INDUCED $\ell_\infty-$ OPTIMAL GAIN-SCHEDULED FILTERING OF TAKAGI-SUGENO FUZZY SYSTEMS

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Abstract: The problem of designing gain-scheduled filters with guaranteed induced ℓ_∞ norm for the estimation of the state-vector of finite dimensional discrete-time parameter-dependent Takagi-Sugeno Fuzzy Systems systems is considered. The design process applies a lemma which was recently derived by the authors of this paper, characterizing the induced ℓ_∞ norm by Linear Matrix Inequalities. The suggested filter has been successfully applied to a guidance motivated estimation problem, where it has been compared to an Extended Kalman Filter.

1 INTRODUCTION

The theory of optimal design of estimators for linear discrete-time systems in a state-space formulation has been first established in (Kalman, 1960). The original problem formulation assumed Gaussian white noise models for both the measurement noise and the exogenous driving process. For this case, the results of (Kalman, 1960) provided the Minimum-Mean-Square Estimator (MMSE). The Kalman filter has found since then many applications (see e.g. (Sorenson, 1985) and the references therein). Following the introduction of H_∞ control theory in (Zames, 1981), a method for designing discrete-time H_∞ optimal estimators within a deterministic framework has been developed in (Yaesh and Shaked, 1991), where the exogenous signals are of finite energy. The case where the driving signal is of finite energy (e.g. piecewise constant for a finite time) ,whereas the measurement noise is white has been recently considered in (Yaesh and Shaked, 2006). However, in some cases the minimization of the maximum absolute value of the estimation error (namely the $\ell_\infty-$norm) rather than the error energy is required where the exogenous signals are also of finite $\ell_\infty-$norm. In such cases, an induced $\ell_\infty-$norm is obtained which is often referred to as an ℓ_1 problem due to the fact that the induced-$\ell_\infty-$norm for a linear system is just the ℓ_1-norm of its impulse response and an upper-bound on the ℓ_1-norm of its transfer function (see (Dahleh and Pearson, 1987)).

In the present paper, the problem of discrete-time optimal state-estimation in the minimum induced $\ell_\infty-$norm sense is considered for a class of Takagi-Sugeno fuzzy systems. The plant model for the systems considered, is described by a collection of 'sample' finite-dimensional linear-time-invariant plants which possess the same structure but differ in their parameters. All possible plant models are then assumed to be convex combinations of these specific plant models (namely a polytopic system where the 'sample' plant models are denoted as its vertices (Boyd et al., 1994)). The solution of the estimation problem is characterized by LMIs (Linear Matrix Inequalities) based on the quadratic stability assumption. We note that more recent developments of (Geromel et al., 2000) include gain scheduled filter synthesis for the cases of linear (Tuan et al., 2001) and nonlinear (Hoang et al., 2003) dependence on the parameters with parameter dependent Lyapunov functions. We also note that in (Salcedo and Martinez, 2008) related results appear where the continuous-time fuzzy output feedback and filtering were considered in parallel to the discrete-time results of the present paper.

The paper is organized as follows. In Section 2, the problem is formulated and a key lemma character-

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izing the induced $\ell_\infty-$norm in terms of LMIs is presented. In Section 3 the filter design inequalities are obtained. Section 4 considers a numerical example dealing with a robust gain scheduled tracking problem. Finally, Section 5 brings some concluding remarks.

Notation: Throughout the note the superscript 'T' stands for matrix transposition, $\mathcal{R}^n$ denotes the n dimensional Euclidean space, $\mathcal{R}^{n\times m}$ is the set of all $n\times m$ real matrices, and the notation $P>0$, for $P\in\mathcal{R}^{n\times n}$ means that P is symmetric and positive definite. The space of square summable functions over $[0 \quad \infty]$ is denoted by $l_2[0 \quad \infty]$, and $||.||_2$ stands for the standard l_2-norm,$||u||_2 = (\Sigma_{k=0}^{\infty} u_k^T u_k)^{1/2}$. We also use $||.||_\infty$ for the l_∞-norm namely, $||u||_\infty^2 = sup_k(u_k^T u_k)$. The convex hull of a and b is denoted by $\mathcal{C}o\{a,\ b\}$, I_n is the unit matrix of order n, and $0_{n,m}$ is the $n\times m$ zero matrix and $I_{m,n}$ is a version of I_n with last $n-m$ rows omitted.

2 PROBLEM FORMULATION AND PRELIMINARIES

We consider the following linear system:

$$\begin{array}{l} x(k+1) = A(k)x(k) + Bw(k), \quad x(0) = x_0 \\ y(k) = C(k)x(k) + Dw(k) \\ z(k) = L(k)x(k) \end{array} \tag{1}$$

where $x\in\mathcal{R}^n$ is the system states, $y\in\mathcal{R}^r$ is the measurement, $w\in\mathcal{R}^q$ includes the driving process and the measurement noise signals and it is assumed to have bounded $\ell_\infty-$norm. The sequence $z\in\mathcal{R}^m$ is the state combination to be estimated and A, B, C, D and L are matrices of the appropriate dimensions.

We assume that the system parameters lie within the following polytope

$$\Omega := \begin{bmatrix} A & B & C & D & L \end{bmatrix} \tag{2}$$

which is described by its vertices. That is, for

$$\Omega_i := \begin{bmatrix} A_i & B_i & C_i & D_i & L_i \end{bmatrix} \tag{3}$$

we have

$$\Omega = \mathcal{C}o\{\Omega_1, \Omega_2, ..., \Omega_N\} \tag{4}$$

where N is the number of vertices. In other words:

$$\Omega = \sum_{i=1}^{N} \Omega_i f_i \quad , \sum_{i=1}^{N} f_i = 1 \quad , f_i \geq 0. \tag{5}$$

Assuming that f_i are exactly known, the above system is just a Tagaki-Sugeno fuzzy system. To see this, one may introduce new parameters $s_i(t), i=1,2,...,p$ (so called premise variables, see (Tanaka and Wang, 2001)) possibly depending on the state-vector $x(t)$, external disturbances and/or time (Tanaka and Wang, 2001) and rewrite (1) as :

IF s_1 is M_{i1} and s_2 is M_{i2} and ... s_p is M_{ip} **THEN**

$$\begin{array}{l} x(k+1) = A_i(k)x(k) + B_i w(k), \quad x(0) = x_0 \\ y(k) = C_i(k)x(k) + D_i w(k) \\ z(k) = L_i(k)x(k) \\ i = 1,2,...,N \end{array} \tag{6}$$

where M_{ij} is the fuzzy set and N is the number of model rules. Defining $s(t) = col\{s_1(t), s_2(t), ..., s_p(t)\}$,

$$\omega_i(s(t)) = \Pi_{j=1}^{p} M_{ij}(s_j(t))$$

and

$$f_i(s(t)) = \frac{\omega_i(t)}{\Sigma_{i=1}^{N}\omega_i(s(t))}$$

we readily get the representation of (1). We, therefore, assume indeed that the p premise scalar variables $s_i(t), i=1,2,...,p$, and, consequently f_i are exactly known and consider the following filter:

$$\hat{x}(k+1) = A\hat{x}(k) + K(k)(y - C\hat{x}),\ \hat{z}(k) = L\hat{x}(k) \tag{7}$$

where the filter gain is given by the following:

$$K = \sum_{i=1}^{N} K_i f_i \tag{8}$$

and where $A = \sum_{i=1}^{N} A_i f_i$ and $L = \sum_{i=1}^{N} L_i f_i$. We will differently treat, in the sequel, the case where C is constant and the case where $C = \sum_{i=1}^{N} C_i f_i$.

Our aim is to find the filter parameters K_i so that the following induced $\ell_\infty-$norm condition is satisfied.

$$sup_{w\in\ell_\infty} ||z-\hat{z}||_\infty / ||w||_\infty < \gamma \tag{9}$$

To solve this problem we will first define another polytopic system :

$$\bar{\Omega} := \begin{bmatrix} \bar{A} & \bar{B} & \bar{C} & \bar{D} \end{bmatrix} \tag{10}$$

which is described by the vertices:

$$\bar{\Omega}_i := \begin{bmatrix} \bar{A}_i & \bar{B}_i & \bar{C}_i & \bar{D}_i \end{bmatrix},\ i=1,...,N \tag{11}$$

The system of (10)-(11) will represent, in the sequel, the dynamics of the estimation error for the system (1). The following technical lemma will be needed in order to provide convex characterization of the induced $\ell_\infty-$ norm of the estimation error system:

Lemma 1. The system

$$\begin{array}{l} \bar{x}(k+1) = \bar{A}(k)\bar{x}(k) + \bar{B}w(k), \quad x(0) = x_0 \\ z(k) = \bar{C}(k)\bar{x}(k) + \bar{D}\bar{w}(k) \end{array} \tag{12}$$

satisfies

$$sup_{w\in\ell_\infty} ||z||_\infty / ||w||_\infty < \gamma \tag{13}$$

if the following matrix inequalities are satisfied for $i = 1,2,\ldots,N$:

$$\begin{bmatrix} \bar{A}_i^T P \bar{A}_i + \lambda P - P & \bar{A}_i^T P \bar{B}_i \\ \bar{B}_i^T P \bar{A}_i & -\mu I + \bar{B}_i^T P \bar{B}_i \end{bmatrix} < 0 \quad (14)$$

and

$$\begin{bmatrix} \lambda P & 0 & \bar{C}_i^T \\ 0 & (\gamma - \mu) I & \bar{D}_i^T \\ \bar{C}_i & \bar{D}_i & \gamma I \end{bmatrix} > 0 \quad (15)$$

so that $P > 0$, $\mu > 0$ and $\lambda < 1$.

The proof of this lemma is given in (Shaked and Yaesh, 2007) and is also provided, for the sake of completeness, in Appendix A.

Remark. Note that (14) can be written, using Schur complements ((Boyd et al., 1994)), as follows:

$$\begin{bmatrix} P - \lambda P & 0 & \bar{A}_i^T P \\ 0 & \mu I & \bar{B}_i^T P \\ P\bar{A}_i & P\bar{B}_i & P \end{bmatrix} > 0 \quad (16)$$

or equivalently as

$$\begin{bmatrix} P - \lambda P & 0 & \bar{A}_i^T \\ 0 & \mu I & \bar{B}_i^T \\ \bar{A}_i & \bar{B}_i & P^{-1} \end{bmatrix} > 0 \quad (17)$$

The fact that the inequality (16) is affine in $\bar{A}_i$ and $\bar{B}_i$ will be utilized in the sequel to obtain convex characterization (i.e. in LMI form) of the filter parameters K_i.

3 GAIN SCHEDULED FILTERING

Defining the state estimation error to be:

$$e(k) = x(k) - \hat{x}(k) \quad (18)$$

we readily have for the case where f_i are available for the estimation process, that

$$e(k+1) = (A - K(k)C)e(k) + (B - K(k)D)w(k) \quad (19)$$

and

$$z(k) - \hat{z}(k) = Le(k) \quad (20)$$

We substitute $\bar{A}_i = A_i - K_iC$, $\bar{B}_i = B_i - K_iD$ and $\bar{C} = L_i$ in (14) and (15) where we restrict our attention to the case where C and D are not vertex dependent (i.e. $C_i = C, D_i = D, i = 1,2,\ldots,N$). In this case, we define $Y_i = PK_i$ and readily obtain from (16) and (15) that

$$\begin{bmatrix} P - \lambda P & 0 & A_i^T P - C^T Y_i^T \\ 0 & \mu I & B_i^T P - D^T Y_i^T \\ PA_i - Y_iC & PB_i - Y_iD & P \end{bmatrix} > 0 \quad (21)$$

and

$$\begin{bmatrix} \lambda P & 0 & L_i^T \\ 0 & (\gamma - \mu) I & 0 \\ L_i & 0 & \gamma I \end{bmatrix} > 0, \quad \lambda < 1 \quad (22)$$

We, therefore, obtain the following result:

Theorem 1. Consider the estimator of (12) for the system of (1) with $C_i = C, D_i = D, i = 1,2,\ldots,N$. The estimation error satisfies (9) if (21) and (22) are satisfied for $i = 1,2,\ldots,N$ so that $P > 0$, $\mu > 0$ and $\lambda < 1$.

We next address the problem where C and D are vertex dependent. To this end we consider a version of Lemma 1 which can be written in terms of Ω rather than Ω_i, namely we replace (16) and (14) by:

$$\begin{bmatrix} P - \lambda P & 0 & \bar{A}^T P \\ 0 & \mu I & \bar{B}^T P \\ P\bar{A} & P\bar{B} & P \end{bmatrix} > 0 \quad (23)$$

and

$$\begin{bmatrix} \lambda P & 0 & \bar{C}^T \\ 0 & (\gamma - \mu) I & \bar{D}^T \\ \bar{C} & \bar{D} & \gamma I \end{bmatrix} > 0 \quad (24)$$

and substitute $\bar{A} = \sum_{i,j=1}^N (A_i - K_iC_j) f_i f_j, \bar{B}_i = \sum_{i,j=1}^N (B_i - K_iD_j) f_i f_j$ and $\bar{C} = \sum_{i=1}^N L_i f_i$. We obtain defining $Y_i = PK_i$:

$$\sum_{i,j=1}^N G_{ij} f_i f_j > 0 \quad (25)$$

where

$$G_{ij} := \begin{bmatrix} P - \lambda P & 0 & A_i^T P - C_j^T Y_i^T \\ 0 & \mu I & B_i^T P - D_j^T Y_i^T \\ PA_i - Y_iC_j & PB_i - Y_iD_j & P \end{bmatrix} \quad (26)$$

and

$$\sum_{i=1}^N \begin{bmatrix} \lambda P & 0 & L_i^T \\ 0 & (\gamma - \mu) I & 0 \\ L_i & 0 & \gamma I \end{bmatrix} f_i > 0 \quad (27)$$

Since, however (see (Tanaka and Wang, 2001)) equation (25) can be also written as

$$\sum_{i,j=1}^N G_{ij} f_i f_j = \sum_{i=1}^N G_{ii} f_i^2 + 2 \sum_{i=1}^N \sum_{i<j} \frac{G_{ij} + G_{ji}}{2} f_i f_j \quad (28)$$

Defining a simple transformation of the convex coordinates f_k so that for $k = 1,2,\ldots,N$ we set $h_k = f_k^2$ where as the remaining h_k for $k = N+1, N+2, \ldots, N + \frac{N(N-1)}{2}$ are defined by $h_k = 2 f_i f_j,\ j = 1,2\ldots,N, i < j$. Since obviously $\sum_{k=1}^{N+\frac{N(N-1)}{2}} h_k = 1$ where $h_k \geq 0$ they can serve as convex coordinates. We, therefore, define the following LMIs inspired by (Tanaka and Wang, 2001),

$$G_{ii} > 0, i = 1,2,\ldots N \text{ and } G_{ij} + G_{ji} > 0, i < j \quad (29)$$

and obtain the following result:

Theorem 2. Consider the estimator (12) for the system (1). The estimation error satisfies (9) if (29) and (22) for $i = 1,2,...,N$ are satisfied so that $P > 0$, $\mu > 0$ and $\lambda < 1$.

The solution offered above, for the case where C and D are uncertain and are known to reside in a given polytope, seeks a single matrix P that solves the LMIs for $\frac{N(N+1)}{2}$ vertices, instead of the N vertices that were solved for in the case of known C and D. A solution for such large number of vertices by a single P entails a significant overdesign. Even the relaxation offered by e.g. (Shaked, 2003) to reduce the overdesign by allowing different $P_i, i = 1,2,..., \frac{N(N+1)}{2}$ for the $\frac{N(N+1)}{2}$ vertices still suffers from a considerable conservatism. Moreover, the computational complexity of the solution also rapidly increases as a function of the number of vertices.

In many cases, C resides in some uncertainty polytope, while D is fixed and known. In such a case, an alternative way to deal with the problem is to define $\xi(k) = col\{x(k),\ y(k)\}$ and $\tilde{w}(k) = col\{w(k),\ w(k+1)\}$so that the augmented system becomes:

$$\begin{array}{l} \xi(k+1) = \tilde{A}(k)x(k) + \tilde{B}\tilde{w}(k) \\ y(k) = \tilde{C}(k)\xi(k) + \tilde{D}\tilde{w}(k) \\ z(k) = \tilde{L}(k)\xi(k) \\ \text{where} \\ \tilde{A} = \begin{bmatrix} A & 0 \\ CA & 0 \end{bmatrix} \quad \tilde{B} = \begin{bmatrix} B & 0 \\ CB & D \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} 0 & I_r \end{bmatrix}, \\ \quad \tilde{L} = \begin{bmatrix} L & 0 \end{bmatrix}, \text{ and } \tilde{D} = \begin{bmatrix} D & 0 \end{bmatrix} \end{array} \tag{30}$$

In (30) the uncertainties appear in $\bar{A}$ and $\bar{B}$ only and, therefore, Theorem 1 above may be invoked. We, therefore, obtain the following result which offers reduced conservatism with respect the corresponding continuous-time results of (Salcedo and Martinez, 2008):

Theorem 3. Consider the estimator of (12) for the system of (1) for $D_i = D, i = 1,2,...,N$. The estimation error satisfies (9) with γ replaced by $\sqrt{2}\gamma$ if (21) and (22) are satisfied for $i = 1,2,...,N$ so that $P > 0$, $\mu > 0$ and $\lambda < 1$ with A,B,C,L replaced by $\tilde{A},\tilde{B},\tilde{C},\tilde{L}$ of (30).

4 EXAMPLE

We consider the dynamic model of guidance in a plane:

$$\begin{array}{c} \dot{\tilde{x}} = \nu cos(\tilde{\psi}) + w_1 \\ \dot{\tilde{y}} = \nu sin(\tilde{\psi}) + w_2 \\ \dot{\tilde{\psi}} = \tilde{\phi} \\ \dot{\phi} = -\phi/\tau + u/\tau \end{array} \tag{31}$$

where $\tilde{x}$ and $\tilde{y}$ are the first two coordinates of a flight vehicle cruising in a constant altitude, in a local level north-east-down system, $\tilde{\psi}$ is the vehicle body angle with respect to the north (i.e. azimuth angle) and ϕ is the vehicle's roll angle assumed to be governed by a first-order low-pass filter dynamics having a time-constant of τ seconds, driven by the roll-angle command u. The wind velocities at the north and east directions respectively are denoted by w_1 and w_2 whereas ν is the true-air-speed. Our aim is to filter the noisy measurements of $\tilde{x}$, $\tilde{y}$ and ϕ and to estimate $\tilde{\psi}$. Defining,

$$x = col\{\tilde{x}, \tilde{y}, vsin(\tilde{\psi}), vcos(\tilde{\psi}), \phi\}$$

the measurements vector which consists of noisy measurements of the position components $\tilde{x}$ and $\tilde{y}$ and the roll angle ϕ is given by

$$y = Cx + R^{1/2}v$$

where v is the measurement noise which is taken in the simulations in the sequel as a 3-vector of zero-mean unity variance white noise sequences but for all practical purposes is assumed to be $v \in \ell_\infty$. The noise level is set by

$$R = diag\{25, 25, 0.1\}$$

and the measurement matrix is

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Note that

$$\begin{array}{c} \dot{x_1} = x_4 \\ \dot{x_2} = x_3 \\ \dot{x_3} = x_4 x_5 \\ \dot{x_4} = -x_3 x_5 \\ \dot{x_5} = -x_5/\tau + u/\tau \end{array} \tag{32}$$

namely we have a bilinear system rather than a linear one. Following (Tanaka and Wang, 2001) with a series of simple manipulations, this system can be represented as a Takagi-Sugeno fuzzy system, namely as a convex combination of linear systems where the convex coordinates are online measured. To achieve such a representation we recall that $x_5 = \phi$ is measured on line, and define $s_1 = x_5$ while neglecting the small enough noise in measuring ϕ. The validity of the latter assumption will be verified in the sequel by the estimation quality we will obtain. Assuming $x_5 \in [-\phi_{max}, \phi_{max}]$ we define $f_1 = \frac{s_1 - s_{1,min}}{s_{1,max} - s_{1,min}} =$

$\frac{x_5+\phi_{max}}{2\phi_{max}}$, $f_2 = 1 - f_1$, $\alpha_1 = \phi_{max}$ and $\alpha_2 = -\phi_{max}$. We readily see that $s_1 = \phi_{max} f_1 - \phi_{max} f_2 := \alpha_1 f_1 + \alpha_2 f_2$. We note then that the system is then governed by $\dot{\xi} = A_c(\xi)\xi + B_c w$ where $\xi = col\{x_1, x_2, x_3, x_4, x_5\}$ and

$$A_c = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & s_1 & 0 \\ 0 & 0 & -s_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1/\tau \end{bmatrix}$$

Therefore, $A_c = A_{c,1}\alpha_1 + A_{c,2}\alpha_2$ where $A_{c,1}$ is obtained from A_c by replacing s_1 by α_1 and $A_{c,2}$ is similarly obtained from A_c by replacing s_1 by α_2. We also define $w = col\{w_1, w_2, \nu_1, \nu_2, \nu_3\}$ and complete the remaining matrices needed for the representation of our problem (1)-(3) by applying a zero-order-hold discrete-time equivalent of our continuous-time system, where we have chosen a sampling time of $h = 0.02$. Due to the small enough h we have chosen, we have $e^{Ah} = I + Ah + O(h^2)$ and we, therefore, readily obtain that the system is governed by (1) and (3)-(5) where $A = A_1\alpha_1 + A_2\alpha_2 + O(h^2)$ where

$$A_1 = \begin{bmatrix} 1.0000 & -0.0002 & 0.0200 & 0 & \\ 0 & 1.0000 & 0.0200 & 0.0002 & 0 \\ 0 & 0 & 0.9998 & 0.0209 & 0 \\ 0 & 0 & -0.0209 & 0.9998 & 0 \\ 0 & 0 & 0 & 0 & 0.9231 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1.0000 & 0 & 0.0002 & 0.0200 & 0 \\ 0 & 1.0000 & 0.0200 & -0.0002 & 0 \\ 0 & 0 & 0.9998 & -0.0209 & 0 \\ 0 & 0 & 0.0209 & 0.9998 & 0 \\ 0 & 0 & 0 & 0 & 0.9231 \end{bmatrix}$$

$$B_1 = B_2 = B = \begin{bmatrix} 0.2000 & 0 & 0 & 0 & 0 \\ 0 & 0.2000 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$D_1 = D_2 = D = \begin{bmatrix} 0 & 0 & 2.2361 & 0 & 0 \\ 0 & 0 & 0 & 2.2361 & 0 \\ 0 & 0 & 0 & 0 & 0.1000 \end{bmatrix}$$

We note at his point that, in order to minimize the design conservatism stemming from the quadratic stability assumption, we applied a parameter dependent Lyapunov function (Boyd et al., 1994), $max(x^T P_1 x, x^T P_2 x)$. Minimization of γ subject the the LMis that are obtained with this function to replace (21) and (22) (see Appendix B), using *fminsearch.m* from the optimization toolbox of $MATLAB^{TM}$ and (Lagarias et al., 1998) to search $\lambda_i,\ i = 1,2,3,4,\ \rho_1,\ \rho2_2,\ \theta_1$ and θ_2, has resulted in $\gamma = \gamma_0 = 10.2732$ and $\lambda = 2.41 \times 10^{-7}$. The following gain matrices K_1 and K_2 have been obtained for $\gamma = \gamma_0$ were obtained:

$$K_1 = \begin{bmatrix} 0.7574 & -0.0007 & 0.0000 \\ -0.0018 & 0.7593 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & -0.0000 & 0.0000 \\ -0.0000 & -0.0000 & 0.0003 \end{bmatrix}$$

$$K_2 = \begin{bmatrix} 0.7558 & -0.0024 & 0.0000 \\ -0.0021 & 0.7613 & -0.0000 \\ -0.0000 & 0.0000 & -0.0000 \\ 0.0000 & 0.0000 & 0.0000 \\ -0.0000 & 0.0000 & 0.0003 \end{bmatrix}$$

This ℓ_∞ filter will be compared to an Extended Kalman Filter (EKF, see (Jazwinsky, 1970)) based on the nonlinear model of (31). Note that higher complexity filters such as the particle filter (e.g. (Oshman and Carmi, 2006) are out the scope of the present paper. For the simulations we take $\nu = 100m/s$ and try to control the vehicle to follow a constant command at $y = 5m$, in spite of a wind step at w_2 of $10m/s$. The estimation results are used to control the vehicle, using the simple law $u = -\begin{bmatrix} 0.0200 & 4.0000 \end{bmatrix} \begin{bmatrix} \hat{y} - 5 & \hat{\psi} \end{bmatrix}^T$ where all components of the initial state-vector are taken as zero, besides $y_0 = 20m$. The EKF and the ℓ_∞ estimated $\tilde{x} - \tilde{y}$ trajectory results are compared in Fig. 1 to the true trajectory. One can notice the bias in the EKF estimate. In Fig. 2, the true $\tilde{\psi}$ and the estimated values for $\tilde{\psi}$ that are obtained by using the EKF and the ℓ_∞ filter are depicted. We clearly see in this figure that the ℓ_∞ filter outperforms the EKF which assumes a white noise w_2 but leads to a bias when w_2 has a bias. In contrast, the $\tilde{\psi}$ estimate of the ℓ_∞ filter is barely separable from the true values. Moreover, the ℓ_∞ filter does not require the on-line numerical solution of a Riccati equation of order 4 and the gains are obtained there by a simple convex interpolation on K_1 and K_2. The fact that K_1 and K_2 are close to each other is somewhat surprising. Our experience shows that for a larger γ (i.e. suboptimal values), a larger $||K_1 - K_2||$ is obtained.

5 CONCLUSIONS

The problem of designing robust gain-scheduled filters with guaranteed induced $\ell_\infty-$norm has been considered. The solution has been derived using a recently developed bounded-real-lemma like condition for bounding the induced ℓ_∞ norm of a system. This result has been applied to derive the robust induced $\ell_\infty-$filter (or equivalently robust ℓ_1 filter) in terms of LMIs. These LMIs have been applied to a guidance motivated estimation example. In this example, the

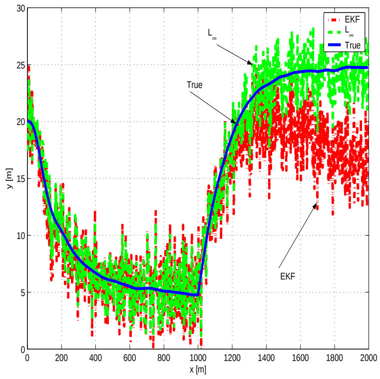

Figure 1: True, Extended Kalman Filter and ℓ_∞-Filter Estimated Trajectories - $\tilde{x}(t)$ versus $\tilde{y}(t)$.

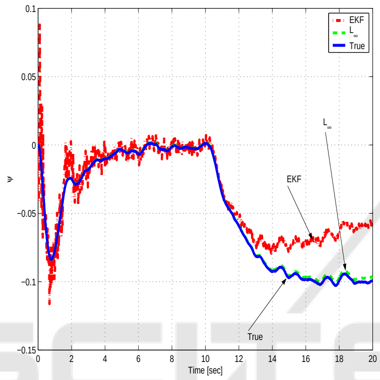

Figure 2: True, Extended Kalman Filter and ℓ_∞-Filter Estimated Azimuth Angle - $\tilde{\psi}$ versus t.

superiority of the induced ℓ_∞ filter over the Extended Kalman Filter has been demonstrated, both in terms of performance and simplicity of implementation.

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APPENDIX A - PROOF OF LEMMA 1

Consider the system

$$x_{k+1} = \bar{A}x_k + \bar{B}w_k, \quad z_k = \bar{C}x_k + \bar{D}w_k$$

and define, following (Abedor et al., 1996),

$$\xi_k = x_{k+1}^T P x_{k+1} - x_k^T P x_k + \lambda x_k^T P x_k - \mu w_k^T w_k.$$

Namely,

$$\xi_k = (x_k^T \bar{A}^T + w_k^T \bar{B}^T) P (\bar{A}x_k + \bar{B}w_k) - x_k^T P x_k + \lambda x_k^T P x_k - \mu w_k^T w_k.$$

Collecting terms we have

$$\xi_k = x_k^T(\bar{A}^T P\bar{A} + \lambda P - P)x_k + x_k^T(\bar{A}^T P\bar{B})w_k + w_k^T(\bar{B}^T P\bar{A})x_k + w_k^T(-\mu I + \bar{B}^T P\bar{B})w_k.$$

Therefore, (14) guarantees $\xi_k < 0$ for all w_k and x_k.

Defining $\zeta_k = x_k^T P x_k$ and assuming $x_0 = 0$ and $w_k^T w_k < 1$ we have that $\zeta_{k+1} - \zeta_k + \lambda\zeta_k - \mu w_k^T w_k < 0$. Namely, $\zeta_k < \bar{\zeta}_k$ where $\bar{\zeta}_{k+1} = (1-\lambda)\bar{\zeta}_k + \mu w_k^T w_k$. However, using $\rho := 1 - \lambda$ we have

$$\bar{\zeta}_k = \sum_{j=0}^{k-1} \rho^{k-j-1}\mu w_j^T w_j = \rho^{k-1}\sum_{j=0}^{k-1}(\rho^{-1})^j \mu w_j^T w_j$$

$$< \rho^{k-1}\frac{(1-\rho^{-1})^k}{1-\rho^{-1}}\mu = \mu\frac{1-\rho^k}{1-\rho}. \qquad \textbf{A.1}$$

From (15) we have using Schur complements that

$$[x^T \ w^T]\left(\begin{bmatrix}\lambda P & 0\\ 0 & (\gamma-\mu)I\end{bmatrix} - \gamma^{-1}\begin{bmatrix}\bar{C}^T\\ \bar{D}^T\end{bmatrix}\begin{bmatrix}\bar{C} & \bar{D}\end{bmatrix}\right)\begin{bmatrix}x\\ w\end{bmatrix} > 0$$

Namely,

$$z_k^T z_k < \gamma[(\gamma-\mu)w_k^T w_k + \lambda x_k^T P x_k] < \gamma[(\gamma-\mu) + \lambda\bar{\zeta}_k] \quad \textbf{A.2}$$

Substituting A.1 we readily see that

$$z_k^T z_k < \gamma[(\gamma-\mu) + (1-\rho)\mu\frac{1-\rho^k}{1-\rho}] = \gamma[\gamma-\mu+\mu-\mu\rho^k].$$

Since $0 < \rho < 1$ we obtain that

$$z_k^T z_k < \gamma[\gamma-\mu+\mu] = \gamma^2.$$

APPENDIX B - PARAMETER DEPENDENT RESULTS

In order to reduce conservatism, we replace in the proof of Lemma 1 in Appendix A, the parameter-independent Lyapunov function $V(x,P) = x_k^T P x_k$ by the parameter-dependent Lyapunov function ((Boyd et al., 1994)) $V(x,P_1,P_2) = max(x_k^T P_1 x_k, x_k^T P_2 x_k)$. To ensure $V(x_k,P_1,P_2) > 0$ we have to satisfy $x_k^T P_1 x_k > 0$ whenever $x_k^T P_1 x_k > x_k^T P_2 x_k$ and $x_k^T P_2 x_k > 0$ whenever $x_k^T P_1 x_k < x_k^T P_2 x_k$. Applying the S-procedure (Boyd et al., 1994) we readily obtain that a sufficient condition or these requirements to hold, is the existence of constants $\rho_1 > 0$ and $\rho_2 > 0$ so that

$$P_1 - \rho_1(P_1 - P_2) > 0 \text{ and } P_2 - \rho_2(P_2 - P_1) > 0.$$

We also require that if $x_k^T P_1 x > x_k^T P_2 x_k$ then

$$\xi_k^{\{1\},\bar{A}} := x_k^T(\bar{A}^T P_1\bar{A} + \lambda P_1 - P_1)x_k + x_k^T(\bar{A}^T P_1\bar{B})w_k + w_k^T(\bar{B}^T P_1\bar{A})x_k + w_k^T(-\mu I + \bar{B}^T P_1\bar{B})w_k < 0,$$

and if $x_k^T P_2 x > x_k^T P_1 x_k$ then

$$\xi_k^{\{2\},\bar{A}} := x_k^T(\bar{A}^T P_2\bar{A} + \lambda P_2 - P_2)x_k + x_k^T(\bar{A}^T P_2\bar{B})w_k + w_k^T(\bar{B}^T P_2\bar{A})x_k + w_k^T(-\mu I + \bar{B}^T P_2\bar{B})w_k < 0.$$

Since these conditions are required to be satisfied throughout the polytope, we readily obtain, using again the S-procedure, that in addition to the constant $\lambda > 0$, the existence of six additional constants $\lambda_i > 0$, $i = 1,2,3,4$, $\theta_1 > 0$, $\theta_2 > 0$ establishes a sufficient condition for the above inequalities to hold, if

$$-\xi_k^{\{1\},\bar{A}_i} - \lambda_1(P_1 - P_2) > 0, i = 1,2$$

and

$$-\xi_k^{\{1\},\bar{A}_i} - \lambda_2(P_2 - P_1) > 0, i = 1,2.$$

Following the lines of proof of Theorem 1 above, we readily obtain the following LMIs for $i = 1,2$ to replace (21) and (22):

$$\begin{bmatrix} P_1 - \lambda P - \lambda_i(P_1 - P_2) & 0 & A_i^T P_1 - C^T Y_i^T\\ 0 & \mu I & B_i^T P_1 - D^T Y_i^T\\ P_1 A_i - Y_i C & P_1 B_i - Y_i D & P_1 \end{bmatrix} > 0,$$

$$\begin{bmatrix} \lambda P_1 - \theta_1(P_1 - P_2) & 0 & L_i^T\\ 0 & (\gamma-\mu)I & 0\\ L_i & 0 & \gamma I \end{bmatrix} > 0, \quad \lambda < 1$$

and

$$\begin{bmatrix} P_2 - \lambda P - \lambda_{i+2}(P_2 - P_1) & 0 & A_i^T P_2 - C^T Y_i^T\\ 0 & \mu I & B_i^T P_2 - D^T Y_i^T\\ P_2 A_i - Y_i C & P_2 B_i - Y_i D & P_2 \end{bmatrix} > 0,$$

$$\begin{bmatrix} \lambda P_2 - \theta_2(P_2 - P_1) & 0 & L_i^T\\ 0 & (\gamma-\mu)I & 0\\ L_i & 0 & \gamma I \end{bmatrix} > 0, \quad \lambda < 1.$$

We note that we have also replaced (A.2) with:

$$z_k^T z_k < \gamma[(\gamma-\mu)w_k^T w_k + \lambda \times max(x_k^T P_1 x_k, x_k^T P_2 x_k)].$$

Namely, if $x^T P_1 x > x^T P_2 x$ we require

$$z_k^T z_k < \gamma[(\gamma-\mu)w_k^T w_k + \lambda x_k^T P_1 x],$$

whereas if $x^T P_2 x > x^T P_1 x$ we require

$$z_k^T z_k < \gamma[(\gamma-\mu)w_k^T w_k + \lambda x_k^T P_2 x].$$

Using again the $\mathcal{S}$-Procedure with additional tuning constants $\theta_1 > 0$ and $\theta_2 > 0$, which add up to the previously introduced 7 tuning constants $\rho_1 > 0$, $\rho_2 > 0$, $\lambda > 0$ and $\lambda_i > 0, i = 1,2,3,4$ the above results are obtained. Note that a similar approach can be applied also on the continuous-time results of (Salcedo and Martinez, 2008) to reduce conservatism.