GRAPH CUTS AND APPROXIMATION OF THE EUCLIDEAN METRIC ON ANISOTROPIC GRIDS

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Abstract: Graph cuts can be used to find globally minimal contours and surfaces in 2D and 3D space, respectively. To achieve this, weights of the edges in the graph are set so that the capacity of the cut approximates the contour length or surface area under chosen metric. Formulas giving good approximation in the case of the Euclidean metric are known, however, they assume isotropic resolution of the underlying grid of pixels or voxels. Anisotropy has to be simulated using more general Riemannian metrics. In this paper we show how to circumvent this and obtain a good approximation of the Euclidean metric on anisotropic grids directly by exploiting the well-known Cauchy-Crofton formulas and Voronoi diagrams theory. Furthermore, we show that our approach yields much smaller metrication errors and most interestingly, it is in particular situations better even in the isotropic case due to its invariance to mirroring. Finally, we demonstrate an application of the derived formulas to biomedical image segmentation.

1 INTRODUCTION

Graph cuts were originally developed as an elegant tool for interactive image segmentation (Boykov and Funka-Lea, 2006) with applicability to N-D problems and allowing integration of various types of regional or geometric constraints. Nevertheless, they quickly emerged as a general technique to solve diverse computer vision and image processing problems (Boykov and Veksler, 2006). Particularly, graph cuts are suitable to find global minima of certain classes of energy functionals (Kolmogorov and Zabih, 2004) frequently used in computer vision in polynomial time. Among others, these may include energy terms dependent on contour length or surface area. This is due to (Boykov and Kolmogorov, 2003) who proved that despite their discrete nature graph cuts can approximate any Euclidean or Riemannian metric with arbitrarily small error and derived the required formulas for edge weights.

In the following text we focus on the Euclidean metric as it is essential for graph cut based minimization of many popular energy functionals such as the Chan-Vese model for image segmentation (Chan and Vese, 2001) (Zeng et al., 2006). The formulas derived in (Boykov and Kolmogorov, 2003) assume isotropic resolution of the underlying grid of pixels/voxels which is a limitation in some fields. For instance, volumetric images produced by optical microscopes often have notably lower resolution in the *z* axis than in the *xy* plane. Hence, before processing it is necessary to either upsample the *z* direction which substantially increases computational demands or downsample the *xy* plane which causes loss of information. Last option is to simulate the anisotropy using the more general Riemannian metrics. Unfortunately, it turns out that the corresponding formulas have significantly larger approximation error that once again can be reduced only for the price of slower and more memory intensive computation taking into account larger neighbourhood.

In this paper we show how to solve the above mentioned problem and derive the weights required for the approximation of the Euclidean metric on anisotropic grids directly. For this purpose we follow (Boykov and Kolmogorov, 2003) and exploit the well-known Cauchy-Crofton formulas from integral geometry. However, several amendments allow us to obtain a better approximation. Namely, we employ Voronoi diagrams theory to calculate the partitioning of angular orientations of lines which is required during the discretization of the Cauchy-Crofton formulas. This among other things makes our approximation invariant to image mirroring. Moreover, we show that our approach has much smaller metrication error, especially in the case of small neighbourhood or large

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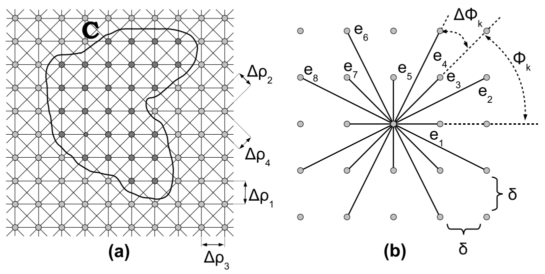

Figure 1: (a) 8-neighbourhood 2D grid graph. (b) 16-neighbourhood system on a grid with isotropic resolution.

anisotropy and that under specific conditions it is better even in the isotropic case.

The paper is structured as follows. The notation and known results are briefly reviewed in Section 2. In Section 3 we present our contribution and derive the formulas approximating the Euclidean metric on both 2D and 3D grids with anisotropic resolution. Section 4 contains detailed discussion of the approximation error and gives example of an application of our results to biomedical image segmentation. We conclude the paper in Section 5.

2 CUT METRICS

Consider an undirected graph $\mathcal{G}$ embedded in a regular orthogonal 2D grid with all nodes having topologically identical neighbourhood system and with isotropic spacing δ between the nodes. Example of such graph with 8-neighbourhood system is depicted in Fig.1a. Further, let the neighbourhood system $\mathcal{N}$ be described by a set of vectors $\mathcal{N} = \{e_1, \ldots, e_n\}$. We assume that the vectors are listed in the increasing order of their angular orientation $0 \leq \phi_k < \pi$. We also assume that vectors e_k are undirected (we do not differentiate between e_k and $-e_k$) and shortest possible in given direction, e.g. 16-neighbourhood would be represented by a set of 8 vectors $\mathcal{N}_{16} = \{e_1, \ldots, e_8\}$ as depicted in Fig.1b. Finally, we define the distance between the nearest lines generated by vector e_k in the grid as ρ_k (for 8-neighbourhood these are depicted in Fig.1a).

Lets assume each edge e_k is assigned particular weight w_k and imagine we are given a contour as shown in Fig.1a. This contour divides the nodes of the graph into two groups based on whether they lie inside or outside the contour. A cut $\mathcal{C}$ is defined as the set of all edges joining the inner nodes with the outer ones. The cut capacity $|\mathcal{C}|_{\mathcal{G}}$ is the sum of the weights of the cut edges. The question stands whether it is possible to set weights w_k so that the capacity of the cut approximates the Euclidean length $|\mathcal{C}|_{\varepsilon}$ of the contour. Since algorithms for finding minimal cuts constitute well studied part of combinatorial optimization (Boykov and Kolmogorov, 2004) this would allow us to effectively find globally minimal contours or surfaces satisfying certain criterion.

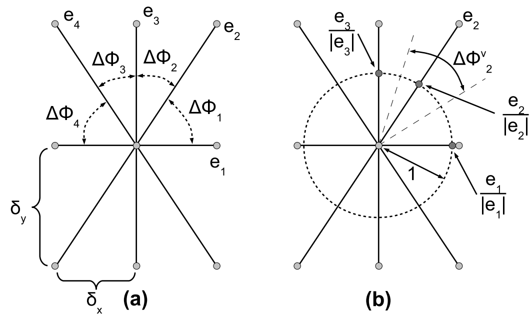

Figure 2: (a) 8-neighbourhood system on a grid with anisotropic resolution. (b) Computation of $\Delta\phi_2^v$.

The technical result of (Boykov and Kolmogorov, 2003) answers the question positively. Based on the Cauchy-Crofton formula from integral geometry the weights for a 2D grid should be set to:

$$w_k = \frac{\delta^2 \Delta\phi_k}{2|e_k|} \tag{1}$$

The whole derivation of the formula is omitted here, so is the extension to 3D grids. Both are being explained in more detail in the remaining text. Nevertheless, as already suggested in the introduction Euclidean metric is not the only one that can be approximated using graph cuts. The complete discussion can be found in (Kolmogorov and Boykov, 2005).

3 EUCLIDEAN METRIC ON ANISOTROPIC GRIDS

Up until now we assumed that the grid of graph nodes has isotropic resolution δ. In this section we investigate the anisotropic case and adjust the edge weight formulas appropriately.

3.1 2D Grids

Consider an undirected graph $\mathcal{G}$ embedded in a regular orthogonal 2D grid with all nodes having topologically identical neighbourhood system. However, let the spacing of the grid nodes be δ_x and δ_y in horizontal and vertical directions, respectively. Otherwise the whole notation remains unchanged. Example of an 8-neighbourhood system over an anisotropic grid is depicted in Fig. 2a.

Now, consider the Cauchy-Crofton formula that links Euclidean length $|\mathcal{C}|_\varepsilon$ of contour $\mathcal{C}$ with a measure of a set of lines intersecting it:

$$|\mathcal{C}|_\varepsilon = \frac{1}{2}\int n_c \, d\mathcal{L} \tag{2}$$

where $\mathcal{L}$ is the space of all lines and $n_c(l)$ is the number of intersections of line l with contour $\mathcal{C}$. Every line in a plane is uniquely identified by its angular orientation ϕ and distance ρ from the origin. Thus, the formula can be rewritten in the form:

$$|\mathcal{C}|_\varepsilon = \int_0^\pi \int_{-\infty}^{+\infty} \frac{n_c(\phi,\rho)}{2}\, d\rho\, d\phi \tag{3}$$

and discretized by partitioning the space of all lines according to the neighbourhood $\mathcal{N} = \{e_1, \ldots, e_n\}$:

$$|\mathcal{C}|_\varepsilon \approx \sum_{k=1}^{n} \left(\sum_i \frac{n_c(k,i)}{2} \Delta\rho_k \right) \Delta\phi_k \tag{4}$$

where i enumerates lines generated by vector e_k. Further, let $n_c(k) = \sum_i n_c(k,i)$ be the total number of intersections of contour $\mathcal{C}$ with all lines generated by vector e_k. We obtain:

$$|\mathcal{C}|_\varepsilon \approx \sum_{k=1}^{n} n_c(k) \frac{\Delta\rho_k \Delta\phi_k}{2} \tag{5}$$

From the last equation it can be seen, that if we set

$$w_k = \frac{\Delta\rho_k \Delta\phi_k}{2} \tag{6}$$

then (proof omitted):

$$|\mathcal{C}|_\mathcal{G} \xrightarrow[\sup \Delta\phi_k \to 0, \sup |e_k| \to 0]{\delta_x, \delta_y \to 0} |\mathcal{C}|_\varepsilon \tag{7}$$

Finally, the distance between the closest lines generated by vector e_k in the grid equals to:

$$\Delta\rho_k = \frac{\delta_x \delta_y}{|e_k|} \tag{8}$$

and if we substitute Eq.8 into Eq.6 we obtain the above mentioned Eq.1.

So far we have followed the method of (Boykov and Kolmogorov, 2003). However, when $\delta_x \neq \delta_y$ this approach has a serious flaw. One may notice that in the example depicted in Fig. 2a edges e_2 and e_4 will be assigned different weights because $\Delta\phi_2 \neq \Delta\phi_4$ and $\Delta\rho_2 = \Delta\rho_4$. But this means that if we mirror the contour horizontally we will obtain different cut capacity. Hence, edge weights derived this way are not invariant to mirroring, which is rather inconvenient property causing additional bias of the approximation. In fact, this bias is present in the isotropic case as well, but not for all neighbourhoods. For instance, in the 16-neighbourhood depicted in Fig.1b edges e_2 and e_8 will be assigned different weights and it indeed has a negative effect on the approximation as we will show in the following section

The solution lies in different partitioning of the unit circle of angular orientations. We do not utilize $\Delta\phi_k$ in the way it has been used so far. Instead we introduce new symbol $\Delta\phi_k^v$ which from a probabilistic point of view can be interpreted as a measure of lines closest to e_k in terms of their angular orientation. The computation is done as follows. Let $\mathcal{S} = \{\frac{e_1}{|e_1|}, \ldots, \frac{e_n}{|e_n|}\}$ be a set of points lying on a unit circle. We calculate the Voronoi diagram of $\mathcal{S}$ on the 1D circle manifold and define $\Delta\phi_k^v$ to be the size of the Voronoi cell corresponding to point $\frac{e_k}{|e_k|}$. The whole process is depicted in Fig. 2b. It reduces to the following formula:

$$\Delta\phi_k^v = \frac{\Delta\phi_k + \Delta\phi_{k-1}}{2} \tag{9}$$

Putting this together with Eq.6 and Eq.8 the final edge weights for a 2D grid with anisotropic resolution are calculated as:

$$w'_k = \frac{\Delta\rho_k \Delta\phi_k^v}{2} = \frac{\delta_x \delta_y (\Delta\phi_k + \Delta\phi_{k-1})}{4|e_k|} \tag{10}$$

Such edge weights still follow the distribution of the angular orientations of lines generated by vectors in $\mathcal{N}$ but in a smarter way causing the approximation to be invariant to contour mirroring while not breaking the convergence of the original approach at the same time.

3.2 3D Grids

In three dimensions the contour $\mathcal{C}$ is replaced by a surface $\mathcal{C}^2$ and the graph $\mathcal{G}$ is embedded in a regular orthogonal 3D grid with δ_x, δ_y and δ_z spacing between the nodes in x, y and z directions, respectively, with all nodes having topologically identical 3D neighbourhood system $\mathcal{N} = \{e_1, \ldots, e_n\}$ (e.g. 6-, 18- or 26-neighbourhood).

This time $\Delta\rho_k$ expresses the "density" of lines generated by vector e_k. It is calculated by intersecting these lines with a plane perpendicular to them and computing the area of cells in the obtained 2D grid of points. It can be easily computed using this formula:

$$\Delta\rho_k = \frac{\delta_x \delta_y \delta_z}{|e_k|} \tag{11}$$

Each vector e_k is now determined by two angular orientations φ_k and ψ_k with $\Delta\phi_k$ corresponding to the partitioning of the unit sphere among the angular orientations of vectors in $\mathcal{N}$. In fact, this formulation is rather vague and it is unclear how to calculate $\Delta\phi_k$ the way it is being described in (Boykov

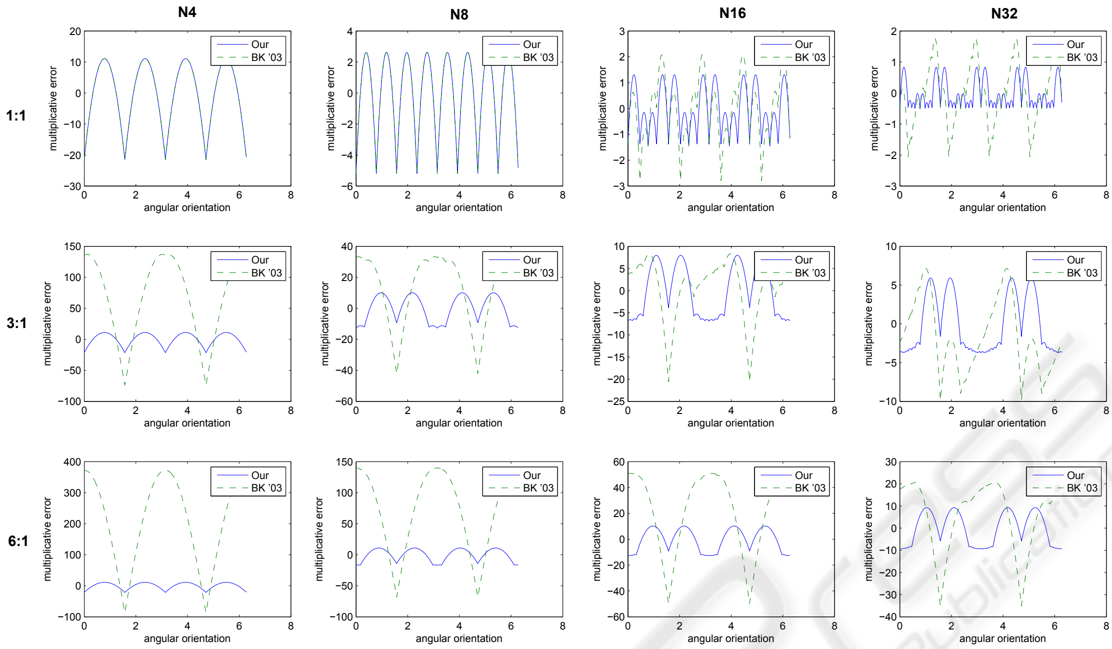

Figure 3: Metrication error (in percents) in 2D for several combinations of neighbourhood system and anisotropy ratio.

and Kolmogorov, 2003). Particularly because for almost all common 3D neighbourhoods (e.g. 18- or 26-neighbourhood) the distribution of the angular orientations is not uniform (this stems from the fact, that it is not possible to create a Platonic solid for such number of points).

The capacity of a cut is analogously defined as the sum of the weights of the edges joining grid nodes enclosed by the surface $\mathcal{C}^2$ with those lying outside and the goal is to set the weights w_k so that the capacity of the cut approximates the area of the surface under Euclidean metric. The Cauchy-Crofton formula for surface area in 3D is:

$$|\mathcal{C}^2|_\varepsilon = \frac{1}{\pi}\int n_c \, d\mathcal{L} \tag{12}$$

and using the same derivation steps as in the case of 2D grids yields the following edge weights:

$$w_k = \frac{\Delta\rho_k \Delta\phi_k}{\pi} = \frac{\delta_x\delta_y\delta_z\Delta\phi_k}{\pi|e_k|} \tag{13}$$

The problem with the clarity of $\Delta\phi_k$ is addressed easily by extending our concept of Voronoi diagram based weights $\Delta\phi_k^v$. Let $\mathcal{S} = \{\frac{e_1}{|e_1|}, \dots, \frac{e_n}{|e_n|}\}$ be a set of points this time lying on a unit sphere. We calculate the Voronoi diagram of $\mathcal{S}$ on the 2D sphere surface manifold and define $\Delta\phi_k^v$ to be the area of the Voronoi cell corresponding to point $\frac{e_k}{|e_k|}$. This is a general and explicit approach that can be used for any type of neighbourhood. Unfortunately, the spherical case can not be reduced to a simple formula. To compute the spherical Voronoi diagram we recommend to use the convex hull based method described in (Brown, 1979). Putting this all together we end up with the final formula for 3D anisotropic grids:

$$w'_k = \frac{\Delta\rho_k \Delta\phi_k^v}{\pi} = \frac{\delta_x\delta_y\delta_z\Delta\phi_k^v}{\pi|e_k|} \tag{14}$$

To conclude this section, this approach can be theoretically extended to any number of dimensions. In the general N-D case one would have to calculate Voronoi diagram of points on a hypersphere to get $\Delta\phi_k^v$ weights. The adjustment of $\Delta\rho_k$ is straightforward.

4 EXPERIMENTAL RESULTS

4.1 Approximation Error

To benchmark the approximations we chose to measure the multiplicative error they give under particular angular orientations in 2D. Graphs of the error are available in Fig. 3. The figure contains 12 graphs where each column corresponds to a particular 2D neighbourhood and each row to a particular anisotropy ratio. We compared our approximation with the method described in (Boykov and Kolmogorov, 2003). To simulate the anisotropy we had to embed it into a Riemannian metric in the latter case. According to the referenced paper the weights for a

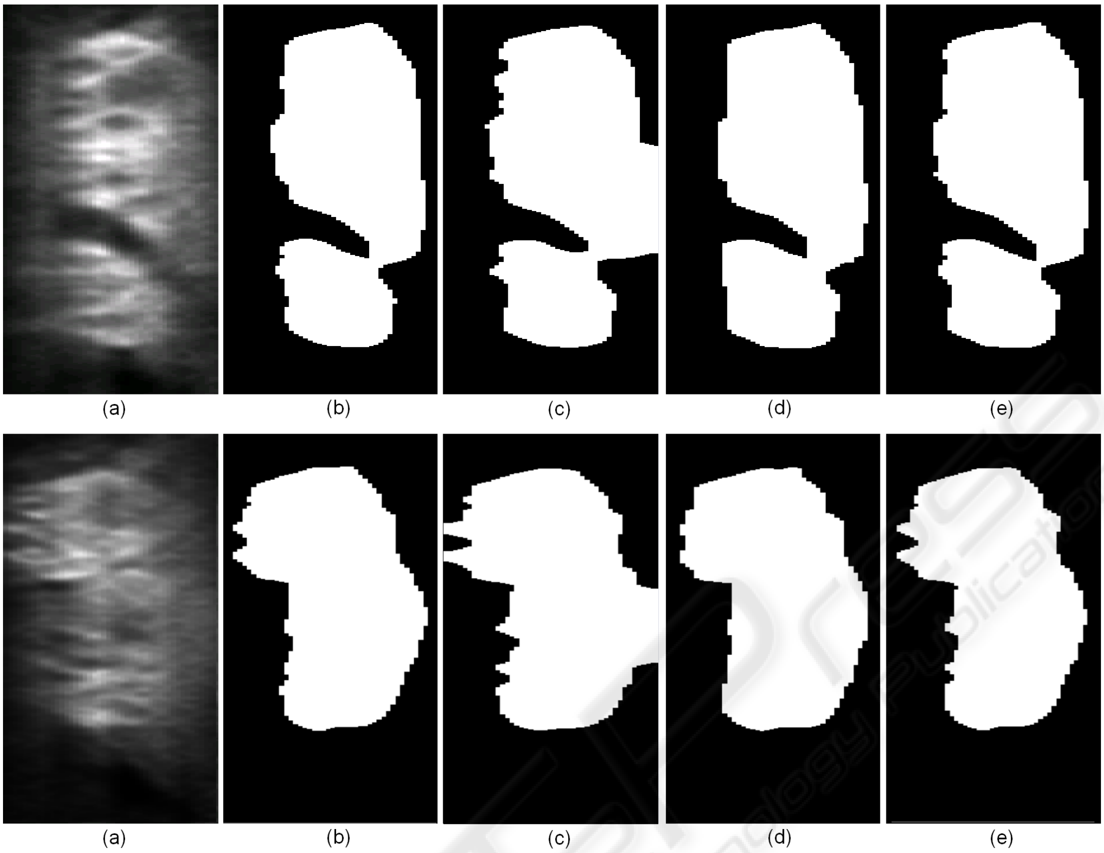

Figure 4: Two examples of biomedical image segmentation using the Chan-Vese model. (a) *yz* cross-section of the segmented image. (b) Level-set based method. (c) Graph cuts with edge weights for isotropic resolution. (d) Graph cuts with anisotropy embedded into the Riemannian metric. (e) Graph cuts with our edge weights.

Riemannian metric with a constant metric tensor D over an isotropic 2D grid should be set to:

$$w_k^{\mathcal{R}} = \frac{\delta^2 \Delta\phi_k}{2|e_k|} \cdot \frac{\det D}{(u_k^T \cdot D \cdot u_k)^{3/2}} \qquad (15)$$

where $u_k = \frac{e_k}{|e_k|}$ and to:

$$w_k^{\mathcal{R}} = \frac{\delta^3 \Delta\phi_k}{\pi|e_k|} \cdot \frac{\det D}{(u_k^T \cdot D \cdot u_k)^2} \qquad (16)$$

in case of a 3D grid. Resolution change corresponds to a constant metric tensor with eigenvectors aligned with the coordinate system and eigenvalues δ_x^2 and δ_y^2. Hence, the metric tensor simulating the anisotropic grid has the following form:

$$D = \begin{pmatrix} \delta_x^2 & 0 \\ 0 & \delta_y^2 \end{pmatrix} \qquad (17)$$

Notice that if D is the identity matrix the second term in Eq.15 and Eq.16 vanishes and the formulas reduce to the isotropic case.

The multiplicative error measures in percents the difference between the approximated value and the factual length, i.e. zero is the ideal meaning no error. As can be seen from Fig. 3 both approaches perform equivalently in the isotropic case for 4- and 8-neighbourhood. For larger neighbourhoods our approach is almost two times better and its invariance to mirroring is also apparent as the graph is symmetrical around values $\frac{\pi}{2}$, π and $\frac{3\pi}{2}$. With increasing anisotropy the gap widens and especially for smaller neighbourhoods the difference is really huge. However, note that the maximal error depends primarily on $\sup \Delta\phi_k$ and that this value increases with increasing anisotropy. Thus, for high anisotropy ratios using larger neighbourhood is inevitable.

4.2 Applications to Image Segmentation

In this subsection we show the practical impact of our results and evaluate the benefits of the improved approximation in biomedical image segmentation. We chose the Chan-Vese segmentation model (Chan and Vese, 2001) that is being very popular in this field for its robust segmentation of highly degraded data. The Chan-Vese model is a binary segmentation model which corresponds to piecewise-constant specialization of the well-known Mumford-Shah energy functional. In simple terms, it segments the image into two regions trying to minimize the length of the frontier between them and their intensity variance. This functional can be minimized using graph cuts (Zeng et al., 2006) and as it tries to minimize the boundary length it obviously depends on the Euclidean metric approximation.

To test the improvement of our approximation over the previous approach also in 3D we plugged the derived formulas into the algorithm and used it to segment low-quality volumetric images of cell clusters acquired by an optical microscope. The *yz* cross-sections of the segmented images are depicted in Fig.4a. The dimensions of the images are $280 \times 360 \times 50$, with resolution in the *xy* plane being about 4.5 times the resolution in the *z* direction. We used 26-neighbourhood to segment the images.

In Fig. 4b is the Chan-Vese segmentation computed using level-sets. This technique was much slower than the graph cuts, however, it does not suffer from the metrication errors so we used its results as the ground truth. Figure 4c shows the graph cut based segmentation when the anisotropy is ignored. The results obtained using the Riemannian metric and our weights are depicted in Fig. 4d and Fig. 4e, respectively. Clearly, our method gives a result closest to the level-sets. On the other hand, the segmentation based on the Riemannian metric seems too flat or chopped. Based on the results from the previous subsection it could be probably greatly improved using a larger neighbourhood, but at the cost of higher computational demands.

5 CONCLUSIONS AND FUTURE WORK

In this paper we addressed the problem of approximation of the Euclidean metric on 2D and 3D anisotropic grids via graph cuts. We derived the required formulas and showed that our approach has a significantly smaller metrication error than the previous one and that it is invariant to image mirroring. Using the presented results it is possible to exploit graph cut based energy minimization dependent on contour length or surface area over images with anisotropic resolution directly without the need to resample them or to use large neighbourhoods for better precision. A possible application of the results was demonstrated on a biomedical image segmentation.

As explained in Section 4.1 anisotropic grids correspond to a special case of the Riemannian metric with a constant metric tensor with eigenvectors aligned with the coordinate system. However, the general case of this metric is also being widely used in several fields including image segmentation. Taking into account the relatively high error of the current formulas we would like to make use of the presented results and focus on better approximations of the general case of the Riemannian metric in our future work.

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