REPRESENTATION THEOREM FOR FUZZY FUNCTIONS
Graded Form

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Abstract: In this contribution, we will extend results relating to representability of a fuzzy function using a crisp function. And additionally, we show for which functions there exist fuzzy function of a specific form. Our notion of fuzzy function has a graded character. More precisely, any fuzzy relation has a property of being a fuzzy function that is expressed by a truth degree. And it consists of two natural properties: extensionality and functionality. We will also provide a separate study of these two properties.

1 INTRODUCTION

Historically, fuzzy functions took many forms by their definition and they live contemporaneously with their applications. As an example, we put here two mostly known interpretations of this notion: 1'st – it is any mapping that assign a fuzzy set to a fuzzy set, see e.g., (Novák et al., 1999); 2'nd – it is a fuzzy relation specified by various properties (see e.g., (Demirci, 1999a; Demirci, 2001; Demirci and Recasens, 2004)), which gave rise to notions such as partial, perfect, strong fuzzy function etc.

In this work, we will turn our attention to the second class of the interpretation, i.e., we will explore fuzzy relations that meet some special requirements. Wide overview together with applications can be found in the following exemplary sources (Klawonn, 2000; Demirci, 2001; Bělohlávek, 2002). As noted in (Demirci, 2000), not all notions of the fuzzy function do coincide with the classical notions for crisp functions. In this paper, we will try to avoid this problem and all the subsequent definitions will be consistent with the classical notions whenever applied on crisp input. As a basis for our work we will take Demirci's definition of fuzzy function (Demirci, 2001) adjusted to our framework.

Our framework and methodology stems from (Běhounek and Cintula, 2006) and it can be characterized by the following items:

1. We will work inside a specific fuzzy logic.
2. Fuzzy sets (relations) will be handled as objects in formal language defined by a formula without direct interpretation using truth values (over some chosen structure).
3. Statements about the objects of the interest will be in the form of graded theorems, which means that instead of the usual statement of the problem

$$\text{If} \vdash \varphi \text{ then } \vdash \psi \quad \text{– Classical theorem} \qquad (1)$$

we search for more informative and general form (not equivalent) of this statement

$$\vdash \varphi^n \to \psi \quad \text{– Graded theorem.} \qquad (2)$$

Practically, we analyze how many times we need to incorporate an antecedent φ to prove the consequent ψ and we code the result into the degree n.

Graded theorems may become very difficult for non-experienced reader therefore, each section will be equipped by paragraphs that translate the most important formulae. Translation will be given first into the language of models for some special theories and second into the special language that mathematicians can use whenever they work over some "fuzzy logic"[1]. Those who prefer formal notation may skip these parts of the text and concentrate only on the technical aspects of the addressed problems.

The subsequently proposed reading of the graded theorem will be completely analogous to the classical case (using classical mathematical logic (CML))

[1] The notion Fuzzy logic does not represent here a wide range of applications as it is usual in engineers papers, but it denotes formal logics (of specified order and type) having syntax and semantics.

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and we will distinguish between them using a special typeface. For the classical two-valued logic we will reserve the following typeface:

Language of CML	Reading
$\varphi \Rightarrow \psi$	If φ then ψ
$\varphi \Leftrightarrow \psi$	φ iff ψ (if and only if)
$\varphi \wedge \psi$	φ and ψ
$\varphi \vee \psi$	φ or ψ

And for the chosen fuzzy logic, we set

Language of FL	Reading
$\varphi \rightarrow \psi$	IF φ THEN ψ
$\varphi \leftrightarrow \psi$	φ IFF ψ
$\varphi \wedge \psi$	φ AND ψ
$\varphi \vee \psi$	φ OR ψ
$\varphi \& \psi$	φ and ψ
$\varphi \oplus \psi$	φ or ψ

Indeed, meta-language that we are using when speaking about provable formulae over some fuzzy logic is the formal language of classical mathematical logic (obviously by the fact that a given formula is either provable or not). Hence, using the completeness of the used background logic, we can carefully reinterpret known results proved in an algebraic setting onto syntactical level and we obtain classical theorem. And further, we can try it for graded theorems.

The main difference between readings of a classical theorem and a graded theorem is as follows. Having a classical theorem, we use language of classical logic to read propositions that include provable formulae. And in the case of graded theorem, we read directly a formula using the generalized language. We can say:

"We write classically, but we think in grades."

As an example of classical theorem (1), we can assume that φ can be interpreted as "relation is extensional" and ψ as "relation is Lipschitz continuous". In this case, we can read (1) as

"If a relation is extensional then it is Lipschitz continuous."

Here, the given relation must be extensional to the degree 1 to deduce that it is Lipschitz continuous to the same degree. It can be shown that there is also graded theorem (2) with $n = 1$ for this statement that can be read in the analogous way

"IF a relation is extensional THEN it is Lipschitz continuous."

In this case, we have incorporated also additional hidden grades for both properties. Hence, a relation is Lipschitz continuous (e.g. to the degree b) as much as it is extensional (to the degree a such that $a \leq b$).

The paper will be organized in the following way:

- First we introduce our logical framework, basic notions and overview related results.
- The following two sections will be devoted to the notions of extensionality and functionality, respectively.
- And finally, Section 4 will provide a deep insight into the "theory" of fuzzy functions.

2 LOGICAL FRAMEWORK

Let us work in the framework of fuzzy class theory (FCT) (Běhounek and Cintula, 2005), which is a schematic extension of a background logic (that contain crisp equality $=$ and Baaz-delta Δ) by axiom of comprehension and extensionality axiom. Provability in FCT will be denoted simply by the same shortcut in front of $\vdash$ or it will be explicitly written. The background logic may be various (so that completeness theorem is valid for FCT) due to our actual requirements. In our case, the weakest background logic will be a many-sorted first order involutive monoidal t-norm based logic (IMTL) and we will deal only with this logic throughout the whole text.

The language J consists of the following set of basic connectives $(\&, \rightarrow, \wedge)$, involutive negation[2] $\neg$. The quantifier $\forall$, truth constants $\top, \bot$ and variables of the specific sorts.

Standardly, we introduce the following connectives and quantifier:

$$x \vee y \equiv_{df} ((x \rightarrow y) \rightarrow y) \wedge ((y \rightarrow x) \rightarrow x), \quad (3)$$

$$x \oplus y \equiv_{df} \neg(\neg x \& \neg y), \quad (4)$$

$$x \leftrightarrow y \equiv_{df} (x \rightarrow y) \& (y \rightarrow x), \quad (5)$$

$$(\exists x)\varphi \equiv_{df} \neg(\forall x)\neg\varphi. \quad (6)$$

IMTL extends MTL by the following schemata of axioms:

$$\text{(INV)} \quad \neg\neg\varphi \rightarrow \varphi.$$

Let us summarize properties of MTL and its various extensions.

Proposition 1. *IMTL proves*

$$\varphi \leftrightarrow \neg\neg\varphi, \quad (7)$$

$$(\varphi \rightarrow \psi) \leftrightarrow (\neg\psi \rightarrow \neg\varphi). \quad (8)$$

Interpretation of the connectives is given by the corresponding operations $\{*, \rightarrow_*, \wedge_Ł, \vee_Ł, \sim\}$, and

[2]Mostly, $\neg$ is reserved for residual negation and $\sim$ is usual symbol for involutive negation. In this paper, we work only with involutive negation, hence, there is no danger of confusion between these two various notations.

the constant $\perp$ is interpreted as 0, which together form an IMTL-algebra denoted by $\mathcal{L}$.

An $\mathcal{L}$-structure for the language J is of the form

$$\mathcal{M} = \langle (X_i)_{for\, s_i}, (r_P)_{P-predicate}, (m_c)_{c-constant} \rangle,$$

$i = 1, 2, \ldots, n$, where each X_i is non-empty set of objects, r_P is an L-fuzzy relation of the respective type and m_c belongs to D_i provided that c is of the type s_i.

2.1 Basic Notions

Let $F, \approx_1, \approx_2$ be predicates of the type $(s_1, s_2), (s_1, s_1), (s_2, s_2)$, respectively, i.e. their are interpreted as $\mathbf{F} \underset{\sim}{\subseteq} X_1 \times X_2, \boldsymbol{\approx_1} \underset{\sim}{\subseteq} X_1 \times X_1, \boldsymbol{\approx_2} \underset{\sim}{\subseteq} X_2 \times X_2$; $\underline{0}$ be a constant denoting an atomic individual of the domain of discourse. For the better orientation, we will use the same terminology on the syntax as well as on the semantical level. Moreover, we will omit the specification of sorts, whenever it will be clear from the concept.

Moreover, let R, S are fuzzy relations or fuzzy sets of the same type and $\bar{x}$ includes all free variables of R, S then we define the following properties:

- Reflexivity:
$$\mathrm{Refl}_R \equiv_{df} (\forall x)Rxx$$
- Symmetry:
$$\mathrm{Sym}_R \equiv_{df} (\forall x, y)(Rxy \to Ryx)$$
- Transitivity:
$$\mathrm{Trans}_R \equiv_{df} (\forall x, y, z)[(Rxy \& Ryz) \to Rxz]$$
- Similarity:
$$\mathrm{Sim}_R \equiv_{df} \mathrm{Refl}_R \,\&\, \mathrm{Sym}_R \,\&\, \mathrm{Trans}_R$$
- Subsethood:
$$R \subseteq S \equiv_{df} (\forall x, y)(R(x, y) \to S(x, y))$$
- Strong set-similarity:
$$R \cong S \equiv_{df} (\forall x, y)(R(x, y) \leftrightarrow S(x, y))$$
- Set-similarity:
$$R \approx S \equiv_{df} (R \subseteq S) \& (S \subseteq R)$$
- Totality:
$$\mathrm{Tot}_R \equiv_{df} (\forall x \exists y)Rxy$$
- Surjectivity:
$$\mathrm{Sur}_R \equiv_{df} (\forall y \exists x)Rxy$$
- Injectivity:
$$\mathrm{Inj}_{\approx_1} R \equiv_{df} (\forall x, x')[(\exists y)(Rxy \& Rx'y) \to (x \approx_1 x')]$$

Moreover, the following set operations can be introduced:

$A \cup B =_{df} \{x \mid (x \in A) \oplus (x \in B)\}$	strong union
$A \cap B =_{df} \{x \mid (x \in A) \& (x \in B)\}$	strong intersec.
$A \sqcup B =_{df} \{x \mid (x \in A) \vee (x \in B)\}$	union
$A \sqcap B =_{df} \{x \mid (x \in A) \wedge (x \in B)\}$	intersection

We will additionally deal with relational compositions defined using a class notation. A systematic study can be find in (Bělohlávek, 2002). We will use three basic relational compositions.

- sup-T composition:
$$R \circ S =_{df} \{xy \mid (\exists z)(Rxz \& Szy)\}$$
- BK-subproduct:
$$R \triangleleft S =_{df} \{xy \mid (\forall z)(Rxz \to Szy)\}$$
- BK-superproduct:
$$R \triangleright S =_{df} \{xy \mid (\forall z)(Rxz \leftarrow Szy)\}$$

3 EXTENSIONALITY AND ITS PROPERTIES

The extensionality is one of the most important properties of fuzzy relations that are used in fuzzy control. Indeed, only such relations are considered for approximation by fuzzy rules, see (Hájek, 1998; Daňková, 2007). It is defined by the following formula:

$$\mathrm{Ext}_{\approx_{1,2}} F \equiv_{df} (\forall x, x', y, y') \\ [(x \approx_1 x') \& (y \approx_2 y') \& F(x, y) \to F(x', y')].$$

Using $\approx_{1(2)}$ we capture a relationship between elements of the input (output) space. When assuming two concrete individuals, the requirement says the following: closer are the individuals $\{a, b, c, d\}$ then more equivalent are the degrees to which the proposition fires for these individuals, we can read it symbolically as

"IF $a \in U(b)_{\approx_1}$ and $c \in U(d)_{\approx_2}$ and $F(a, c)$ THEN $F(b, d)$",

where

$$U(x)_{\approx} =_{df} \{y \mid x \approx y\}$$

expresses a neighbourhood of the element x.

3.1 Extensionality of Set Operations and Relational Compositions

Let us summarize properties relating to extensionality.

Proposition 2. FCT*proves*

$$\mathrm{Ext}_{\approx_{1,2}} F \,\&\, \mathrm{Ext}_{\approx_{1,2}} E \to \mathrm{Ext}_{\approx^2_{1,2}}(F \cap E), \tag{9}$$

$$\mathrm{Ext}_{\approx_{1,2}} F \,\&\, \mathrm{Ext}_{\approx_{1,2}} E \to \mathrm{Ext}_{\approx^2_{1,2}}(F \cup E), \tag{10}$$

$$\mathrm{Ext}_{\approx_{1,2}} F \wedge \mathrm{Ext}_{\approx_{1,2}} E \to \mathrm{Ext}_{\approx_{1,2}}(F \sqcap E), \tag{11}$$

$$\mathrm{Ext}_{\approx_{1,2}}(F \sqcup E) \to \mathrm{Ext}_{\approx_{1,2}} F \vee \mathrm{Ext}_{\approx_{1,2}} E. \tag{12}$$

Readings of the above results:
(9) – "IF F and E are extensional THEN their strong intersection is extensional."
(10) – "IF F and E are extensional THEN their strong union is extensional."
(11) – "IF F AND E are extensional THEN their intersection is extensional."
(12) – "IF the union of F and E is extensional THEN F is extensional OR E is extensional."

In the following, the extensionality of superset and similar set is studied.

Proposition 3. FCT*proves*

$$(F \subseteq E)^2 \to [\mathrm{Ext}_{\approx_{1,2}} F \to \mathrm{Ext}_{\approx_{1,2}} E], \tag{13}$$

$$(F \approx E)^2 \to [\mathrm{Ext}_{\approx_{1,2}} F \leftrightarrow \mathrm{Ext}_{\approx_{1,2}} E]. \tag{14}$$

Readings of the results:
(13) – "IF F is a subset of E (we need this requirement twice) and F is extensional THEN E is extensional."
(14) – "IF F and E are similar sets (we need this requirement twice) THEN F is extensional IFF E is extensional."

The above formulae together with properties of the relational compositions produces a long list of consequences.

Corollary 1. *Let*

$$C_1 \equiv_{df} (E_1 \subseteq E_2)^2,$$
$$C_2 \equiv_{df} (F_1 \subseteq F_2)^2,$$
$$C_3 \equiv_{df} (E_1 \approx E_2)^2,$$
$$C_4 \equiv_{df} (F_1 \approx F_2)^2.$$

Then FCT*proves*

$$C_1 \to [\mathrm{Ext}_{\approx_{1,2}}(F \circ E_2) \to \mathrm{Ext}_{\approx_{1,2}}(F \circ E_1)],$$
$$C_1 \to [\mathrm{Ext}_{\approx_{1,2}}(F \lhd E_1) \to \mathrm{Ext}_{\approx_{1,2}}(F \lhd E_2)],$$
$$C_1 \to [\mathrm{Ext}_{\approx_{1,2}}(F \rhd E_2) \to \mathrm{Ext}_{\approx_{1,2}}(F \rhd E_1)],$$

$$C_2 \to [\mathrm{Ext}_{\approx_{1,2}}(F_2 \lhd E) \to \mathrm{Ext}_{\approx_{1,2}}(F_1 \lhd E)],$$
$$C_2 \to [\mathrm{Ext}_{\approx_{1,2}}(F_1 \rhd E) \to \mathrm{Ext}_{\approx_{1,2}}(F_2 \rhd E)],$$

$$C_3 \to [\mathrm{Ext}_{\approx_{1,2}}(F \circ E_2) \leftrightarrow \mathrm{Ext}_{\approx_{1,2}}(F \circ E_1)],$$
$$C_3 \to [\mathrm{Ext}_{\approx_{1,2}}(F \lhd E_2) \leftrightarrow \mathrm{Ext}_{\approx_{1,2}}(F \lhd E_1)],$$
$$C_3 \to [\mathrm{Ext}_{\approx_{1,2}}(F \rhd E_1) \leftrightarrow \mathrm{Ext}_{\approx_{1,2}}(F \rhd E_2)],$$

$$C_4 \to [\mathrm{Ext}_{\approx_{1,2}}(F_2 \lhd E) \leftrightarrow \mathrm{Ext}_{\approx_{1,2}}(F_1 \lhd E)],$$
$$C_4 \to [\mathrm{Ext}_{\approx_{1,2}}(F_1 \rhd E) \leftrightarrow \mathrm{Ext}_{\approx_{1,2}}(F_2 \rhd E)],$$

Intersection:

$$\mathrm{Ext}_{\approx_{1,2}}\Big(\Big(\bigcap_{F \in \mathcal{A}} F\Big) \circ E\Big) \to \mathrm{Ext}_{\approx_{1,2}}\Big(\bigcap_{F \in \mathcal{A}} (F \circ E)\Big),$$
$$\mathrm{Ext}_{\approx_{1,2}}\Big(\bigcup_{F \in \mathcal{A}} (F \lhd E)\Big) \to \mathrm{Ext}_{\approx_{1,2}}\Big(\Big(\bigcap_{F \in \mathcal{A}} F\Big) \lhd E\Big),$$
$$\mathrm{Ext}_{\approx_{1,2}}\Big(\bigcup_{E \in \mathcal{A}} (F \lhd E)\Big) \to \mathrm{Ext}_{\approx_{1,2}}\Big(F \lhd \bigcup_{E \in \mathcal{A}} E\Big).$$

Union:

$$\mathrm{Ext}_{\approx_{1,2}}\Big(\Big(\bigcup_{F \in \mathcal{A}} F\Big) \circ E\Big) \leftrightarrow \mathrm{Ext}_{\approx_{1,2}}\Big(\bigcup_{F \in \mathcal{A}} (F \circ E)\Big),$$
$$\mathrm{Ext}_{\approx_{1,2}}\Big(\bigcap_{F \in \mathcal{A}} (F \lhd E)\Big) \leftrightarrow \mathrm{Ext}_{\approx_{1,2}}\Big(\Big(\bigcup_{F \in \mathcal{A}} F\Big) \lhd E\Big),$$
$$\mathrm{Ext}_{\approx_{1,2}}\Big(\bigcap_{E \in \mathcal{A}} (F \lhd E)\Big) \leftrightarrow \mathrm{Ext}_{\approx_{1,2}}\Big(F \lhd \bigcap_{E \in \mathcal{A}} E\Big).$$

3.2 Duality between Extensionality and 1-Lipschitz Continuity

The duality between pseudo-metrics and similarities based on the additive generator of a t-norm has been used to prove the duality between extensionality (on the model where & is interpreted as continuous archimedean t-norm) and Lipschitz continuity in the induced pseudo-metric space (Mesiar and Novák, 1999) (later in (Perfilieva, 2004)). Below, we provide a graded form of the result from (Daňková, 2010).

By the above observations, the mutual inversion between extensionality and Lipschitz continuity obviously follows.

Theorem 1. *Let*

$$\mathrm{Lipschitz}_{d,d'} F \equiv_{df} (\forall \overline{x}, \overline{y})[d(F(\overline{x}), F(\overline{y})) \to d'(\overline{x}, \overline{y})],$$

and moreover, Define

$$d'(\overline{x}, \overline{y}) =_{df} \neg(x_1 \approx_1 y_1) \oplus \neg(x_2 \approx_2 y_2), \tag{15}$$

$$d(x, y) = \neg(x \leftrightarrow y). \tag{16}$$

Then FCT*proves*

$$\mathrm{Ext}_{\approx_{1,2}} F \leftrightarrow \mathrm{Lipschitz}_{d,d'} F.$$

Reading of the result:

"F is extensional w.r.t. $\approx_{1,2}$ IFF
F is Lipschitz continuous w.r.t. d, d'."

Whenever $\approx_{1,2}$ are not similarities, we should not speak about an analogy with Lipschitz continuity. There, we obtain only one pseudo-metric dual to equivalence, i.e. d, and a correct interpretation of $\mathrm{Lipschitz}_{d,d'} F$ can be expressed as a domination of pseudo-metric d applied on values F by relation d'.

4 FUNCTIONALITY AND ITS PROPERTIES

In this section, we will introduce a property of fuzzy relation called functionality that is a direct generalization of the related crisp notion.

Let the functionality property be given by the following formula (semantic version in (Demirci, 1999b)):

$$\mathrm{Func}_{\approx_{1,2}} F \equiv_{df} (\forall x, x', y, y')$$
$$[(x \approx_1 x') \& F(x,y) \& F(x',y') \to (y \approx_2 y')]. \quad (17)$$

It is not an obvious generalization of the classical definition of functionality axiom:

$$\text{If } F(x,y) \text{ and } F(x,y') \text{ then } y = y',$$

represented by

$$F(x,y) \& F(x,y') \to y = y',$$

whenever F is crisp. The classical functionality can be tested on fuzzy relations as well and it can be expressed within the fuzzy logic using Baaz-delta Δ as follows:

$$\Delta F(x,y) \& \Delta F(x,y') \to y = y',$$

thus for crisp relations, the form of functionality axiom does not differ from its origin.

The most natural generalization stands in removing all crisp constrains to F and the equality, which is now replaced by an equivalence relation determining which elements are indistinguishable on the universe of discourse

$$F(x,y) \& F(x,y') \to y \approx_2 y', \quad (18)$$

and it seems that we are done at that point. However, taking closer look at the above formula, we uncover the hidden crisp equality there relating to the variable x. Since we work in the setting where $\approx_1$ possesses what is distinguishable on the input space, therefore we should incorporate this fact by modifying left side of the implication in the functionality formula

$$x \approx_1 x' \& F(x,y) \& F(x',y') \to y \approx_2 y'.$$

Observe that when assuming reflexivity of $\approx_1$, we have (18) as the special instance of the above formula and so of the formula defining $\mathrm{Func}_{\approx_{1,2}} F$.

Example 1. *Duality between extensionality and Let* $\mathcal{L} = \langle [0,1], \otimes, \Rightarrow, \vee, \wedge, 0, 1 \rangle$ *be the standard Łukasiewicz algebra,*

$$x \sim y = (x \Leftrightarrow y)^3 = (x \Leftrightarrow y) \otimes (x \Leftrightarrow y) \otimes (x \Leftrightarrow y).$$

The following is the example of a relation that is functional to the degree 1 *w.r.t.* $=,=$*:* $F_1(x,y) = \{{}^{0.5}/\sin(x)\}$.

Considering similarity relation $\sim$*, we find out that*

1. $F_2(x,y) = y \sim \sin(x)$ *is functional to the degree 1 w.r.t.* $\sim, \sim$.
2. $F_3(x,y) = y \sim f(x)$*, where*
$$f(x) = \begin{cases} 1.7x^2, & x \in [0, 0.5]; \\ \cos(0.9x) - 0.5, & \textit{otherwise}. \end{cases}$$
is functional to the degree 1 w.r.t. $=, \sim$.
3. $F_4(x,y) = F_2 \vee F_3$ *is functional to the degree 0 w.r.t.* $\sim, \sim$*. Take* $x = x' = 1$ *and* $y = 0.1$, $y' = 0.8$ *then* $F(x,y) \otimes F(x',y') = 1$*, but* $y \sim y' = 0$.

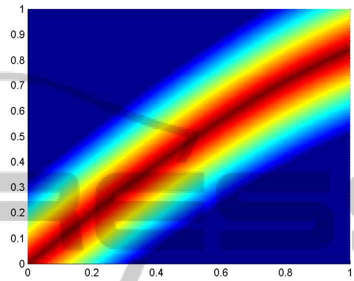

Figure 1: Functional relation F_2 w.r.t. $\sim, \sim$.

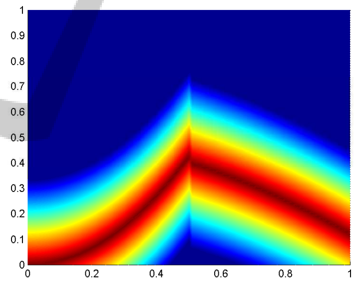

Figure 2: Functional relation F_3 w.r.t. $\sim, \sim$.

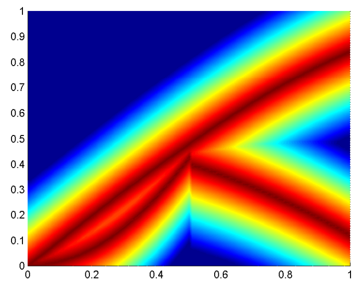

Figure 3: Non-functional relation F_4 w.r.t. $\sim, \sim$.

4.1 Functionality of Set Operations and Relational Compositions

Let us summarize properties relating to functionality.

Proposition 4. FCT*proves*

$$\mathrm{Func}_{\approx_{1,2}} F \,\&\, \mathrm{Func}_{\approx_{2,3}} S \to \mathrm{Func}_{\approx_{1,3}}(F \circ S), \tag{19}$$

$$\mathrm{Func}_{\approx_{1,2}} F \,\&\, \mathrm{Func}_{\approx_{1,2}} S \to \mathrm{Func}_{\approx^2_{1,2}}(F \cap S), \tag{20}$$

$$\mathrm{Func}_{\approx_{1,2}} F \wedge \mathrm{Func}_{\approx_{1,2}} S \to \mathrm{Func}_{\approx_{1,2}}(F \sqcap S), \tag{21}$$

$$\mathrm{Func}_{\approx_{1,2}}(F \sqcup S) \to \mathrm{Func}_{\approx_{1,2}} F \wedge \mathrm{Func}_{\approx_{1,2}} S. \tag{22}$$

We can also introduce the following properties of relations analogous to the classical notions.

Proposition 5. FCT*proves*

$$(S \subseteq F)^2 \to [\mathrm{Func}_{\approx_{1,2}} F \to \mathrm{Func}_{\approx_{1,2}} S], \tag{23}$$

$$(F \approx S)^2 \to [\mathrm{Func}_{\approx_{1,2}} F \leftrightarrow \mathrm{Func}_{\approx_{1,2}} S]. \tag{24}$$

Similarly as in the previous section, we may generate a long list of corollaries for the relational compositions.

Note that we have decided to omit the reading of the above results because it is in a direct analogy with reading of the results in Section 3.1.

5 FUZZY FUNCTIONS AND THEIR RELATION TO CRISP FUNCTIONS AND VICE-VERSA

When having crisp functions, we can always express this special relational dependency by $y = f(x)$, which follows from the functionality property throughout the compatibility property w.r.t. $=,=$ defined by (25). If we exchange $=$ by $\approx_2$, i.e. we assume that each x is mapped to a neighbourhood of $f(x)$, which can be represented using $\approx_2$ as $y \approx_2 f(x)$. This relation is the fuzzy functions w.r.t. $=,\approx_2$, provided that $\approx_2$ is reflexive.

Let us first explore how does the relation between compatibility and functionality/extensionality look like for a specially chosen relation $F_f(x,y) := y \approx_2 f(x)$.

Lemma 1. *Let us define*

$$\mathrm{Comp}_{\approx_1,\approx_2} f \equiv_{df} (\forall x,y)(x \approx_1 y) \to (f(x) \approx_2 f(y)), \tag{25}$$

$$F_f(x,y) \equiv_{df} y \approx_2 f(x), \tag{26}$$

$$C \equiv_{df} \mathrm{Sym}_{\approx_2} \& (\mathrm{Trans}_{\approx_2})^2. \tag{27}$$

Then FCT*proves*

$$\mathrm{Tot}\, f \to \mathrm{Tot} F_f, \tag{28}$$

$$C \to [\mathrm{Comp}_{\approx_1,\approx_2} f \to \mathrm{Func}_{\approx_1,\approx_2} F_f], \tag{29}$$

$$C \to [\mathrm{Comp}_{\approx_1,\approx_2} f \to \mathrm{Ext}_{\approx_1,\approx_2} F_f], \tag{30}$$

When defining fuzzy function (property Function defined by (31) assigned to a relation F), it is usually assumed the extensionality and functionality together. Extensionality says that we can substitute the original inputs (x,y) by the indistinguishable one (x',y'). The formula representing the functionality is the exact analogy with the classical definition, where we assume that the images of indistinguishable elements are indistinguishable. Relating to this interpretation, we must still keep on mind that $\approx_{1,2}$ represent the granularity of the input (output) space, which means that they are coarsest relations in our system $(\forall R)(\approx_{1(2)} \subseteq R)$ enabling us to distinguish elements of the universe.

Theorem 2. *Let us define*

$$\mathrm{Function}_{\approx_{1,2}} F \equiv_{df} \mathrm{Ext}_{\approx_{1,2}} F \wedge \mathrm{Func}_{\approx_{1,2}} F, \tag{31}$$

$$\mathrm{Definition}_{F,f} \equiv_{df} (\forall x)[F(x, f(x)) \leftrightarrow (\exists y) F(x,y)], \tag{32}$$

F_f be defined by $F_f(x,y) \equiv_{df} y \approx_2 f(x)$, where f is some unary functional symbol.

Then FCT*proves*

$$C \,\&\, \mathrm{Comp}_{\approx_1,\approx_2} f \to \mathrm{Function}_{\approx_1,\approx_2} F_f, \tag{33}$$

$$\mathrm{Tot}\, f \,\&\, \mathrm{Definition}_{F_f, f_{F_f}} \to (f_{F_f} \approx f). \tag{34}$$

Reading of the results:
(33) – "IF $\approx_2$ is symmetric and transitive (we need transitivity twice) and f is compatible THEN $y \approx_2 f(x)$ is fuzzy function."
(34) – "IF f is compatible and total and f_{F_f} is so that for an arbitrary $x : [f_{F_f}(x) \approx_2 f(x)$ IFF there exists $y : y \approx_2 f(x)]$ THEN f_{F_f} is similar to f."

Now, let us address the reverse problem: consider a fuzzy relation F and let us find a crisp function f_F such that it is compatible with $(\approx_1, \approx_2)$ and its extension to fuzzy relation F_{f_F} is similar to F.

Theorem 3. *Let $D \equiv_{df}$* $\mathrm{Tot} F \,\&\, \mathrm{Definition}_{F, f_F}$.
Then FCT*proves*

$$D \& (\mathrm{Tot} F) \& \mathrm{Func}_{\approx_{1,2}} F \to \mathrm{Comp}_{f_F}, \tag{35}$$

$$D \,\&\, \mathrm{Function}_{\approx_{1,2}} F \,\&\, \mathrm{Refl}_{\approx_1} \to (F_{f_F} \approx F). \tag{36}$$

Reading of the results:
(35) – "IF F is functional and total (we need totality twice) and f_F is so that for an arbitrary $x : [F(x, f_F(x)$ IFF there exists $y : F(x,y)]$ THEN f_F is compatible function."
(36) – "IF F is a total fuzzy function and f_F is as above and $\approx_1$ is reflexive THEN F_{f_F} is similar to F."

Let us recall the original representation theorem (translated into our framework) to provide a comparison with our graded variant. For the semantical version see e.g. (Bělohlávek, 2002).

Theorem 4. *Let*

$$\begin{aligned}
&\mathrm{FEqual}_R \equiv_{df} \mathrm{Sim}_R \,\&\, (\forall x,y)[\Delta R(x,y) \to (x=y)],\\
&T_1 = \{\mathrm{Sim}_{\approx_1}, \mathrm{FEqual}_{\approx_2}, \mathrm{Comp}_{\approx_1,\approx_2} f\}\\
&T_2 = \{\mathrm{Sim}_{\approx_1}, \mathrm{FEqual}_{\approx_2}, \mathrm{Tot}F, \mathrm{Function}_{\approx_{1,2}} F,\\
&\qquad \mathrm{Definition}_{F,f_F}\},\\
&T_3 = \{\mathrm{Definition}_{F_f, f_{F_f}}\} \cup T_1.
\end{aligned}$$

Then

$$\mathrm{FCT} \cup T_1 \vdash \mathrm{Function}_{\approx_{1,2}} F_f, \tag{37}$$

$$\mathrm{FCT} \cup T_2 \vdash \mathrm{Comp}_{\approx_{1,2}} f_F. \tag{38}$$

Moreover,

$$\mathrm{FCT} \cup T_3 \vdash f_{F_f} = f, \tag{39}$$

$$\mathrm{FCT} \cup T_2 \vdash F_{f_F} = F. \tag{40}$$

Reading of the above formulae:
(37) – "If $\approx_1$ is a similarity and $\approx_2$ is a fuzzy equality and f is compatible w.r.t. $\approx_{1,2}$ then F_f is the fuzzy function."
(39) – "If we additionally assume that $F_f(x, f_{F_f}(x)) = \bigvee_{y\in Y} F_f(x,y)$, for an arbitrary x, then f_{F_f} is equal to f."
(38)+(40) – "If $\approx_1$ is a similarity and $\approx_2$ is a fuzzy equality and F is a total fuzzy function then f_F satisfying $F(x, f_F(x)) = \bigvee_{y\in Y} F(x,y)$ is compatible function and moreover, F_{f_F} and F are identical."

We see that the readings do not differ significantly. The main difference is in gradualness of the new results, which reflects also in similarity of f_{F_f} and F_{f_F} with their pre-images, respectively.

An intended class of applications of the introduced theory of fuzzy functions is connected with fuzzy rules, especially, the implicative fuzzy rules. Originally, implicative fuzzy rules where introduced as an approximation of fuzzy relations that are functional (to the degree 1). In this case, we obtain an upper approximation of the original relation and moreover, a precision of such approximation can be estimated. But if we have at the disposal only partially functional relation (to the degree $0 < \alpha < 1$) then we cannot expect that implicative rules will provide an upper approximation. The graded approach gives at least an estimation of a residua between the implicative rules and the original relation. The following example illustrates an approximation of partially functional fuzzy relations using implicative rules.

Example 2. *Let us assume the standard Łukasiewicz algebra $\mathcal{L}_Ł$ as an interpretation for the connectives, i.e., $\mathcal{L}_Ł = \langle [0,1], \otimes, \to_\otimes, \wedge, \vee, \sim\rangle$, where*

$$\begin{aligned}
x \otimes y &= \max(0, x+y-1),\\
x \to_\otimes y &= \max(1, 1-x+y),\\
x \wedge y &= \min(x,y),\\
x \vee y &= \max(x,y),\\
\sim x &= 1-x.
\end{aligned}$$

In this case, we have $x \leftrightarrow_\otimes y = \max(0, 1-|x-y|)$ and

$$\begin{aligned}
x \leftrightarrow_\otimes^k y &= \underbrace{(x \leftrightarrow_\otimes y)\ldots(x \leftrightarrow_\otimes y)}_{k-times}\\
&= \max(0, 1-k|x-y|).
\end{aligned}$$

In (Mesiar and Novák, 1999), it has been shown that Lipschitz continuity of a function $f : [0,1] \mapsto [0,1]$ w.r.t. the standard metric and with the Lipschitz constant k is equivalent with compatibility of f w.r.t. $\leftrightarrow_\otimes^k, \leftrightarrow_\otimes$ over Łukasiewicz algebra, i.e. the interpretation of $\mathrm{Comp}_{\leftrightarrow^k,\leftrightarrow} f$ over $\mathcal{L}_Ł$ is equal to 1, we write symbolically $||\mathrm{Comp}_{\leftrightarrow^k,\leftrightarrow} f|| = 1$. Formula (29) says that $||\mathrm{Comp}_{\leftrightarrow^k,\leftrightarrow} f|| \le ||\mathrm{Func}_{\leftrightarrow^k,\leftrightarrow} F_f||$. Moreover, it can be proved that if $||\mathrm{Func}_{\leftrightarrow^k,\leftrightarrow} F_f|| = 1$ then $\forall x,y \in [0,1]$:

$$F_f(x,y) \le Rules_f(x,y), \text{ where} \tag{41}$$

$$Rules_f(x,y) = \bigwedge_{c\in M} [(x \leftrightarrow_\otimes^k c) \to_\otimes (y \leftrightarrow_\otimes f(c))], \tag{42}$$

and $M \subset [0,1]$. And even more

$$\mathrm{FCT} \vdash \mathrm{Func}_{\leftrightarrow^k,\leftrightarrow} F_f \to (F_f \subseteq Rules_f), \tag{43}$$

which gives the following estimation:

$$||\mathrm{Func}_{\leftrightarrow^k,\leftrightarrow} F_f|| \le ||F_f \subseteq Rules_f||. \tag{44}$$

Let us fix $k = 1$. Then we obtain the following table of degrees of compatibility for the specific functions:

f	$\|\|\mathrm{Comp}_{\leftrightarrow,\leftrightarrow} f\|\|$
$\sin(x)$	1
$\sin(2x)$	$0.6576 = e_2$
$\sin(3x)$	$0.4675 = e_3$

Figures 4–6 depict the sample relations F_f, where f is from the above table. Additionally, Figures 7–9 visualize implicative rules with 11 equidistantly distributed nodes $M = \{0, 0.1, \ldots, 0.9, 1\}$.

From (44) and (29), we conclude that

$$\begin{aligned}
&F_{\sin(x)}(x,y) \le Rules_{\sin(x)}(x,y),\\
&e_2 \le \min_{(x,y)\in[0,1]^2} F_{\sin(2x)}(x,y) \to_\otimes Rules_{\sin(2x)}(x,y),\\
&e_3 \le \min_{(x,y)\in[0,1]^2} F_{\sin(3x)}(x,y) \to_\otimes Rules_{\sin(3x)}(x,y).
\end{aligned}$$

Figures 10 and 11 show residua of $F_{\sin(ix)}$ from $Rules_{\sin(ix)}$, where the visualized z-coordinate is set to $[e_i, 1]$ for $i = 2,3$.

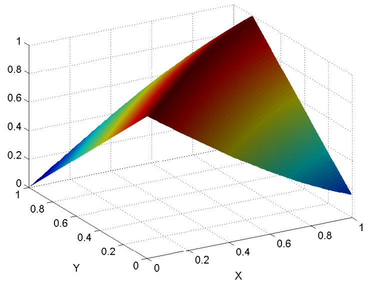

Figure 4: $F_{\sin(x)}$.

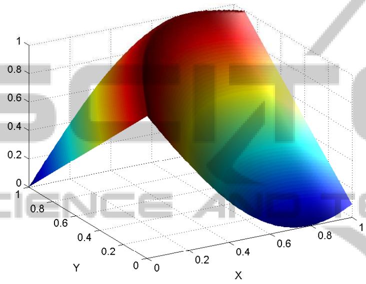

Figure 5: $F_{\sin(2x)}$.

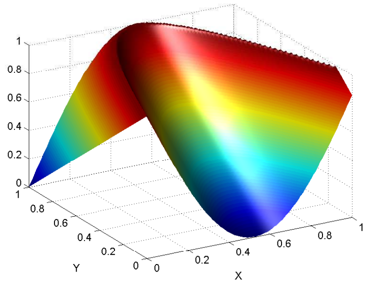

Figure 6: $F_{\sin(3x)}$.

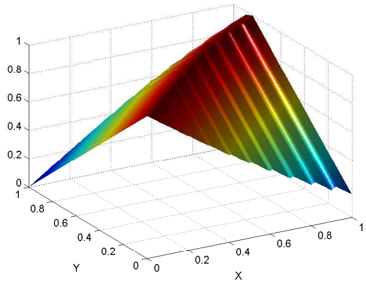

Figure 7: $Rules_{\sin(x)}$.

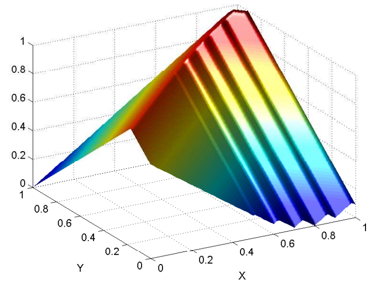

Figure 8: $Rules_{\sin(2x)}$.

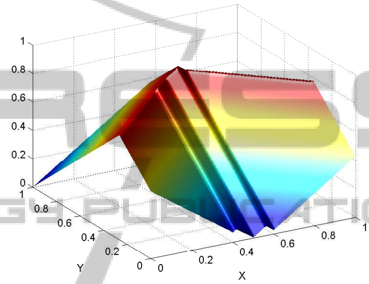

Figure 9: $Rules_{\sin(3x)}$.

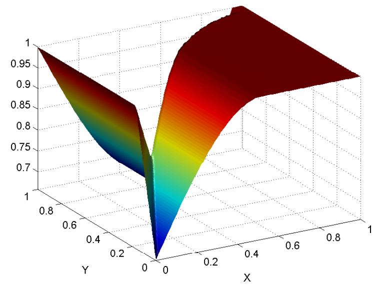

Figure 10: $F_{\sin(2x)} \rightarrow_{\otimes} Rules_{\sin(2x)}$.

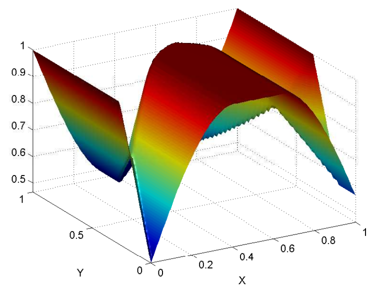

Figure 11: $F_{\sin(3x)} \rightarrow_{\otimes} Rules_{\sin(3x)}$.

6 CONCLUSIONS

In this contribution, we have generalized the well known representation theorem for fuzzy function into the graded form. The main advantage of our approach is that it incorporates the whole scale for degrees of truth, while in the original approach the results were applied to properties valid in the degree 1. Moreover, evaluation of the degrees of the antecedents in our graded theorems provides an additional information about a "precision" of the consequent. E.g., if $\approx$ is similarity relation then an evaluation of D in (34) estimates closeness of $f_{F_f} \approx f$, or in other words, a distance (dual to $\approx$) between f_{F_f} and f.

Hence, we have shown that graded theorems bring a new light into the already well established theory of fuzzy functions. And additionally, the logical framework provides an unified approach to mathematics of fuzzy logic that corresponds with the classical one (also in the notational standards).

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