AN APPROXIMATION OF GAUSSIAN PULSES

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Abstract: A new technique for generating an approximate replica of Gaussian pulses with good accuracy is proposed and investigated. The Gaussian function is approximated with a waveform that results from the convolution of two triangles. The proposed pulse performs better than other previously reported pulse. The results show good agreement not only for the Gaussian pulse but also for its first and second derivatives. As the triangular pulse generator is standard and widely used, the proposed technique needs besides it an appropriate filter.

1 INTRODUCTION

Gaussian pulses are widely used in communications, as they show maximum steepness of transition with no overshoot and minimum group delay. Several applications are mobile telephony (GSM) where Gaussian Minimum Shift Keying (GMSK) signals are used and ultra-wideband (UWB) communications, where ultra-short pulses based on the Gaussian shape are generated.

The Ultra-wideband (UWB) technology (Guofeng, 2003)(Xiliang, 2003) is a new technology for short range, high data rate wireless communication and it was investigated for use in high data-rate indoor wireless networks. UWB can also be used for Personal Area Networks (PAN) as it can deliver data speeds of 480 Mbps at distances of 2-3 meters. A UWB communication system transmits pulses which occupy several GHz of spectrum (from near DC).

As it occupies a very large bandwidth, UWB technology is subjected to very strict spectral and power constraints in order to coexist with other existing communication systems. There are stringent regulations on the radiated energy in order to avoid interference, set by the Federal Communications Commission (FCC) (FCC, 2002). As a consequence, the spectral shape of UWB signals is an important implementation aspect, adhering to constraints and still maximizing available signal power, to enable the targeted high data rate applications (Guofeng, 2003). One should maximize the total transmitted power across the band while complying with the imposed spectral mask.

If the spectral properties are not optimized, the output power has to be lowered to fulfil the mask requirements in every frequency band. Since the ultra-short pulses used are generated with analog components, e.g., the Gaussian Monocycle, their spectral shape is not easy to design. Replacing the analog pulses with digital designs is prohibited by the huge bandwidth and the resulting sampling rates (Berger, 2006).

The most frequently used pulse signals in digital communications are:

1. rectangular pulse;
2. cosine pulse (MSK);
3. raised cosine pulse (quadrature overlapped raised-cosine - QORC);
4. Gaussian pulse (GMSK, UWB).

A Gaussian pulse is a good choice of shaping function since it provides a particularly compact frequency domain spectrum. In general the improvement stems from the elimination of the broad pattern of side lobes characteristic of a

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rectangular pulse, which extends to a surprisingly large distance from the centre frequency (Berger, 2006)(Dou, 2000).

The frequency-domain representation and Fourier transform of the Gaussian pulse are:

$$H(f) = \frac{1}{\sqrt{2\pi}\sigma} \cdot e^{-\frac{f^2}{2\sigma^2}} \quad (1)$$

$$h(t) = \sqrt{2\pi}\sigma\, e^{-2\pi^2\sigma^2 t^2} \quad (2)$$

A true Gaussian pulse has theoretically an infinite extent, so, one has to truncate the tails in time domain and investigate the consequences in the frequency domain. In GMSK the pre-modulation filter is Gaussian and has a transfer function

$$H(f) = A \cdot e^{-\left(\frac{f}{B}\right)^2 \cdot \frac{\ln 2}{2}} \quad (3)$$

where B is the 3 dB band of the filter and A is a constant. If,

$$\alpha = \frac{\sqrt{\ln 2}}{\sqrt{2}B} = \frac{0.5887}{B} \quad (4)$$

the impulse response of the filter becomes

$$h(t) = \frac{\sqrt{\pi}}{\alpha} \cdot e^{-\frac{\pi^2}{\alpha^2} \cdot t^2} \quad (5)$$

In GMSK the Gaussian filter makes smooth the phase trajectory of the MSK signal and limits the variations of the instantaneous frequency of the signal. The impulse response of the filter to a rectangular signal of duration T is (Murota, 1981)

$$g_s(t) = A \cdot \sqrt{\frac{2\pi}{\ln 2}} \cdot B \cdot T \int_{t/T-1/2}^{t/T+1/2} e^{-\frac{2\pi^2 \cdot (B \cdot T)^2 u^2}{\ln 2}} du \quad (6)$$

It can also be expressed as equation (7):

$$g_s(t) = K \cdot \left\{ \mathrm{erf}\left[cB\left(t + \frac{T}{2} \right) \right] - \mathrm{erf}\left[cB\left(t - \frac{T}{2} \right) \right] \right\} \quad (7)$$

or

$$g_s(t) = \frac{1}{2T} \cdot \left[Q\left(2\pi B \frac{t - T/2}{\ln 2} \right) - Q\left(2\pi B \frac{t + T/2}{\ln 2} \right) \right] \quad (8)$$

where K is a constant chosen in order that the area of the impulse be equal to $1/2$, c is a constant $c = \pi\sqrt{2/\ln 2}$ and B is the 3 dB bandwith.

In the sequel we will concentrate on obtaining a good approximation of the Gaussian pulse and its first derivative.

2 FREQUENCY SPECTRUM

A frequently used signalling waveform in digital communication is the rectangular pulse, as it can be produced easily, even at high speeds. However, the rectangular pulse shows a power spectrum that decays slowly.

The power spectral density (p.s.d.) of a polar NRZ-L transmission using equiprobable data bits ($p = 1 - p = 0.5$) (Bennett, 1958), is given by

$$W(f) = \frac{1}{T}\left| G(f) \right|^2 \quad (9)$$

where *G(f)* denotes the Fourier transform of the signaling pulse *g(t)*, and *T* is the duration of the bit interval.

The rectangular pulse of amplitude *A* and duration *T* has a Fourier transform

$$G(f) = AT \cdot \frac{\sin \pi fT}{\pi fT} = AT \cdot \frac{\sin(\pi f / f_0)}{\pi f / f_0} \quad (10)$$

where $f_0 = 1/T$ is the signalling frequency (data rate) and f / f_0 is the normalized frequency with respect to the data rate. The Fourier transform decays rather slowly as $1/f$, taking into account the discontinuous character of the signalling waveform (rectangular pulse). As a consequence, its p.s.d. will decay as f^{-2}.

A well-known theorem in the theory of Fourier transform states that if the signalling waveform $g(t)$ is continuous and equal to zero at the ends of the signalling interval ($\pm T/2$), and has a number of $k-1$ derivatives that are continuous and equal to zero at the ends of the signalling interval, then the Fourier transform will decay as $f^{-(k+1)}$ (Beaulieu, 2004)(Alexandru, 2009). Accordingly, the p.s.d. will decay as $f^{-2(k+1)}$. We will denote this as the continuity feature of $(k-1)-th$ order.

Let us consider a raised cosine (RC) pulse described by equation (11):

$$g(t) = \begin{cases} \frac{1}{2}(1 + \cos 2\pi t) & |t| \le \frac{T}{2} \\ 0 & \text{elsewhere} \end{cases} \quad (11)$$

It satisfies

$$g(t)\big|_{t=\pm T/2} = 0 \quad (12)$$

Its first derivative is

$$g'(t) = -\pi \sin 2\pi t \big|_{t=\pm T/2} = 0 \quad (13)$$

As

$$g''(t)\big|_{t=\pm T/2} \neq 0 \tag{14}$$

the signalling pulse g(t) has $k-1=1$ derivatives that are continuous and equal to zero at the ends of the signalling interval ($\pm T/2$), k = 2 and the p.s.d. will decay as f^{-6}, as seen in Figure 1 in comparison with the spectrum of a rectangular pulse that decays as f^{-2}.

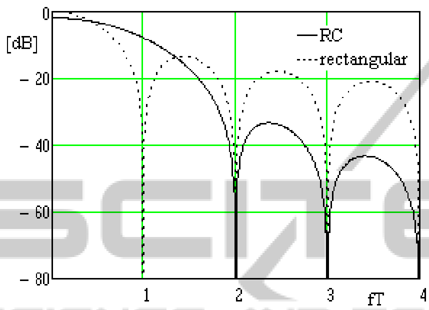

Figure 1: Power spectral density of rectangular and RC pulse.

The Fourier transform of *g(t)* is given by

$$G(f) = \frac{1}{2}\frac{\sin\pi fT}{f\pi(1-f^2T^2)} \tag{15}$$

The RC pulse with a width *T* results from the convolution of a rectangular pulse of width *T*/2 with a cosine lobe of width *T*/2.

3 PULSES RESULTED FROM CONVOLUTION

To exhibit better spectral properties the signalling waveform *g(t)* should be continuous and equal to zero at the ends of the signalling interval ($\pm T/2$) and possess a large number of derivatives that are continuous and equal to zero at the ends of the signalling interval.

This condition is easily met if the signalling pulses are obtained as a result of convolution. Let us assume that a pulse *g(t)* is obtained from the convolution of two pulses *x(t)* and *y(t)*

$$g(t) = x(t) * y(t) = \int x(t-\tau)y(\tau)d\tau \tag{16}$$

$$G(f) = X(f)\cdot Y(f) \tag{17}$$

If *x(t)* and *y(t)* possess continuity of $(k-1)-th$ and $(l-1)-th$ order, respectively, then the p.s.d. of *G(f)* will show a fast roll-off proportional to $f^{-2(k+l+2)}$, which corresponds to a $(k+l)-th$ order of continuity.

As an example let us consider the RC pulse given by equation (18):

$$g(t) = \begin{cases} \frac{1}{2}(1+\cos(\pi t/T)) & |t| \le T \\ 0 & \text{elsewhere} \end{cases} \tag{18}$$

which results from the convolution of a rectangular pulse with a cosine pulse, both of duration *T*. The resulted pulse has duration of 2*T*. In Figure 1 we represented the p.s.d. for a rectangular and a RC pulse. As seen, the spectral roll-off rate is bigger for RC pulse, as it exhibits better continuity properties. We shall characterize a pulse signalling waveform by *CnDm*; *n* = 0, 1 and *m* = 0, 1, 2, 3, …

n=0 means that the signalling waveform is not continuous and equal to zero at the ends of the signalling interval and *n*=1 denotes the opposite.

m is the number of the derivatives that satisfy the *m*-th order of continuity condition.

A rectangular pulse can be characterized as *C0D0* and a cosine pulse as *C1D0*. The RC pulse that results from their convolution is described by *C1D1*. A few classes of signalling pulses are described in Table 1.

Table 1: Classes of signalling pulses produced by convolution.

I *x(t)*	II *y(t)*	III *x(t)*y(t)*
C_0D_0 Rectangle	C_0D_0 Rectangle	C_1D_0 Triangle
C_0D_0 Rectangle	C_1D_0 Triangle Cosine lobe	C_1D_1 C_1D_1 Raised cosine
C_0D_0 Rectangle	C_1D_1 Raised cosine	C_1D_2
C_1D_0 Triangle Cosine lobe	C_1D_0 Triangle Cosine lobe	C_1D_2
C_1D_0 Triangle Cosine lobe	C_1D_1 Raised cosine	C_1D_3
C_1D_1 Raised cosine	C_1D_1 Raised cosine	C_1D_4
C_1D_1 Raised cosine	C_1D_2 Cos*Cos	C_1D_5
C_1D_1 Raised cosine	C_1D_4 Rcos*Rcos	C_1D_7
C_1D_2 Cos*Cos	C_1D_2 Cos*Cos	C_1D_6

4 APPROXIMATIONS OF GAUSSIAN PULSE

An approximation of Gaussian pulse in the interval $(-3, 3)$ (Dimitrov, 1991) used the method of linear voltage integration. The Gaussian function was approximated for $\sigma = 1$ and was normalized to obtain $h(0) = 1$. The Gaussian characteristic is replaced by a piece-wise parabolic approximation using polynomials of power 2 (quadratic parabolas) of the type:

$$y(t) = c_0 + c_1 x_n + c_2 {x_n}^2 \tag{19}$$

where x_n is a discrete variable. When x_n changes gradually during the calculations with a step of 0.01, (Dimitrov, 1991) the approximation function that minimizes the relative error of approximation is given by the reunion of three parabolic pieces, as

$$y(x) = \begin{cases} 1.3276 + 0.8848\, x + 0.1472\, x^2, & -3 \le x \le -1 \\ 0.99089 - 0.40548\, x^2, & -1 \le x \le 1 \\ 1.3276 - 0.8848\, x + 0.1472\, x^2, & 1 \le x \le 3 \\ 0, & \textit{elsewhere} \end{cases} \tag{20}$$

5 PROPOSED SOLUTION

By convolving a rectangular pulse of width T with itself, a triangular pulse of width $2T$ is obtained. We shall use the waveforms resulting from the convolutions of triangular waveforms defined by

$$g(t,T) = \begin{cases} \left(1 + \dfrac{t}{T}\right) & t \in [-T, 0] \\ \left(1 - \dfrac{t}{T}\right) & t \in [0, T] \end{cases} \tag{21}$$

The convolution of two triangles of width $2T$, which is equivalent to convolving four rectangular pulses of width T (Alexandru, 1998), results in an impulse of width $4T$, which is defined by

$$s(t,T) = g(t,T) * g(t,T) = \begin{cases} T\left(\dfrac{2}{3} - \left(\dfrac{t}{T}\right)^2 + \dfrac{1}{2}\left(\dfrac{|t|}{T}\right)^3\right) & -T \le t \le T \\ \dfrac{T}{6}\left(2 - \dfrac{|t|}{T}\right)^3 & T \le |t| \le 2T \end{cases} \tag{22}$$

Figure 2 displays the *s(t)* waveform together with a Gaussian pulse with mean value 0 and dispersion $\sigma = 0.266$. The resemblance is obvious. Figure 3 illustrates the Gaussian pulse with $\sigma = 1$ and its approximation $s(t, T)$ with T=1.67.

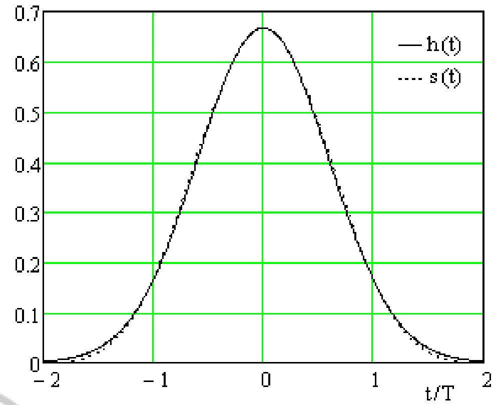

Figure 2: Gaussian pulse h(t) with $\sigma = 0.266$ and its approximation *s(t)* with $T = 1$.

The first derivative of the Gaussian pulse is given by:

$$h'(t) = -4\sqrt{2}\,\sigma^3 t \pi^{5/2} e^{-2\pi^2 \sigma^2 t^2} \tag{23}$$

The first derivative of *s(t)* pulse is given by equation (24):

$$s'(t,T) = \begin{cases} \dfrac{1}{2}\left(2 + \dfrac{t}{T}\right)^2 & -2T \le t \le T \\ -\left(\dfrac{3t^2}{2T^3} + \dfrac{2t}{T^2}\right)T & -T \le t \le 0 \\ \left(\dfrac{3t^2}{2T^3} - \dfrac{2t}{T^2}\right)T & 0 \le t \le T \\ -\dfrac{1}{2}\left(2 - \dfrac{t}{T}\right)^2 & T \le t \le 2T \end{cases} \tag{24}$$

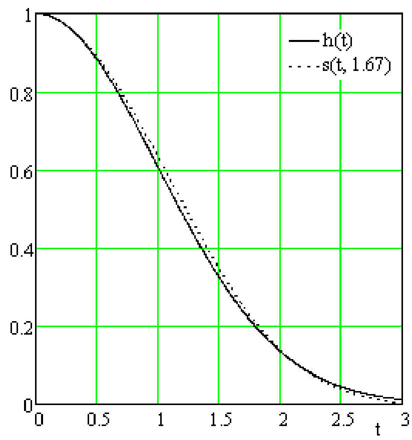

Figure 3: Normalized Gaussian pulse *h(t)* with $\sigma = 1$ and its approximation *s(t, T)* with $T = 1.67$.

Figure 4 illustrates the first derivatives of the Gaussian pulse and of its approximation $s'(t)$.

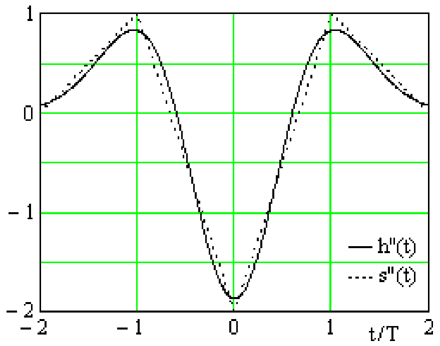

Figure 4: First order derivatives of Gaussian pulse *h(t)* for $\sigma = 0.266$ and its approximation *s'(t).*

The second derivative of Gaussian pulse *h(t)* is given by equation (25):

$$h''(t) = 4\sqrt{2}\sigma^3\pi^{5/2}\, e^{-2\pi^2\sigma^2 t^2}\left(4\pi^2\sigma^2 t^2 - 1\right) \tag{25}$$

and

$$s'(t,T) = \begin{cases} \dfrac{(2 + t/T)}{T} & -2T \le t \le T \\ -\left(\dfrac{3t}{T^3} + \dfrac{2}{T^2}\right)T & -T \le t \le 0 \\ \left(\dfrac{3t}{T^3} - \dfrac{2}{T^2}\right)T & 0 \le t \le T \\ \dfrac{(2 - t/T)}{T} & T \le t \le 2T \end{cases} \tag{26}$$

They are illustrated in Figure 5.

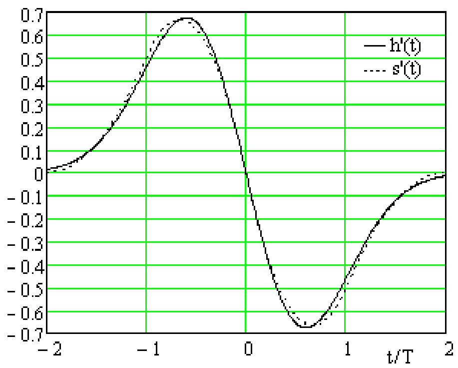

Figure 5: Second order derivatives of Gaussian pulse *h(t)* for $\sigma = 0.266$ and its approximation *s''(t).*

The squared value of the approximation error:

$$e^2(t) = (s(t) - h(t))^2 \tag{27}$$

is illustrated in Figure 6 in logarithmic representation for $T = 1$ and $\sigma = 0.266$ and proves to be quite small.

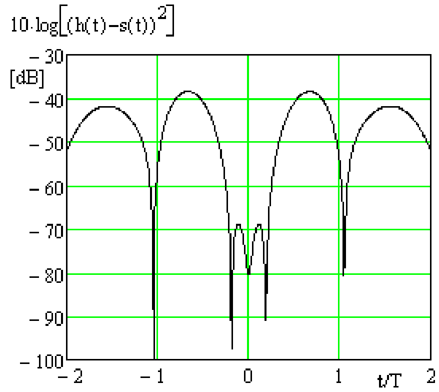

Figure 6: Mean square value of the approximation error in logarithmic representation.

Figure 7 illustrates the relative error of approximation for Dimitrov's pulse and the proposed pulse, for which *t/T* was substituted by *x*. One can see that the proposed pulse performs better than the Dimitrov's pulse.

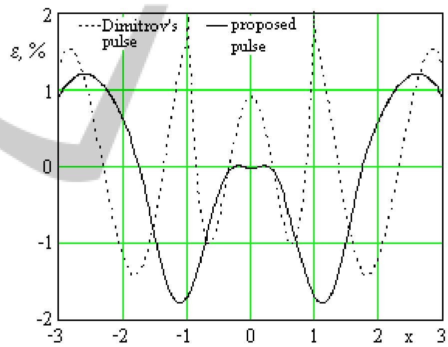

Figure 7: Relative error of approximation.

6 CONCLUSIONS

An approximation of the Gaussian pulse based on a waveform resulted from the convolution of four rectangles or equivalently of two triangles was proposed. A closed-form expression was derived for it implying polynomials of third degree in *t*.

The relative approximation error is quite small, so this makes it a good substitute for the Gaussian pulse. A better performance was obtained with respect to other proposed approximation (Dimitrov, 1991). This technique can be used for generation of Gaussian pulses in communication systems. As the triangular pulse generator is standard and widely used, the proposed technique needs besides it an appropriate filter.

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