A Globally Exponentially Convergent Immersion and Invariance Speed Observer for n Degrees of Freedom Mechanical Systems with constraints

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Abstract. The problem of velocity estimation for mechanical systems is of great practical interest. Although many partial solutions have been reported in the literature the basic question of whether it is possible to design a globally convergent speed observer for general n degrees of freedom mechanical systems remains open. In this paper an affirmative answer to the question is given by proving the existence of a $3n+1$–dimensional globally *exponentially* convergent speed observer. Instrumental for the construction of the speed observer is the use of the Immersion and Invariance technique, in which the observer design problem is recast as a problem of rendering attractive and invariant a manifold defined in the extended state–space of the plant and the observer.

Notation. For general mappings $S : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^q$, $(x, \zeta) \mapsto S$ we define $\nabla_x S(x,\zeta) := \frac{\partial S(x,\zeta)}{\partial x}$ and $\nabla_\zeta S(x,\zeta) := \frac{\partial S(x,\zeta)}{\partial \zeta}$. For brevity, when clear from the context, the subindex of ∇ and, in general, the arguments of all the functions are omitted.

1 Problem Formulation

We consider general n degree of freedom mechanical systems with nonholonomic constraints described in Lagrangian form by [11], [13],

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + \nabla U(q) = G(q)u + A(q)\lambda, \tag{1}$$

$$A^\top(q)\dot{q} = 0, \tag{2}$$

where $q(t), \dot{q}(t) \in \mathbb{R}^n$ are the generalized positions and velocities, respectively, $u(t) \in \mathbb{R}^m$ is the control input, $A(q)\lambda$ are the constraint forces with $A : \mathbb{R}^n \to \mathbb{R}^{n\times k}$, $\lambda \in \mathbb{R}^k$, $G : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ is the input matrix, $M : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ is the mass matrix with

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$M = M^\top > 0$ and $U : \mathbb{R}^n \to \mathbb{R}$ is the potential energy function. $C(q, \dot{q})\dot{q}$ is the vector of Coriolis and centrifugal forces, with the (ik)–th element of the matrix $C : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$ defined by

$$C_{ik}(q, \dot{q}) = \sum_{j=1}^{n} C_{ijk}(q)\dot{q}_j,$$

where $C_{ijk} : \mathbb{R}^n \to \mathbb{R}$ are the Christoffel symbols of the first kind defined as

$$C_{ijk}(q) := \frac{1}{2}\left\{\frac{\partial M_{\rm ik}}{\partial q_{\rm j}} + \frac{\partial M_{\rm jk}}{\partial q_{\rm i}} - \frac{\partial M_{\rm ij}}{\partial q_{\rm k}}\right\}. \tag{3}$$

We consider $q(t)$ to be measurable and assume that the input $u(t)$ is such that $q(t), \dot{q}(t)$ exist for all time, that is, the system is forward complete. Our objective is to design a globally asymptotically convergent observer for $\dot{q}(t)$.

Speed observation is a longstanding problem whose complete theoretical solution has proven highly elusive. The first results were reported in 1990 in the fundamental paper [14], and many interesting partial solutions have been reported afterwards. Particular attention has been given to the case in which the system (1) can be rendered linear in the unmeasurable velocities via partial changes of coordinates, see, *e.g.*, [6, 16]. An intrinsic local observer, exploiting the Riemannian structure of the system, has been recently proposed in [1] (see also [2] for a Lyapunov analysis and [7] for a generalization). A solution for a class of two degrees of freedom systems has been recently reported in [8]. The reader is referred to the recent books [5, **?**,**?**] for an exhaustive list of references.

A complete solution to the problem is given by the proposition below. As will become clear in the proof, the construction of the observer relies on the use of the Immersion and Invariance (I&I) technique—first reported in [4] and further developed in [3, 10]. In I&I the observer design is recast as a problem of rendering attractive a suitably selected invariant manifold defined in the extended state–space of the plant and the observer. It should be mentioned that the observers in [8, 16] are also based on the I&I approach.

2 Main Result

Proposition 1. *Consider the system (1), and assume u is such that trajectories exist for all $t \geq 0$. There exist smooth mappings $A : \mathbb{R}^{3n-2k+1} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{3n-2k+1}$ and $B : \mathbb{R}^n \to \mathbb{R}^{(n-k)\times(3n-2k+1)}$ such that the dynamical system*

$$\dot{\chi} = A(\chi, q, u) \tag{4}$$

with state $\chi(t) \in \mathbb{R}^{3n-2k+1}$, inputs $q(t)$ and $u(t)$, and output

$$\eta = B(q)\chi, \tag{5}$$

has the following property.

All trajectories of the interconnected system (1), (2), (4) are such that

$$\lim_{t\to\infty} e^{\alpha t}[\mathcal{N}(q)\dot{q}(t) - \eta(t)] = 0, \tag{6}$$

for some $\alpha > 0$ *and for all initial conditions* $(q(0), \dot{q}(0), \chi(0)) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{3n-2k+1}$, *where* $\mathcal{N}(q) : \mathbb{R}^n \to \mathbb{R}^{n-k} \times \mathbb{R}^n$ *is a left invertible matrix. That is, (4), (5) is a globally exponentially convergent speed observer for the mechanical system (1)-(2).*

Remark 1. For the special case of a mechanical system with no nonholonomic constraints, it is clear that $k = 0$ and subsequently the matrix $\mathcal{N}(q)$ becomes an invertible square matrix.

3 A Preliminary Lemma

Before giving the proof of the main result, we recall that the system (1)-(2) can be written in the port-Hamiltonian form [13] as

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{pmatrix} \nabla_q H(q,p) \\ \nabla_p H(q,p) \end{pmatrix} + \begin{bmatrix} 0 \\ G(q) \end{bmatrix} u + \begin{bmatrix} 0 \\ A(q) \end{bmatrix} \lambda, \tag{7}$$

$$A^\top(q)\lambda = 0, \tag{8}$$

where $p = M(q)\dot{q}$ are the generalized momenta and

$$H(q,p) = \frac{1}{2} p^\top M^{-1}(q) p + U(q)$$

represents the total energy stored in the system. Further, as per [13], the system (7)-(8) when restricted to the constrained space

$$\mathcal{X}_c = \{(q, \dot{q}) | A^\top(q)\dot{q} = 0\},$$

takes the form

$$\begin{bmatrix} \dot{q} \\ \dot{\tilde{p}} \end{bmatrix} = \begin{bmatrix} 0 & \tilde{S}(q) \\ -\tilde{S}^\top(q) & J(q,\tilde{p}) \end{bmatrix} \begin{pmatrix} \nabla_q H(q,\tilde{p}) \\ \nabla_{\tilde{p}} H(q,\tilde{p}) \end{pmatrix} + \begin{bmatrix} 0 \\ B_c(q) \end{bmatrix} u + \begin{bmatrix} 0 \\ A(q) \end{bmatrix} \lambda, \tag{9}$$

$$H(q,\tilde{p}) = V(q) + \frac{1}{2}\tilde{p}^\top \tilde{M}^{-1}(\tilde{q})\tilde{p}, \tag{10}$$

with $\tilde{p} \in \mathbb{R}^{n-k}$ being given as $\tilde{p} = \tilde{S}^\top(q)p$ where $\tilde{S}(q) \in \mathbb{R}^{n \times n-k}$ is the full rank annihilator of the matrix $A(q)$ satisfying the condition $A^\top(q)\tilde{S}(q) = 0$. The matrix $J(q,\tilde{p})$ is skew-symmetric and is given by

$$J_{ij}(q,\tilde{p}) = -p^\top[\tilde{S}_i, \tilde{S}_j], \tag{11}$$

where $[\tilde{S}_i, \tilde{S}_j]$ denotes the standard Lie bracket of the column vectors S_i, S_j and the matrix $\tilde{M}(q) \in \mathbb{R}^{n-k \times n-k}$ is symmetric positive-definite.

In order to streamline the presentation in this section, we introduce a factorization of the mass matrix

$$\tilde{M}^{-1}(q) = T^\top(q)T(q), \tag{12}$$

where $T : \mathbb{R}^n \to \mathbb{R}^{n-k\times n-k}$ is a full rank matrix[4] and define the mappings $L : \mathbb{R}^n \to \mathbb{R}^{n\times n-k}$ and $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{n-k}$ as

$$L(q) = \tilde{S}(q)T^\top(q), \tag{13}$$

$$F(q,u) = L^\top(q)[B_c(q)u - \nabla U(q)]. \tag{14}$$

Notice that, since q and u are measurable, these mappings are known. We next state the following proposition.

Lemma 1. *The system dynamics (9)-(10) when expressed in the new coordinates* $(y, x) = (q, T(q)\tilde{p})$*, admits a state space representation of the form*

$$\dot{y} = L(y)x, \tag{15}$$

$$\dot{x} = S(y,x)x + F(y,u), \tag{16}$$

where

$$S = TJT^\top + \sum_{i=1}^{n}(\{\frac{\partial T}{\partial y_i}T^{-1}x\}\{L^\top e_i\}^\top - \{L^\top e_i\}\{\frac{\partial T}{\partial y_i}T^{-1}x\}^\top), \tag{17}$$

and e_i *is the* i^{th} *basis vector of* $\mathbb{R}^{n-k}$.

Proof. We directly obtain (15) by differentiating y and by using (9), (10), (13). We next compute the following,

$$\dot{x} = \dot{T}\tilde{p} + T\dot{\tilde{p}}, \tag{18}$$

$$= \dot{T}\tilde{p} - T\tilde{S}^\top\frac{\partial}{\partial y}(\frac{1}{2}\tilde{p}^\top\tilde{M}^{-1}(\tilde{q})\tilde{p}) - T\tilde{S}^\top\nabla U + T\tilde{S}^\top B_c u + TJT^\top x, \tag{19}$$

$$= \dot{T}\tilde{p} - L^\top\frac{\partial}{\partial y}(\frac{1}{2}\tilde{p}^\top\tilde{M}^{-1}(\tilde{q})\tilde{p}) + F + TJT^\top x, \tag{20}$$

where we have made use of (10), (13) and (14). We now compute that,

$$\dot{T}\tilde{p} = \sum_{\mathrm{i}=1}^{\mathrm{n}}(\frac{\partial T}{\partial y_{\mathrm{i}}}\tilde{p})(e_{\mathrm{i}}^\top\tilde{S}\tilde{M}^{-1}\tilde{p}),$$

$$= \sum_{\mathrm{i}=1}^{\mathrm{n}}(\frac{\partial T}{\partial y_{\mathrm{i}}}\tilde{p})(e_{\mathrm{i}}^\top L)x, \tag{21}$$

and further obtain

$$\frac{\partial}{\partial y}\{\frac{1}{2}\tilde{p}^\top\tilde{M}^{-1}\tilde{p}\} = \frac{\partial}{\partial y}\{\frac{1}{2}\tilde{p}^\top T^\top Tp\}$$

$$= \sum_{\mathrm{i}=1}^{\mathrm{n}} e_i\{\frac{\partial T}{\partial y_{\mathrm{i}}}\tilde{p}\}^\top x. \tag{22}$$

[4] Since M is positive definite this factorization always exists. It may be taken to be the (univocally defined) Cholesky factorization, as proposed in [9].

Substituting (21) and (22) in (20) we obtain the dynamics of x as,

$$\begin{aligned}\dot{x} &= \sum_{i=1}^{n}(\{\frac{\partial T}{\partial y_i}T^{-1}x\}\{L^\top e_i\}^\top - \{L^\top e_i\}\{\frac{\partial T}{\partial y_i}T^{-1}x\}^\top)x + TJT^\top x + F, \\ &= Sx + F, \end{aligned} \tag{23}$$

where we have used (17) to obtain the equation (23). This concludes the proof.

Remark 2. It can be verified easily that matrix $S(y, x)$ defined in (17) satisfies the following properties:

(i) S is skew–symmetric, that is,

$$S + S^\top = 0.$$

(ii) S is linear in the second argument, that is,

$$S(y, a_1 x + a_2 \bar{x}) = a_1 S(y, x) + a_2 S(y, \bar{x}),$$

for all $y, x, \bar{x} \in \mathbb{R}^n$, and $a_1, a_2 \in \mathbb{R}$.

(iii) There exists a mapping $\bar{S} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n \times n}$ such that

$$S(y, x)\bar{x} = \bar{S}(y, \bar{x})x,$$

for all $y, x, \bar{x} \in \mathbb{R}^n$.

Remark 3. Lemma 1 implies that the speed observer problem for system (1)-(2) can be recast as an observer problem for system (15)-(16) with output y.

Remark 4. For the special case of no nonholonomic constraints, we have $k = 0$ and

$$J(q, \tilde{p}) = 0, \tilde{S}(q) = I, \tilde{M}(q) = M(q), L(q) = T^\top(q).$$

This subsequently simplifies the expression for $S(y, x)$ as

$$S(y, x) = \sum_{i=1}^{n}(\{\frac{\partial T}{\partial y_i}T^{-1}x\}\{Te_i\}^\top - \{Te_i\}\{\frac{\partial T}{\partial y_i}T^{-1}x\}^\top). \tag{24}$$

It can be shown (refer to [16]) that the (jk)–th element of S is $S_{jk} = -\{T^{-1}x\}^\top[\mathcal{T}_j, \mathcal{T}_k]$.

4 Proof of the Main Result

The observer is constructed in four steps.

(S1) Following the I&I procedure [3], we define a manifold (in the extended state-space of the plant and the observer) that should be rendered attractive and invariant[5]. As is well–known, to achieve the latter objective a partial differential equation (PDE) should, in principle, be solved.

[5] The manifold should be such that the unmeasurable part of the state can be reconstructed from the function that defines the manifold.

(S2) To avoid the need to solve the PDE the "approximation" technique proposed in [10] is adopted. Using this approximation induces some errors in the observer error dynamics.

(S3) Always borrowing from [10], we introduce a dynamic scaling that dominates—in a Lyapunov–like analysis—the effect of the aforementioned disturbance terms, proving that the scaled observer error converges to zero.

(S4) To prove that the dynamic scaling factor is bounded and, consequently, that the actual observer error converges, exponentially, to zero, high gain terms are introduced in the observer dynamics to, again, dominate sign–indefinite terms in a Lyapunov–like analysis.

Step 1. (Definition of the manifold) For the system (15)-(16), we propose the manifold

$$\mathcal{M} := \{(y, x, \xi, \hat{y}, \hat{x}) : \xi - x + \beta(y, \hat{y}, \hat{x}) = 0\} \subset \mathbb{R}^{5n-3k}, \tag{25}$$

where $\xi \in \mathbb{R}^{n-k}, \hat{y} \in \mathbb{R}^{n-k}, \hat{x} \in \mathbb{R}^{n}$ are (part of) the observer state, the dynamics of which are defined below, and the mapping $\beta : \mathbb{R}^{3n-2k} \to \mathbb{R}^{n-k}$ is also to be defined.

To prove that the manifold $\mathcal{M}$ is attractive and invariant it is shown that the off–the–manifold coordinate

$$z = \xi - x + \beta(y, \hat{y}, \hat{x}), \tag{26}$$

the norm of which determines the distance of the state to the manifold $\mathcal{M}$, is such that:

(C1) $z(0) = 0 \Rightarrow z(t) = 0$, for all $t \geq 0$ (invariance);

(C2) $z(t)$ asymptotically (*exponentially*) converges to zero (attractivity).

Notice that, if $z(t) \to 0$, an asymptotic estimate of x is given by $\xi + \beta$.

To obtain the dynamics of z differentiate (26), yielding

$$\begin{aligned}\dot{z} &= \dot{\xi} - \dot{x} + \dot{\beta}\\ &= \dot{\xi} - S(y, x)x - F + \nabla_y \beta \dot{y} + \nabla_{\hat{y}} \beta \dot{\hat{y}} + \nabla_{\hat{x}} \beta \dot{\hat{x}}.\end{aligned}$$

Let

$$\dot{\xi} = F - \nabla_{\hat{y}} \beta \dot{\hat{y}} - \nabla_{\hat{x}} \beta \dot{\hat{x}} + S(y, \xi + \beta)(\xi + \beta) - \nabla_y \beta L(y)(\xi + \beta), \tag{27}$$

where $\dot{\hat{y}}$ and $\dot{\hat{x}}$ are to be defined. Replacing (27) in the equation of $\dot{z}$ above, and invoking properties (ii) and (iii) of Lemma 1, yields

$$\begin{aligned}\dot{z} &= -S(y, \xi + \beta - z)(\xi + \beta - z) +\\ &\quad + S(y, \xi + \beta)(\xi + \beta) - \nabla_y \beta L(y) z\\ &= S(y, x)z + S(y, z)(\xi + \beta) - \nabla_y \beta L(y) z\\ &= S(y, x)z + \bar{S}(y, \xi + \beta)z - \nabla_y \beta L(y) z.\end{aligned} \tag{28}$$

From (28) it is clear that condition (C1) above is satisfied. On the other hand, condition (C2) would be satisfied if we could find a function β that solves the PDE

$$\nabla_y \beta = [k_1 I + \bar{S}(y, \xi + \beta)] L^{-1}(y), \tag{29}$$

with $k_1 > 0$, where $L^{-1}(y) : \mathbb{R}^n \to \mathbb{R}^{n-k \times n}$ is the full rank left inverse of the matrix $L(y)$. Indeed, in this case, the z–dynamics reduce to $\dot{z} = (S - k_1)z$, achieving the desired exponential convergence property. Unfortunately, solving the PDE (29) is a daunting task, and we don't even know if a solution exists. Therefore, in the next step of the design we proceed to "approximate" its *solution*.

Step 2. ("Approximate solution" of the PDE) Define the "ideal $\nabla_y \beta$" as

$$H(y, \xi + \beta) := [k_1 I + \bar{S}(y, \xi + \beta)]L^{-1}(y), \tag{30}$$

and denote the columns of this $n - k \times n$ matrix by $H_i : \mathbb{R}^n \times \mathbb{R}^{n-k} \to \mathbb{R}^{n-k}$ for $i = 1, \ldots, n$, that is,

$$H(y, \xi + \beta) = \left[H_1(y, \xi + \beta) \mid \cdots \mid H_n(y, \xi + \beta) \right].$$

Now, mimicking [10], define[6]

$$\beta(y, \hat{y}, \hat{x}) := \int_0^{y_1} H_1([s, \hat{y}_2, \ldots, \hat{y}_n], \hat{x})ds + \cdots + \\ + \int_0^{y_n} H_n([\hat{y}_1, \ldots, \hat{y}_{n-1}, s], \hat{x})ds. \tag{31}$$

From the definition of the mapping β, and adding and substracting $H(y, \xi + \beta)$, we have that $\nabla_y \beta$ can be written as

$$\nabla_y \beta(y, \hat{y}, \hat{x}) = H(y, \xi + \beta) - \Big\{ H(y, \xi + \beta) - \\ \left[H_1(y_1, \hat{y}_2, \ldots, \hat{y}_n, \hat{x}) \ \ldots \ H_n(\hat{y}_1, \ldots, \hat{y}_{n-1}, y_n, \hat{x}) \right] \Big\}.$$

Since the term in brackets is equal to zero if $\hat{y} = y$ and $\hat{x} = \xi + \beta$, and all functions are smooth, there exist mappings $\Delta_y : \mathbb{R}^n \times \mathbb{R}^{n-k} \times \mathbb{R}^n \to \mathbb{R}^{n-k \times n}$, $\Delta_x : \mathbb{R}^n \times \mathbb{R}^{n-k} \times \mathbb{R}^{n-k} \to \mathbb{R}^{n-k \times n}$ such that

$$\nabla_y \beta(y, \hat{y}, \hat{x}) = H(y, \xi + \beta) - \Delta_y(y, \hat{x}, e_y) - \Delta_x(y, \hat{x}, e_x), \tag{32}$$

with

$$e_y := \hat{y} - y, \quad e_x := \hat{x} - (\xi + \beta), \tag{33}$$

and such that

$$\Delta_y(y, \hat{x}, 0) = 0, \quad \Delta_x(y, \hat{x}, 0) = 0, \tag{34}$$

for all $y, \hat{y}, \in \mathbb{R}^n$ and $x, \hat{x} \in \mathbb{R}^{n-k}$.

Replacing (30) and (32) in (28) yields

$$\dot{z} = (S - k_1)z + (\Delta_y + \Delta_x)L(y)z. \tag{35}$$

[6] We attract the readers attention to the particular selection of the arguments used in the integrands. Namely that, with some abuse of notation, the vector $\hat{y}$ has been spelled out into its components.

Recalling that S is skew–symmetric and $k_1 > 0$, it is clear that the mappings Δ_y and Δ_x play the role of disturbances that we will try to dominate with a dynamic scaling in the next step of the design.

Step 3. (Dynamic scaling) Define the scaled off–the–manifold coordinate

$$\eta = \frac{1}{r} z, \tag{36}$$

with r a scaling dynamic factor to be defined below. Differentiating (36), and using (35), yields

$$\begin{aligned} \dot{\eta} &= \frac{1}{r}\dot{z} - \frac{\dot{r}}{r}\eta \\ &= (S - k_1)\eta + (\Delta_y + \Delta_x)L(y)\eta - \frac{\dot{r}}{r}\eta. \end{aligned}$$

Consider the function

$$V_1(\eta) = \frac{1}{2}|\eta|^2,$$

and note that its time derivative is such that

$$\begin{aligned} \dot{V}_1 &= -(k_1 + \frac{\dot{r}}{r})|\eta|^2 - \eta^\top(\Delta_y + \Delta_x)L(y)\eta \\ &\leq -\left(\frac{k_1}{2} + \frac{\dot{r}}{r} - \frac{1}{2k_1}\|[\Delta_y + \Delta_x]L\|^2\right)|\eta|^2 \\ &\leq -\left(\frac{k_1}{2} + \frac{\dot{r}}{r} - \frac{1}{k_1}\left(\|\Delta_y L\|^2 + \|\Delta_x L\|^2\right)\right)|\eta|^2, \end{aligned} \tag{37}$$

where $\|\cdot\|$ is the matrix induced 2—norm and we have applied Young's inequality (with the factor k_1) to get the second bound. Let

$$\dot{r} = -\frac{k_1}{4}(r-1) + \frac{r}{k_1}\left(\|\Delta_y L\|^2 + \|\Delta_x L\|^2\right), \quad r(0) \geq 1. \tag{38}$$

Notice that the set $\{r \in \mathbb{R} \mid r \geq 1\}$ is invariant for the dynamics (38). Replacing (38) in (37) yields the bounds

$$\begin{aligned} \dot{V}_1 &\leq -\left(\frac{k_1}{2} - \frac{k_1}{4}\frac{r-1}{r}\right)|\eta|^2 \\ &\leq -\frac{k_1}{4}|\eta|^2, \end{aligned} \tag{39}$$

where the property $\frac{r-1}{r} \leq 1$ has been used to get the second bound. From (39) we conclude that $\eta(t)$ converges to zero exponentially.

Step 4. (High–gain injection) From (36) and the previous analysis it is clear that $z(t) \rightarrow$

0 if we can prove that $r \in \mathcal{L}_\infty$, which is the property established in this step. To enhance readability the procedure is divided into two parts. First, we make the function

$$V_2(\eta, e_y, e_x) = V_1(\eta) + \frac{1}{2}(|e_y|^2 + |e_x|^2),$$

a strict Lyapunov function. Then, the derivative of the function

$$V_3(\eta, e_y, e_x, r) = V_2(\eta, e_y, e_x) + \frac{1}{2}r^2, \tag{40}$$

is shown to be non–positive—establishing the desired boundedness of r. In both steps the objectives are achieved adding, via a suitable selection of the observer dynamics, negative quadratic terms in η, e_y, e_x in the Lyapunov function derivative. We recall that e_y and e_x are measurable quantities, defined in (33).

Towards this end, define

$$\dot{\hat{y}} = L(y)(\xi + \beta) - \psi_1(y, r)e_y, \tag{41}$$

with $\psi_1 : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+$ a gain function to be defined. The error dynamics, obtained combining (15) and (41), are

$$\dot{e}_y = Lz - \psi_1 e_y. \tag{42}$$

Now, select

$$\dot{\hat{x}} = F + S(y, \xi + \beta)(\xi + \beta) - \psi_2(y, r)e_x, \tag{43}$$

with $\psi_2 : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+$ a gain function to be defined. Recalling (27) the error dynamics for e_x become

$$\dot{e}_x = \nabla_y \beta L z - \psi_2 e_x. \tag{44}$$

Using (39), (42) and (44) and doing some basic bounding, yields

$$\begin{aligned} \dot{V}_2 \leq & -\frac{k_1}{4}|\eta|^2 + r e_y^\top L\eta - \psi_1|e_y|^2 + \\ & + r e_x^\top \nabla_y \beta L\eta - \psi_2|e_x|^2 && (45) \\ \leq & -(\frac{k_1}{4} - 1)|\eta|^2 - \left(\psi_1 - \frac{r^2}{2}\|L\|^2\right)|e_y|^2 - \\ & - \left(\psi_2 - \frac{r^2}{2}\|\nabla_y\beta\|^2\|L\|^2\right)|e_x|^2. && (46) \end{aligned}$$

Selecting

$$\begin{aligned} \psi_1 &= k_2 + \psi_3 + \frac{r^2}{2}\|L\|^2, \\ \psi_2 &= k_3 + \psi_4 + \frac{r^2}{2}\|\nabla_y\beta\|^2\|L\|^2, \end{aligned} \tag{47}$$

with $k_2, k_3 > 0$ and $\psi_3, \psi_4 : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+$ to be defined, we conclude that

$$\dot{V}_2 \leq -\frac{1}{2}(k_1 - 2)|\eta|^2 - k_2|e_y|^2 - k_3|e_x|^2,$$

which, selecting $k_1 > 4$, establishes that $\eta, e_y, e_x \in \mathcal{L}_2 \cap \mathcal{L}_\infty$ and the origin of the (non-autonomous) subsystem with state η, e_y, e_x is uniformly globally exponentially stable.

We are now ready for the *coup de grâce*, namely the selection of ψ_3 and ψ_4 to guarantee that $r \in \mathcal{L}_\infty$. For, recall (34), which ensures the existence of mappings $\bar{\Delta}_y : \mathbb{R}^n \times \mathbb{R}^{n-k} \times \mathbb{R}^n \to \mathbb{R}^{n-k \times n}$, $\bar{\Delta}_x : \mathbb{R}^n \times \mathbb{R}^{n-k} \times \mathbb{R}^{n-k} \to \mathbb{R}^{n-k\times n}$ such that

$$\begin{aligned} \|\Delta_y(y, \hat{x}, e_y)\| &\leq \|\bar{\Delta}_y(y, \hat{x}, e_y)\| \; |e_y| \\ \|\Delta_x(y, \hat{x}, e_x)\| &\leq \|\bar{\Delta}_x(y, \hat{x}, e_x)\| \; |e_x|. \end{aligned} \tag{48}$$

Now, evaluate the time derivative of V_3, defined in (40), replace (47) in (46), and use the bounds (48) to get

$$\begin{aligned} \dot{V}_3 \leq -(\frac{k_1}{4} - 1)|\eta|^2 - \left(\psi_3 - \frac{r^2}{k_1}\|\bar{\Delta}_y\|^2\|L\|^2\right)|e_y|^2 - \\ - \left(\psi_4 - \frac{r^2}{k_1}\|\bar{\Delta}_x\|^2\|L\|^2\right)|e_x|^2. \end{aligned}$$

Fixing

$$\begin{aligned} \psi_3 &= \frac{r^2}{k_1}\|\bar{\Delta}_y\|^2\|L\|^2 \\ \psi_4 &= \frac{r^2}{k_1}\|\bar{\Delta}_x\|^2\|L\|^2 \end{aligned}$$

ensures $\dot{V}_3 \leq 0$, which ensures $r \in \mathcal{L}_\infty$.

To prove condition (6) note that equation (39) implies

$$|\eta(t)| \leq |\eta(0)|e^{-\frac{k_1}{8}t},$$

hence

$$|z(t)| \leq \frac{r(t)}{r(0)}|z(0)|e^{-\frac{k_1}{8}t} \leq \sup_{t\geq 0}\{r(t)\}|z(0)|e^{-\frac{k_1}{8}t},$$

which yields the claim, by boundedness of $r(t)$.

The proof is completed defining the state vector of the observer as $\chi = (\hat{x}, \hat{y}, \xi, r)$, obtaining $A(\chi, q, u)$ from (43), (41), (27), and (38), and defining

$$B(y) := \left[T^{-1}(y) \; 0 \; 0 \; 0 \right].$$

Remark 5. The four components $\hat{x}$, $\hat{y}$, ξ and r of the state vector of the observer can be given the following interpretation. The component $\hat{x}$ is the estimate of x and a filtered version of $\xi + \beta$. The component $\hat{y}$ is a filtered version of the measured variable y. The ξ-dynamics render the set $z = 0$ invariant, regardeless of the selection of the other dynamics, and ξ can be regarded as the *state* of a reduced order observer [7]. Finally, the r-dynamics are used to trade stability of the nominal design for robustness against the *disturbances* Δ_y and Δ_x.

[7] To clarify this point note that, *ideally*, the PDE (29)should have a solution β which is a function of y alone. In this case the variable ξ would play the role of the state of the (reduced) order observer(see the examples in [8]).

Remark 6. Although the analysis of the performance of the proposed observer in the presence of noise is not within the scope of the paper, it is worth noting the following. The Lyapunov argument establishing uniform asymptotic stability of the zero equilibrium of the (η, e_y, e_x)-subsystem yields robustness againsts small additive perturbations on the measured variables u and y. In the presence of such perturbations the variables e_y and e_x do not converge to zero. Nevertheless, as long as they are sufficiently small, equation (38) can be regarded as describing a linear (non-autonomous) scalar differential equation in which, by equations (34), the coefficient of the linear term is uniformly negative. This ensures boundedness of $r(t)$ for all t.

5 Conclusions

A definite affirmative answer has been given to the question of existence of a globally convergent speed observer for general mechanical systems of the form (1). No assumption is made on the existence of an upperbound for the inertia matrix, hence the result is applicable for robots with prismatic joints. Also, no conditions are imposed on the potential energy function. The only requirement is that the system is forward complete, *i.e.*, that trajectories of the system exist for all times $t \geq 0$—which is a rather weak condition.

In some sense, our contribution should be interpreted more as an existence result than an actual, practically implementable, algorithm. Leaving aside the high complexity of the observer dynamics, that can be easily retraced from the proof of Section 4, the difficulty stems from the fact that the key function β is defined via the integrals (31), whose explicit analytic solution cannot be guaranteed *a priori*. Of course, the (scalar) integrations can always be numerically performed leading to a numerical implementation of the observer. Given the recent spectacular advances in computational technology this does not seem to constitute an unsurmountable difficulty.

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