COMPARISON AMONG ITERATIVE ALGORITHMS FOR PHASE RETRIEVAL

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Abstract: Phase retrieval problem is a problem of reconstructing a signal or the phase of Fourier transform of the signal from the magnitude of its Fourier transform. In this paper we address the problem of reconstructing an unknown signal from the magnitude of its Fourier transform and the magnitude of Fourier transform of another signal that is given by the addition of a known reference signal. In addition to a brief summary of the uniqueness conditions under which a signal can be uniquely specified from the given information, we present a simple justification that an iterative algorithm converges to the unknown original signal. And we compare three of the iterative algorithms developed so far.

1 INTRODUCTION

The phase retrieval problem is the problem of reconstruction of a signal from the magnitude of its Fourier transform (or Fourier intensity). This problem arises in a variety of different applications including X-ray crystallography, electron microscopy, astronomy, optics, and signal processing (Hayes 1980, Ramachandran 1970, Hayes 1982). This problem, however, is not easy to solve because this problem does not have a unique solution in general. For example, suppose we have the magnitude of the Fourier transform of a signal. If the signal is one-dimensional, we can make an infinite number of different sequences which have the same magnitude by convolving many different all-pass sequences. Even though we restrict the signal to be finite, we can find many other signals which have the same Fourier transform magnitude by the process called `zero flipping' (Hayes 1980). If the signal is two-dimensional, even though we know that almost all two-dimensional sequences have irreducible z-transforms and can be uniquely defined to within a trivial ambiguity, we cannot present a practical and efficient algorithm to perform the reconstruction.

To overcome the difficulties associated with the reconstruction of a signal from its Fourier intensity, a group of researchers have proposed many different methods by adding additional information (Kim 1990a, Kim 2004, Fiddy 1983) or using window functions (Nakajima 1987, Kim 1993, Kim 2008), or using partial information such as one bit of phase information (Van Hove 1983).

Among these, in (Kim 1990a), the authors considered the phase retrieval using a known additive signal. The authors presented several conditions under which a signal is determined uniquely from the two magnitudes of Fourier transforms: one is the magnitude of the Fourier transform of an unknown, desired signal and the other is the magnitude of the Fourier transform of another signal that is given by the addition of the desired signal and a known ′reference’ signal. Also, the authors presented closed-form algorithms that can determine the unknown, desired signal from the given information. These closed-form algorithms, however, may be very sensitive to noise especially to computational noise, which may cause the propagation of errors because they are derived from the autocorrelations so that they are composed of recursive equations. As results, the noise levels are different at different locations of the solution. On the other hand, iterative algorithms have two important advantages over recursive algorithms. One is that, at each stage of the iterative algorithms, we can put various constraints such as the positivity, the finite region of support, or the magnitude constraints. The other is that with the control over the number of iterations, the iterative algorithms may be terminated at any time before the effects of noise becomes serious.

In (Kim 1990b), the author had presented an iterative algorithm and shown a justification that the

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algorithm converges in the sense that the defined error criterion converges to zero. Also, the paper had presented an example that shows that the solution signal actually converges to the unknown desired signal. However, the paper had not given a proof that the updated signal converges to the exact solution signal.

In this paper we consider the same phase retrieval problem that had been considered in (Kim 1990a). After we mention some of the uniqueness conditions given in (Kim 1990a), we present a corollary that may be used during the development of the algorithm and develop an iterative algorithm. We introduce the iterative algorithm developed in (Kim 1990b) and present a simple justification at the algorithm actually converges to the unknown desired signal. Finally we present performance analysis among the 3 iterative algorithms that can be applied to the phase retrieval problem, i.e., Fienup algorithm (Gerchberg-Saxton algorithm), the algorithm developed in (Kim 1990b), and an adaptive relaxation algorithm.

2 UNIQUENESS

In this section, we present some of the uniqueness conditions in (Kim 1990a). Since the properties of the one-dimensional signals are very different from those of two- or higher dimensional signals, we first mention the one-dimensional signals (Kim 1990a).

2.1 One-dimensional Case

To begin with, we assume that $x(n)$ be an unknown desired signal that is a real and discrete-time sequence of length N. To be more specific, we assume that $x(n)$ has its region of support $R[0, N-1]$, $x(0) \neq 0$, and $x(N-1) \neq 0$. Let $y(n)$ be another sequence which is derived from $x(n)$ by the addition of a known reference sequence $h(n)$, i.e.,

$$y(n) = x(n) + h(n) \tag{1}$$

Now, we assume that the given information is the two magnitudes of Fourier transforms, $|X(e^{j\omega})|$, $|Y(e^{j\omega})|$, and the known signal $h(n)$.

According to Theorem 3 in (Kim 1990a), the unknown desired signal $x(n)$ can be uniquely defined from the given information if the z-transform of the nonlinear phase part of $h(n)$ does not divide $X(z)$ the z-transform of $x(n)$. In mathematical terms, let $H(z)$ be factored as

$$H(z) = A(z)H_{lp}(z), \tag{2}$$

where $H_{lp}(z) = \pm z^{-2n_0} H_{lp}(z)$ is the z-transform of a finite length linear phase signal $h_{lp}(n)$.

This uniqueness condition can be put this way. Let $\hat{x}(n)$ be another signal such that its satisfies all the conditions given to $x(n)$. Then the phase of Fourier transform of $\hat{x}(n)$ is related with the phase of the Fourier transform of $x(n)$ by either $\phi_{\hat{X}}(\omega) = \phi_X(\omega)$ or $\phi_{\hat{X}}(\omega) = 2\phi_H(\omega) - \phi_X(\omega)$. If $x(n)$ is uniquely defined from the given condition such as in Theorem 3 in (Kim 1990a), then $\phi_{\hat{X}}(\omega)$ should satisfy $\phi_{\hat{X}}(\omega) \neq 2\phi_H(\omega) - \phi_X(\omega)$. This can be summarized into the next corollary.

Corollary 1

Let $x(n)$ and $y(n)$ be real, non-symmetric, finite-length sequences such that they satisfy all the conditions given in Theorem 3 in (Kim 1990a) such that $x(n)$ can be determined uniquely from the given conditions. Let $\hat{x}(n)$ and $\hat{y}(n)$ be other signals that satisfy all the conditions given to $x(n)$ and $y(n)$ as in Theorem 3 in (Kim 1990a), respectively. If $x(n)$ is uniquely defined from the given conditions, then

$$\phi_{\hat{X}}(\omega) \neq 2\phi_H(\omega) - \phi_X(\omega), \tag{3}$$

where $\phi_{\hat{X}}(\omega)$, $\phi_X(\omega)$, and $\phi_H(\omega)$ are the phases of the Fourier transforms of $\hat{x}(n)$, $x(n)$, and $h(n)$, respectively.

The proof can be done easily. Let $\hat{x}(n)$ and $\hat{y}(n)$ be other signals that satisfy all the conditions given in the corollary 1. Then, we have

$$|Y(e^{j\omega})|^2 = |X(e^{j\omega})|^2 + |H(e^{j\omega})|^2 + 2|X(e^{j\omega})||H(e^{j\omega})|\cos(\phi_X(\omega) + \phi_H(\omega))$$

$$|\hat{Y}(e^{j\omega})|^2 = |\hat{X}(e^{j\omega})|^2 + |H(e^{j\omega})|^2 + 2|\hat{X}(e^{j\omega})||H(e^{j\omega})|\cos(\phi_{\hat{X}}(\omega) + \phi_H(\omega))$$

Equating the two equations, we get

$$\cos(\phi_X(\omega) + \phi_H(\omega)) = \cos(\phi_{\hat{X}}(\omega) + \phi_H(\omega))$$

If we solve this equation, then we get either

$$\phi_{\hat{X}}(\omega) = \phi_X(\omega)$$

or

$$\phi_{\hat{X}}(\omega) = \phi_X(\omega) - 2\phi_H(\omega).$$

Since the first equation means $\hat{x}(n) = x(n)$, the satisfaction of the second equation means there exists more than one signal that satisfies all the conditions given, which violates the uniqueness condition. Thus $\phi_{\hat{X}}(\omega)$ should be $\phi_{\hat{X}}(\omega) \neq \phi_X(\omega) - 2\phi_H(\omega)$.

As a simple example, suppose that the given additive reference signal $h(n)$ is a point sequence such that

$$h(n) = A\delta(n - n_0) \tag{4}$$

According to Theorem 2 in (Kim 1990a), there exist only two sequences that satisfy the given two magnitudes conditions. There are $x(n)$ and $x(2n_0 - n)$ (Kim 1990a). Furthermore, if we specify the region of support of the sequences, then we can remove this ambiguity. Since without loss of generality we can assume that $x(n)$ is a finite length sequence such that it has its region of support $R[0, N-1]$ with $x(0) \neq 0$ and $x(N-1) \neq 0$. Then the two signals $x(n)$ and $x(2n_0 - n)$ will have the same region of support only when N is odd and $n_0 = (N-1)/2$.

2.2 Two- or Higher-dimensional Case

For the two- or higher-dimensional signals, the uniqueness conditions are very similar to those of the one-dimensional signals except that the properties of two- or higher-dimensional signals are very much different from those of one-dimensional signals. Unlike the z-transforms of one-dimensional signals, the z-transforms of almost all the multi-dimensional signals are irreducible such that almost all of the z-transforms of the two- or higher dimensional signals are composed of only one factor (Hayes 1982b). This means that unless the additive reference signal $h(\vec{n})$ is a point signal, in almost all cases the uniqueness condition can be guaranteed.

3 ITERATIVE ALGORITHM AND ITS CONVERGENCE

Having established conditions under which a signal is uniquely defined in terms of the two magnitudes of Fourier transforms, we consider an iterative algorithm that may reconstruct the desired unknown signal from the given information. The main frame of the algorithm is given in (Kim 1990b). In (Kim 1990b), the author had shown that the algorithm converges in the sense that the defined error criterion $E_k(\omega)$ converges to 0. In this section, we first introduce the algorithm briefly and present a simple justification that the algorithm converges to the desired solution signal.

To develop the iterative algorithm, note from (1) that the magnitude of the Fourier transform $y(n)$ is. related to Fourier transforms of $x(n)$ and $h(n)$ as follows

$$\begin{aligned} |Y(e^{j\omega})|^2 &= |X(e^{j\omega})|^2 + |H(e^{j\omega})|^2 \\ &+ X^*(e^{j\omega})H(e^{j\omega}) + X(e^{j\omega})H^*(e^{j\omega}) \end{aligned} \tag{5}$$

Now we define

$$K(e^{j\omega}) = |Y(e^{j\omega})|^2 - |X(e^{j\omega})|^2 - |H(e^{j\omega})|^2 \tag{6}$$

Then from Eq. (5), we have

$$K(e^{j\omega}) = X^*(e^{j\omega})H(e^{j\omega}) + X(e^{j\omega})H^*(e^{j\omega}) \tag{7}$$

where $K(e^{j\omega})$ can be determined uniquely from the given information, i.e., two Fourier intensities and the given additive reference signal. Using the method of successive approximations, we may then establish the following update equation for the iterative algorithm for finding a solution $X(e^{j\omega})$.

$$\begin{aligned} X_{k+1}(e^{j\omega}) &= X_k(e^{j\omega}) + \beta(\omega)[K(e^{j\omega}) \\ &- X_k^*(e^{j\omega})H(e^{j\omega}) - X_k(e^{j\omega})H^*(e^{j\omega})] \end{aligned} \tag{8}$$

Here, $\beta(\omega)$ is a function that is used to control the convergence of the algorithm. To see how this algorithm works, define the error $E_{k+1}(e^{j\omega})$ at the $(k+1)$ st stage of the iteration as follows

$$\begin{aligned} E_{k+1}(e^{j\omega}) &= K(e^{j\omega}) - X_{k+1}^*(e^{j\omega})H(e^{j\omega}) \\ &- X_{k+1}(e^{j\omega})H^*(e^{j\omega}) \end{aligned} \tag{9}$$

Using (5) in (6) it may be shown that the error at the $(k+1)$ st stage is related to the error at the k th stage by

$$\begin{aligned} E_{k+1}(e^{j\omega}) &= K(e^{j\omega}) - X_{k+1}^*(e^{j\omega})H(e^{j\omega}) - X_{k+1}(e^{j\omega})H^*(e^{j\omega}) \\ &= [1 - 2\cdot \mathrm{Re}\{\beta(\omega)H^*(e^{j\omega})\}]E_k(e^{j\omega}) \end{aligned} \tag{10}$$

Now, if we let $\beta(\omega)$ be a real-valued function of ω, it follows that $E_k(e^{j\omega})$ will converge to 0 provided

$$0 < \mathrm{Re}\{\beta(\omega)H^*(e^{j\omega})\} < 1 \tag{11}$$

Therefore, if we set

$$\beta(\omega) = \mathrm{sgn}[\mathrm{Re}\{H(e^{j\omega})\}]\cdot \beta_0 \tag{12}$$

where β_0 is a positive constant such that

$$0 < \beta_0 < 1/(\max\{|\mathrm{Re}\{H(e^{j\omega})\}|\} \tag{13}$$

then, for each ω, $E_{k+1}(\omega) < E_k(\omega)$ will converge to 0 as $k \to \infty$ and $X_k^*(e^{j\omega})H(e^{j\omega}) + X_k(e^{j\omega})H(e^{j\omega})$ converges to $K(e^{j\omega})$.

Now, we are going to show that $E_k(\omega)$ converges to 0 means that $X_k(e^{j\omega})$ converges to $X(e^{j\omega})$. From Eq. (8), this equation can be rewritten as

$$\begin{aligned} [X_{k+1}(e^{j\omega}) - X_k(e^{j\omega})] &= \beta(\omega)[K(e^{j\omega}) \\ &- X_k^*(e^{j\omega})H(e^{j\omega}) - X_k(e^{j\omega})H^*(e^{j\omega})] \\ &= \beta(\omega)E_k(\omega) \end{aligned} \tag{14}$$

From (14), we get

$$\frac{X_{k21}(e^{j\omega})-X_{k+1}(e^{j\omega})}{X_{k+1}(e^{j\omega})-X_{k}(e^{j\omega})}=\frac{E_{k+1}(\omega)}{E_{k}(\omega)}<1 \qquad (15)$$

Thus, as k goes to infinity, the update equation (8) converges to some signal, say $X_{\infty}(e^{j\omega})$.

Now, we are going to see the converged signal $X_{\infty}(e^{j\omega})$ converges to the desired signal $X(e^{j\omega})$. If we take a limit $k \to \infty$ to (8), we get

$$\begin{aligned} X_{\infty}(e^{j\omega}) &= X_{\infty}(e^{j\omega})+\beta(\omega)[K(e^{j\omega}) \\ &\quad - X_{\infty}^{*}(e^{j\omega})H(e^{j\omega})-X_{\infty}(e^{j\omega})H^{*}(e^{j\omega})] \end{aligned} \qquad (16)$$

or, combining (7), we get

$$\begin{aligned} &|X(e^{j\omega})||H(e^{j\omega})|\cos(\phi_X(\omega)-\phi_h(\omega)) \\ &=|X_{\infty}(e^{j\omega})||H(e^{j\omega})|\cos(\phi_{X_\infty}(\omega)-\phi_h(\omega)) \end{aligned} \qquad (17)$$

where $X_{\infty}(e^{j\omega})=|X_{\infty}(e^{j\omega})|\exp(j\phi_{X_\infty}(\omega))$.

If we solve the equation above, we get either

$$\phi_{X_\infty}(\omega)\to\phi_X(\omega) \qquad (18)$$

or

$$\phi_{X_\infty}(\omega)\to 2\phi_H(\omega)-\phi_X(\omega) \qquad . (19)$$

According to Corollary 1, $x(n)$ can be uniquely determined from the given condition, if $\phi_{X_\infty}(\omega)\neq 2\phi_h(\omega)-\phi_X(\omega)$. This means that $\phi_{X_\infty}(\omega)\to\phi_X(\omega)$ and the update equation $X_k(e^{j\omega})$ converges to $X(e^{j\omega})$, which means $\hat{x}(n)\to x(n)$, in turn.

The block diagram of the developed iterative algorithm is shown in Figure 1.

4 RECONSTRUCTION

In this section, we consider the performance comparison of 3 iterative algorithms. One is the described in Section 3. The second is Gerchberg-Saxton algorithm, also known as Fienup algorithm, which is the most basic and fundamental one to the algorithms in phase retrieval problem area (Gerchberg 1972). The block diagram of the GS algorithm is given in Figure 2. This algorithm has a very simple structure. Basically while we take Fourier transform and inverse Fourier transform the solution signal back and forth, we put various constraints to the solution signal. For example, if the solution is in time-domain, then the constraints of finite-support, non-negativity, real signal are used. If the solution signal is in Fourier domain, then we use magnitude or phase constraints. One of the characteristics of the Gerchberg-Saxton algorithm is that if the problem satisfies the uniqueness as in the magnitude retrieval problems, then this algorithm has a tendency to find the exact solution (Hayes 1980). If not, as in the general phase retrieval problem, this algorithm usually does not converge to the solution.

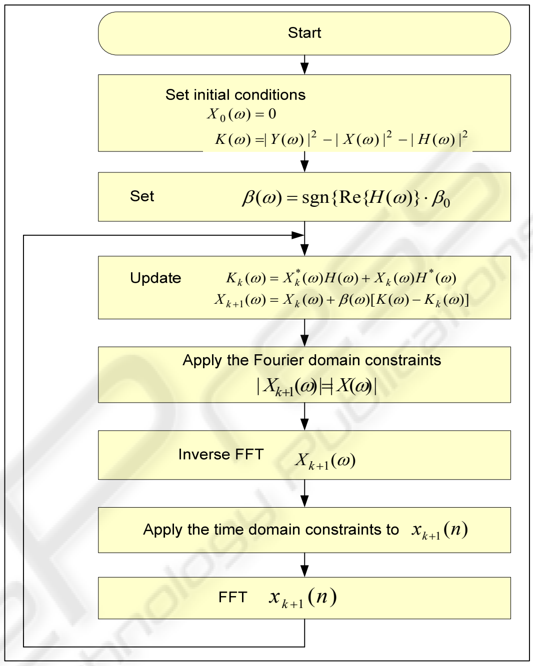


Figure 1: The block diagram of the iterative algorithm.

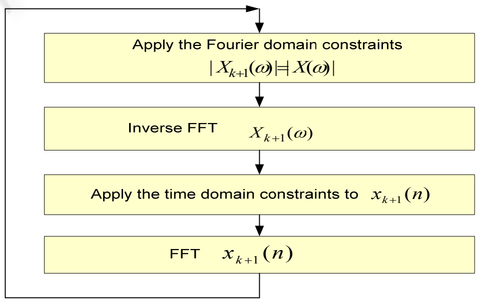


Figure 2: The block diagram of Gerchberg-Saxton algorithm.

The final algorithm is the combination of the first iterative algorithm and adaptive relaxation algorithm. The adaptive relaxation algorithm is an algorithm whose update equation is given as

$$x_{k+1}(n)=(1-\lambda_k)x_k(n)+\lambda_k T\{x_k(n)\} \qquad (20)$$

where $T\{x_k(n)\}$ is an constraint operator such as time-domain or frequency-domain constraint operator as is given in GS algorithm or the iterative algorithm in Figure 1 and λ_k is the relaxation parameter (Hayes 1982a). This parameter can be determined to minimize the error energy such that

$$\lambda_k = -\frac{\sum_F [x_k(T\{x_k\} - x_k)]}{\sum_F [T\{x_k\} - x_k]^2} \qquad (21)$$

where F implies the region that the constraints are applied to (Hayes 1982a). The block diagram of the algorithm is given in Figure 3.

As a simulation, we present a result that shows the performance of the algorithm. Fig. 4 (a) is the picture of a two dimensional original signal having 128 x 128 pixels which is assumed to be unknown and the magnitude of its Fourier transform is given. The additive reference signal is shown (b). The reference signal $h(m,n)$ is assumed to be given as

$$h(m,n) = \begin{cases} 255 & (m,n) \in [50,51] \times [55,56] \\ 0 & otherwise \end{cases}$$

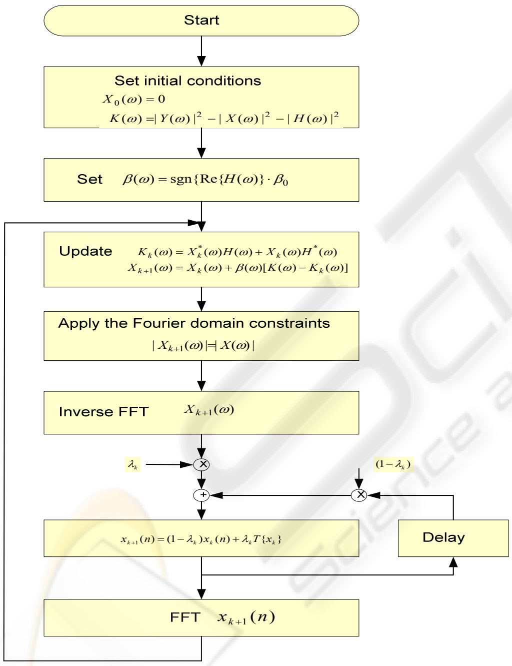


Figure 3: The block diagram of the combination of the iterative algorithm in Figure 2 and adaptive relaxation.

which is a 2x2 square box but may look like a blurred point signal.

Then the another magnitude of the signal

$$y(m,n) = x(m,n) + h(m,n)$$

is assumed to be given. Since the maximum value of $H(e^{j\omega_1}, e^{j\omega_2})$ is 255*2*2 = 1020, the convergence constant β_0 should be greater than zero and less than 1/(255*2*2)= 9.89039×10^{-04} and in this case we picked $\beta_0 = 9.89039 \times 10^{-04}$. Figure (c), (d), and (e) show the images that are reconstructed by Gerchberg-Saxton algorithm , Iterative algorithm, and Adaptive relaxation algorithm, respectively, after 20 iterations. In these pictures, we can see that while the result from Gerchberg-Saxton algorithm does not converge to the desired signal, the signals reconstructed by the other two algorithms looks similar to the desired signal and thus converge to the desired signal.

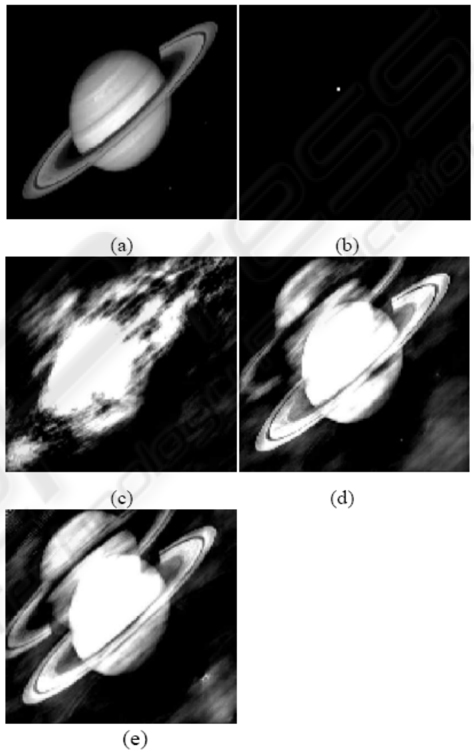


Figure 4: The original signal and the reconstructed signals after 20 iterations; (a). the original signal, (b) the additive reference signal, (c) the reconstructed signal using the GS algorithm, (d) using the iterative algorithm, and (e) the algorithm with adaptive relaxation.

Figure 5 shows the comparison between the performances of the 3 algorithms. The mean squared error here is defined as

$$MSE = \frac{\sum [x_o(m,n) - x_r(m,n)]^2}{M \times N}$$

where $M \times N$ is the number of pixels in $x(m,n)$. As we can see in this picture, the Gerchberg-Saxton algorithm does neither diverge nor converge. On the `other hand, the iterative algorithm converges as iteration goes on constantly. On the other hand, in the beginning part of the simulation, the algorithm with adaptive relaxation converges quickly. However, the algorithm saturated and does not converge any more. Obviously, the optimal algorithm is in the beginning part of the iteration, the algorithm needs to follow the adaptive relaxation property and later to follow the iterative algorithm.

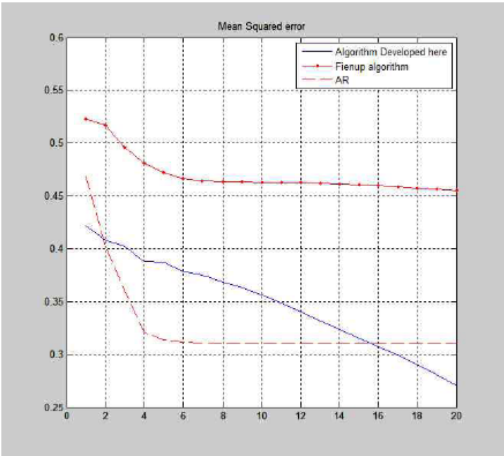
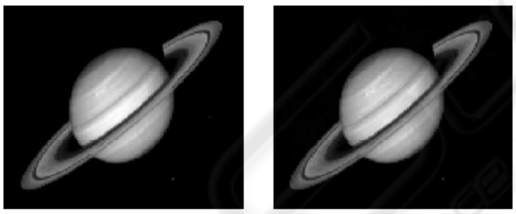


Figure 5: Comparison of the convergence properties between the GS algorithm, the iterative algorithm, and algorithm with adaptive relaxation.

Finally, in Figure 6, we presented the reconstructed algorithm after 100 iterations using the iterative algorithm. As we had given the justification, the algorithm converges and the reconstructed signal actually converges to the desired signal.



(a) (b)

Figure 6: An example that shows the convergence property of the iterative algorithm. (a) The original image and (b) the reconstructed image after 100 iterations using the iterative algorithm.

5 CONCLUSIONS

In this paper we considered the problem of iteratively reconstructing a one-dimensional or a two-dimensional signal from a pair of Fourier intensities: the intensity of the signal along with the intensity of another signal that is related by the addition of a known reference signal. After we present the uniqueness of the solution briefly, we presented a simple proof that the iterative algorithm converges the desired original signal, which is assumed to be unknown. The algorithm combined with the iterative algorithm and the adaptive relaxation algorithm converges fast in the beginning part and however goes saturated fastly also. Future work may be the evaluating the robustness of the algorithms to noise in the measured intensities and methods of improving the convergence properties of the constrained iterative algorithm.

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