis can not reveal nontrivial regimes (e.g., semi-stable or nested limit cycles, hidden oscillations and attractors (Gubar', 1961; Leonov et al., 2008; Leonov et al., 2010c; Leonov et al., 2011a)).

In this paper, following works (Leonov et al., 2011b; Kuznetsov et al., 2011a; Kuznetsov et al., 2011b; Leonov et al., 2010b; Kuznetsov et al., 2009a; Kuznetsov et al., 2009b; Kuznetsov et al., 2008), rigorous mathematical approach to investigation of Costas loop is described. Mathematical model of high frequency signals is considered and nonlinear model of Costas loop is constructed. Investigation of Costas loop behavior is reduced to investigation of PLL with specific phase detector characteristic.

2 THE DESCRIPTION OF COSTAS LOOP IN SIGNAL SPACE

Consider Costas loop at the level of electronic realization (Fig. 1).

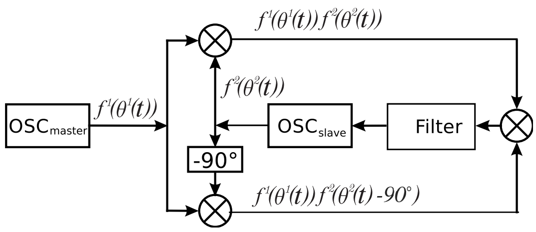

Figure 1: Block diagram of Costas loop at the level of electronic realization.

Here OSC_{master} is a master oscillator, OSC_{slave} is a slave (tunable voltage-controlled) oscillator, which generates oscillations $f^{1,2}(t)$ with high-frequencies $\dot{\theta}^{1,2}(t)$. Block -90^o shifts phase of input signal by $-\frac{\pi}{2}$.

Block $\otimes$ is a multiplier of inputs. The relation between the input $\xi(t)$ and the output $\sigma(t)$ of linear filter has the form $\sigma(t) = \alpha_0(t) + \int\limits_0^t \gamma(t-\tau)\xi(\tau)\,d\tau$. Here $\gamma(t)$ is an impulse transient function of filter, $\alpha_0(t)$ is an exponentially damped function, depending on the initial state of filter at moment $t = 0$.

V. Kuznetsov N., A. Leonov G., V. Yuldashev M. and V. Yuldashev R..
Nonlinear Analysis of Costas Loop Circuit.
DOI: 10.5220/0003976705570560
In *Proceedings of the 9th International Conference on Informatics in Control, Automation and Robotics* (ICINCO-2012), pages 557-560
ISBN: 978-989-8565-21-1

3 COMPUTATION OF PHASE DETECTOR CHARACTERISTIC

Suppose, the phases $\theta^1(t), \theta^2(t)$ of the considered signals are smooth functions with the frequencies $\dot{\theta}^{1,2}(t)$ satisfying the following high-frequency conditions

$$\dot{\theta}^p(\tau) \geq \omega_{min} > 0, \quad p = 1,2 \qquad (1)$$

on the fixed time interval $[0,T]$. Also it is assumed that the frequency difference is uniformly bounded

$$\left|\dot{\theta}^1(\tau) - \dot{\theta}^2(\tau)\right| \leq \Delta\omega, \ \forall \tau \in [0,T], \qquad (2)$$

where $\Delta\omega$ is a certain constant.

Divide the interval $[0,T]$ into small intervals

$$\delta = (\omega_{min})^{-1/2}. \qquad (3)$$

By assumption,

$$\begin{array}{l} |\dot{\theta}^p(\tau) - \dot{\theta}^p(t)| \leq \Delta\Omega, \ p = 1,2, \\ |t-\tau| \leq \delta, \quad \forall \tau, t \in [0,T], \end{array} \qquad (4)$$

where constant $\Delta\Omega$ is independent of t, τ.

The function $\gamma(t)$ is smooth and there exists a constant C such that

$$\begin{array}{l} |\gamma(\tau) - \gamma(t)| \leq C\delta, \\ \forall \tau, t \in [0,T], \quad |t-\tau| \leq \delta. \end{array} \qquad (5)$$

The latter means that on small intervals $[\tau, \tau+\delta]$ the functions $\gamma(t)$ and $\dot{\theta}^{1,2}(t)$ are "almost constant" and the functions $f^{1,2}(t)$ are rapidly oscillating. Obviously, such a condition occurs in the case of high-frequency oscillations.

Consider now harmonic oscillations

$$f^1(\theta^1(t)) = b_1^1 \cos(\theta^1(t)), \ f^2(\theta^2(t)) = b_1^2 \sin(\theta^2(t))$$

and two block diagrams shown in Fig. 2 and Fig. 3.

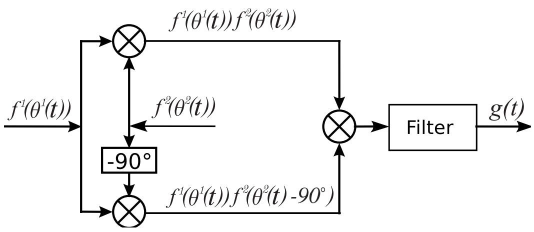

Figure 2: Two inputs and filter output.

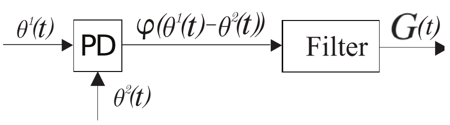

Figure 3: Phase detector and filter.

In Fig. 3 $\theta^{1,2}(t)$ are phases of oscillations $f^{1,2}(\theta^{1,2}(t))$, PD is a nonlinear block with the characteristic $\varphi(\theta)$ (being called a phase detector or discriminator). The phases $\theta^{1,2}(t)$ are the inputs of PD block and the output is the function $\varphi(\theta^1(t) - \theta^2(t))$. It should be noted, that the shape of phase detector characteristic depends on shapes of input signals.

In both diagrams the filters are the same with the same impulse transient function $\gamma(t)$ and the same initial states. The filters outputs are the functions $g(t)$ and $G(t)$, respectively.

A classical Costas loop synthesis for harmonic signals is based on the following result: *For high-frequency harmonic oscillation function $\varphi(\theta)$ has the form $\varphi(\theta) = \frac{1}{8} b_1^1 b_1^2 \sin(2\theta)$ and for the same initial data of filter, the following relation $G(t) - g(t) \approx 0$ is satisfied.*

Further will be considered extension of this result to non-harmonic signals. Consider a partially differentiable odd function $f^1(\theta^1(t))$ in the form of Fourier series

$$f^1(\theta) = \sum_{i=1}^{\infty} b_i^1 \sin(i\theta), \ f^2(\theta) = b_1^2 \sin(\theta). \qquad (6)$$

Here coefficients satisfy the relation $b_i^1 = O(i^{-1})$.

Then the following assertion can be proved.

Theorem 1. *If conditions* (1)–(5) *are satisfied (high-frequency property) and*

$$\varphi(\theta) = \frac{(b_1^2)^2}{8} \left[-(b_1^1)^2 \sin(2\theta) + 2 \sum_{q=1}^{\infty} b_q^1 b_{q+2}^1 \sin(2\theta) \right],$$

then for the same initial state of filter relation

$$G(t) - g(t) = O(\delta), \quad \forall t \in [0,T] \qquad (7)$$

is valid.

This result can be extended to the case of full Fourier series and allows one to compute a phase detector characteristic for standard types of signals.

3.1 Proof of Theorem

Let $t \in [0,T]$. Consider the difference

$$g(t) - G(t) = \int_0^t \gamma(t-s) \times$$
$$\times \left[f^1\big(\theta^1(s)\big) f^2\big(\theta^2(s)\big) f^1\big(\theta^1(s)\big) f^2\big(\theta^2(s) - \tfrac{\pi}{2}\big) - \right.$$
$$\left. -\varphi\big(\theta^1(s) - \theta^2(s)\big) \right] ds.$$

Denote by $m \in \mathbb{N} \cup \{0\}$ a natural number such that $t \in [m\delta, (m+1)\delta]$. From (3) we have $m < T/\delta + 1$. Function $\gamma(t)$ is continuous and, therefore, it is bounded on $[0,T]$, $f^1(\theta), f^2(\theta), \varphi(\theta)$ are also bounded on $\mathbb{R}$. Then

$$\int\limits_t^{(m+1)\delta} \gamma(t-s) f^1\big(\theta^1(s)\big) f^2\big(\theta^2(s)\big) \times$$
$$\times f^1\big(\theta^1(s)\big) f^2\big(\theta^2(s) - \tfrac{\pi}{2}\big) ds = O(\delta),$$
$$\int\limits_t^{(m+1)\delta} \gamma(t-s) \varphi\big(\theta^1(s) - \theta^2(s)\big) ds = O(\delta)$$

and $g(t) - G(t)$ can be rewritten as

$$g(t) - G(t) = \sum_{k=0}^{m} \int\limits_{[k\delta,(k+1)\delta]} \gamma(t-s) \times$$
$$\Big[\times f^1\big(\theta^1(s)\big) f^2\big(\theta^2(s)\big) f^1\big(\theta^1(s)\big) f^2\big(\theta^2(s) - \tfrac{\pi}{2}\big) -$$
$$-\varphi\big(\theta^1(s) - \theta^2(s)\big)\Big] ds + O(\delta). \tag{8}$$

Since (5), it follows that on any interval $[k\delta,(k+1)\delta]$ we have

$$\gamma(t-s) = \gamma(t-k\delta) + O(\delta),$$
$$t > s,\ s \in [k\delta,(k+1)\delta]. \tag{9}$$

where $O(\delta)$ is independent of k. Then by (8), (9) and the boundedness of $f^1(\theta), f^2(\theta), \varphi(\theta)$ we get

$$g(t) - G(t) = \sum_{k=0}^{m} \gamma(t-k\delta) \int\limits_{[k\delta,(k+1)\delta]}$$
$$\Big[f^1\big(\theta^1(s)\big) f^2\big(\theta^2(s)\big) f^1\big(\theta^1(s)\big) f^2\big(\theta^2(s) - \tfrac{\pi}{2}\big) -$$
$$-\varphi\big(\theta^1(s) - \theta^2(s)\big)\Big] ds + O(\delta).$$

Denote $\theta_k^p(s) = \theta^p(k\delta) + \dot\theta^p(k\delta)(s-k\delta)$, $p \in \{1,2\}$. By (4) with $s \in [k\delta,(k+1)\delta]$ we obtain $\theta^p(s) = \theta_k^p(s) + O(\delta)$. Since $\varphi(\theta)$ is bounded and continuous on $\mathbb{R}$, by (2) we have

$$\int\limits_{[k\delta,(k+1)\delta]} \big|\varphi\big(\theta^1(s) - \theta^2(s)\big) - \varphi\big(\theta_k^1(s) - \theta_k^2(s)\big)\big| ds = O(\delta^2).$$

The function $f^2(\theta)$ is smooth while the function $f^1(\theta)$ is partially-differentiable and bounded. If $f^1(\theta)$ is continuous on $\mathbb{R}$, then

$$\int\limits_{[k\delta,(k+1)\delta]} f^1\big(\theta^1(s)\big) f^2\big(\theta^2(s)\big) f^1\big(\theta^1(s)\big) \times$$
$$\times f^2\big(\theta^2(s) - \tfrac{\pi}{2}\big) ds =$$
$$= \int\limits_{[k\delta,(k+1)\delta]} f^1\big(\theta_k^1(s)\big) f^2\big(\theta_k^2(s)\big) f^1\big(\theta_k^1(s)\big) \times$$
$$\times f^2\big(\theta_k^2(s) - \tfrac{\pi}{2}\big) ds + O(\delta^2). \tag{10}$$

Considering sets (10) outside of small neighbourhoods of discontinuity points and using (1)–(5), the proof of theorem is completed. ■

3.2 Example

Consider a triangular signal (Fig. 4)

$$f^1(t) = \frac{8}{\pi^2} \sum_{l=1}^{\infty} \frac{(-1)^{l-1}}{(2l-1)^2} \sin\big((2l-1)\theta^1(t)\big).$$

Then

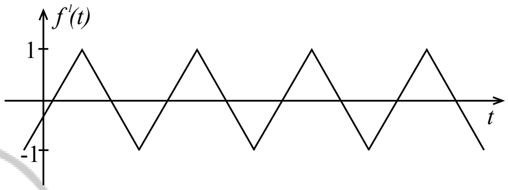

Figure 4: Triangular signal.

$$\varphi(\theta^1 - \theta^2) = \tfrac{8}{\pi^4}\Big[-\sin(2\theta^1 - 2\theta^2) +$$
$$+2\sum_{l=1}^{\infty} \tfrac{1}{(2l-1)^2(2l+1)^2} \sin(2\theta^1 - 2\theta^2)\Big].$$

By $2\sum\limits_{l=1}^{\infty} \frac{1}{(2l-1)^2(2l+1)^2} = \frac{\pi^2}{8} - 1$ we finally get

$$\varphi(\theta^1 - \theta^2) = \Big(\tfrac{1}{\pi^2} - \tfrac{16}{\pi^4}\Big) \sin(2\theta^1 - 2\theta^2).$$

4 PHASE-FREQUENCY MODEL

From Theorem 1 it follows that the block-scheme of Costas loop in signal space (Fig. 1) can be asymptotically changed (for high-frequency generators) by the block-scheme in frequency and phase space (Fig. 5). Here PD is a phase detector with corresponding characteristic computed above.

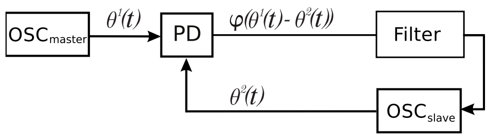

Figure 5: Phase-locked loop with phase detector.

ACKNOWLEDGEMENTS

This work was supported by Academy of Finland, Ministry of Education & Science and Saint-Petersburg State University (Russia).

REFERENCES

Abramovitch, D. (2002). Phase-locked loops: A control centric tutorial. In *Proceedings of the American Control Conference*, volume 1, pages 1–15.

Bragin, V. O., Vagaitsev, V. I., Kuznetsov, N. V., and Leonov, G. A. (2011). Algorithms for finding hidden oscillations in nonlinear, systems. the Aizerman and Kalman conjectures and Chua's circuits. *Journal of Computer and Systems Sciences International*, 50(4):511–543.

Costas, J. (1956). Synchoronous communications. In *Proc. IRE*, volume 44, pages 1713–1718.

Gardner, F. (1966). *Phase–lock techniques*. John Wiley, New York.

Gubar', N. A. (1961). Investigation of a piecewise linear dynamical system with three parameters. *J. Appl. Math. Mech.*, (25):1519–1535.

Kroupa, V. (2003). *Phase Lock Loops and Frequency Synthesis*. John Wiley & Sons.

Kuznetsov, N., Leonov, G., and Seledzhi, S. (2009a). Nonlinear analysis of the costas loop and phase-locked loop with squarer. In *Proceedings of the IASTED International Conference on Signal and Image Processing, SIP 2009*, pages 1–7.

Kuznetsov, N., Leonov, G., Seledzhi, S., and Neittaanmäki, P. (2009b). Analysis and design of computer architecture circuits with controllable delay line. In *ICINCO 2009 - 6th International Conference on Informatics in Control, Automation and Robotics, Proceedings*, volume 3 SPSMC, pages 221–224. doi:10.5220/0002205002210224.

Kuznetsov, N., Leonov, G., Yuldashev, M., and Yuldashev, R. (2011a). Analytical methods for computation of phase-detector characteristics and pll design. In *ISSCS 2011 - International Symposium on Signals, Circuits and Systems, Proceedings*, pages 7–10. doi:10.1109/ISSCS.2011.5978639.

Kuznetsov, N., Neittaanmäki, P., Leonov, G., Seledzhi, S., Yuldashev, M., and Yuldashev, R. (2011b). High-frequency analysis of phase-locked loop and phase detector characteristic computation. In *ICINCO 2011 - Proceedings of the 8th International Conference on Informatics in Control, Automation and Robotics*, volume 1, pages 272–278. doi:10.5220/0003522502720278.

Kuznetsov, N. V., Leonov, G. A., and Seledzhi, S. S. (2008). Phase locked loops design and analysis. In *ICINCO 2008 - 5th International Conference on Informatics in Control, Automation and Robotics, Proceedings*, volume SPSMC, pages 114–118. doi:10.5220/0001485401140118.

Leonov, G. A., Bragin, V. O., and Kuznetsov, N. V. (2010a). Algorithm for constructing counterexamples to the Kalman problem. *Doklady Mathematics*, 82(1):540–542.

Leonov, G. A. and Kuznetsov, N. V. (2011). Algorithms for searching hidden oscillations in the Aizerman and Kalman problems. *Doklady Mathematics*, 84(1):475–481.

Leonov, G. A., Kuznetsov, N. V., and Kudryashova, E. V. (2008). Lyapunov quantities, limit cycles and strange behavior of trajectories in two-dimensional quadratic systems. *Journal of Vibroengineering*, 10(4):460–467.

Leonov, G. A., Kuznetsov, N. V., and Vagaitsev, V. I. (2011a). Localization of hidden Chua's attractors. *Physics Letters A*, 375(23):2230–2233.

Leonov, G. A., Kuznetsov, N. V., Yuldashev, M. V., and Yuldashev, R. V. (2011b). Computation of phase detector characteristics in synchronization systems. *Doklady Mathematics*, 84(1):586–590.

Leonov, G. A., Seledzhi, S. M., Kuznetsov, N. V., and Neittaanmaki, P. (2010b). Asymptotic analysis of phase control system for clocks in multiprocessor arrays. In *ICINCO 2010 - Proceedings of the 7th International Conference on Informatics in Control, Automation and Robotics*, volume 3, pages 99–102. doi:10.5220/0002938200990102.

Leonov, G. A., Vagaitsev, V. I., and Kuznetsov, N. V. (2010c). Algorithm for localizing Chua attractors based on the harmonic linearization method. *Doklady Mathematics*, 82(1):693–696.

Lindsey, W. (1972). *Synchronization systems in communication and control*. Prentice-Hall, New Jersey.