THE GENERALIZED HYBRID AVERAGING OPERATOR AND ITS APPLICATION IN FINANCIAL DECISION MAKING

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Abstract: We present the generalized hybrid averaging (GHA) operator. It is a new aggregation operator that generalizes the hybrid averaging (HA) operator by using the generalized mean. Then, we are able to generalize a wide range of mean operators such as the HA, the hybrid quadratic averaging (HQA), etc. The HA is an aggregation operator that includes the ordered weighted averaging (OWA) operator and the weighted average (WA). Then, with the GHA, we are able to get all the particular cases obtained by using generalized means in the OWA and in the WA such as the weighted geometric mean, the ordered weighted geometric (OWG) operator, the weighted quadratic mean (WQM), etc. We further generalize the GHA by using quasi-arithmetic means. Then, we obtain the quasi-arithmetic hybrid averaging (Quasi-HA) operator. Finally, we apply the new approach in a financial decision making problem.

1 INTRODUCTION

Different types of aggregation operators are found in the literature for aggregating the information. A very common aggregation method is the ordered weighted averaging (OWA) operator (Yager, 1988). It provides a parameterized family of aggregation operators that includes as special cases the maximum, the minimum and the average criteria. Since its appearance, the OWA operator has been used in a wide range of applications (Calvo et al., 2002; Merigó, 2007; Yager, 1993; Yager and Kacprzyk, 1997).

In 2003, Xu and Da introduced the hybrid averaging (HA) operator. It is an aggregation operator that uses the weighted average (WA) and the OWA operator at the same time. Then, it is able to consider in the same problem the attitudinal character of the decision maker and the subjective probablity. For further research on the HA operator, see (Merigó, 2007; Xu, 2004; 2006).

Another interesting aggregation operator is the generalized OWA (GOWA) operator (Karayiannis, 2000; Yager, 2004). It generalizes the OWA operator by using generalized means. Then, it includes as special cases, the maximum, the minimum and the average criteria, and a wide range of other means such as the OWA operator itself, the ordered weighted geometric (OWG) operator, etc. The GOWA operator has been further generalized by using quasi-arithmetic means (Beliakov, 2005) obtaining the Quasi-OWA operator (Fodor et al., 1995). For further research on the GOWA operator, see (Merigó, 2007; Merigó and Casanovas, 2007; Merigó and Gil-Lafuente, 2007).

In this paper, we introduce the generalized hybrid averaging (GHA) operator. It generalizes the HA operator by using generalized means. Then, it includes in the same formulation all the cases coming from the generalized mean. As a result, we obtain new aggregation operators such as the hybrid geometric averaging (HGA) operator, the hybrid quadratic averaging (HQA) operator, etc. We further generalize the GHA operator by using quasi-arithmetic means, obtaining the quasi-HA operator. We also develop an application of the new approach in a financial decision making problem where we can see how it can be implemented in the real life.

In order to do so, this paper is organized as follows. In Section 2, we briefly review some basic aggregation operators. In Section 3, we present the GHA operator. Section 4 studies different families of GHA operators. Section 5 develops an application of the new approach in a financial decision making problem. Finally, in Section 6 we summarize the main conclusions found in the paper.

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2 AGGREGATION OPERATORS

2.1 Hybrid Averaging Operator

The HA operator (Xu and Da, 2003) is an aggregation operator that uses the WA and the OWA operator in the same formulation. It can be defined as follows.

Definition 1. An HA operator of dimension n is a mapping $HA:R^n \to R$ that has an associated weighting vector W of dimension n such that the sum of the weights is 1 and $w_j \in [0,1]$, then:

$$HA(a_1, a_2\ldots, a_n) = \sum_{j=1}^{n} w_j b_j \qquad (1)$$

where b_j is the jth largest of the $\hat{a}_i$ ($\hat{a}_i = n\omega_i a_i$, $i = 1,2,\ldots,n$), $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weighting vector of the a_i, with $\omega_i \in [0, 1]$ and the sum of the weights is 1.

2.2 Generalized OWA Operator

The GOWA operator (Karayiannis, 2000; Yager 2004) is a generalization of the OWA operator by using generalized means. It is defined as follows.

Definition 2. A GOWA operator of dimension n is a mapping $GOWA:R^n \to R$ that has an associated weighting vector W of dimension n such that the sum of the weights is 1 and $w_j \in [0,1]$, then:

$$GOWA(a_1, a_2,\ldots, a_n) = \left(\sum_{j=1}^{n} w_j b_j^{\lambda}\right)^{1/\lambda} \qquad (2)$$

where b_j is the jth largest of the a_i, and λ is a parameter such that $\lambda \in (-\infty, \infty)$.

3 THE GENERALIZED HYBRID AVERAGING OPERATOR

The GHA operator is a generalization of the HA operator by using generalized means. It includes in the same formulation the weighted generalized mean and the GOWA operator. Then, this operator includes the WA, the OWA and the OWG operator as special cases. It is defined as follows.

Definition 3. A GHA operator of dimension n is a mapping $GHA:R^n \to R$ that has an associated weighting vector W of dimension n such that the sum of the weights is 1 and $w_j \in [0,1]$, then:

$$GHA(a_1, a_2\ldots, a_n) = \left(\sum_{j=1}^{n} w_j b_j^{\lambda}\right)^{1/\lambda} \qquad (3)$$

where b_j is the jth largest of the $\hat{a}_i$ ($\hat{a}_i = n\omega_i a_i$, $i = 1,2,\ldots,n$), $\omega = (\omega_1, \ldots, \omega_n)^T$ is the weighting vector of the a_i, with $\omega_i \in [0, 1]$ and the sum of the weights is 1, and λ is a parameter such that $\lambda \in (-\infty, \infty)$.

From a generalized perspective of the reordering step, we can distinguish between the descending GHA (DGHA) operator and the ascending GHA (AGHA) operator. The weights of these operators are related by $w_j = w^*_{n-j+1}$, where w_j is the jth weight of the DGHA and w^*_{n-j+1} the jth weight of the AGHA operator.

The GHA operator is monotonic, commutative and idempotent. Note that this operator is not bounded by the maximum and the minimum because for some special situations it can be higher and lower than them.

Another interesting issue to consider are the measures for characterizing the weighting vector W of the GHA operator such as the attitudinal character, the entropy of dispersion, the divergence of W and the balance operator (Merigó, 2007).

4 FAMILIES OF GHA OPERATORS

In the GHA operator we find different families of aggregation operators. Mainly, we can classify them in two types. The first type represents all the families found in the weighting vector W and the second type, the families found in the parameter λ.

4.1 Analysing the Weighting Vector *W*

By choosing a different manifestation of the weighting vector in the GHA operator, we are able to obtain different types of aggregation operators. For example, we can obtain the hybrid maximum, the hybrid minimum, the generalized mean (GM), the weighted generalized mean (WGM) and the GOWA operator.

The hybrid maximum is obtained if $w_1 = 1$ and $w_j = 0$, for all $j \neq 1$. The hybrid minimum is obtained if $w_n = 1$ and $w_j = 0$, for all $j \neq n$. More generally, if $w_k = 1$ and $w_j = 0$, for all $j \neq k$, we get for any λ, $GHA(a_1, a_2\ldots, a_n) = b_k$, where b_k is the kth largest argument a_i. The GM is found when $w_j = 1/n$, and $\omega_i = 1/n$, for all a_i. The WGM is obtained when $w_j = 1/n$, for all a_i. The GOWA is found when $\omega_i = 1/n$, for all a_i.

Following a similar methodology as it has been developed in (Merigó, 2007; Yager, 1993), we could study other particular cases of the GHA operator such as the step-GHA, the window-GHA, the olympic-GHA, the S-GHA operator, the median-GHA, the maximal entropy GHA weights, the minimal variability GHA, etc.

For example, when $w_{j^*} = 1/m$ for $k \leq j^* \leq k + m - 1$ and $w_{j^*} = 0$ for $j^* > k + m$ and $j^* < k$, we are using the window-GHA operator. Note that k and m must be positive integers such that $k + m - 1 \leq n$.

The olympic-GHA, based on the olympic average (Yager, 1996), is found when $w_1 = w_n = 0$, and for all others $w_{j^*} = 1/(n - 2)$. Note that if $n = 3$ or $n = 4$, the olympic-GHA is transformed in the median-GHA and if $m = n - 2$ and $k = 2$, the window-GHA is transformed in the olympic-GHA.

We note that the median can also be used as GHA operators. For the median-GHA, if n is odd we assign $w_{(n+1)/2} = 1$ and $w_{j^*} = 0$ for all others. If n is even we assign for example, $w_{n/2} = w_{(n/2)+1} = 0.5$ and $w_{j^*} = 0$ for all others.

For the weighted median-GHA, we select the argument b_k that has the kth largest argument such that the sum of the weights from 1 to k is equal or higher than 0.5 and the sum of the weights from 1 to $k - 1$ is less than 0.5.

A further interesting family is the S-GHA operator based on the S-OWA operator (Yager, 1993; Yager and Filev, 1994). It can be subdivided in three classes: the "orlike", the "andlike" and the generalized S-GHA operator. The "orlike" S-GHA operator is found when $w_1 = (1/n)(1 - \alpha) + \alpha$, and $w_j = (1/n)(1 - \alpha)$ for $j = 2$ to n with $\alpha \in [0, 1]$. The "andlike" S-GHA operator is found when $w_n = (1/n)(1 - \beta) + \beta$ and $w_j = (1/n)(1 - \beta)$ for $j = 1$ to $n - 1$ with $\beta \in [0, 1]$. Finally, the generalized S-GHA operator is obtained when $w_1 = (1/n)(1 - (\alpha + \beta)) + \alpha$, $w_n = (1/n)(1 - (\alpha + \beta)) + \beta$, and $w_j = (1/n)(1 - (\alpha + \beta))$ for $j = 2$ to $n - 1$ where $\alpha, \beta \in [0, 1]$ and $\alpha + \beta \leq 1$. Note that if $\alpha = 0$, the generalized S-GHA operator becomes the "andlike" S-GHA operator and if $\beta = 0$, it becomes the "orlike" S-GHA operator.

Other families of GHA operators could be studied such as the centered-GHA, the EZ-GHA weights, the Gaussian GHA weights, the nonmonotonic GHA operator, etc. For more information, see (Merigó, 2007).

4.2 Analysing the Parameter λ

If we analyze different values of the parameter λ, we obtain another group of particular cases such as the usual HA, the hybrid geometric averaging (HGA), the hybrid harmonic averaging (HHA) and the hybrid quadratic averaging (HQA) operator.

When $\lambda = 1$, we get the HA operator.

$$GHA(a_1, a_2\ldots, a_n) = \sum_{j=1}^{n} w_j b_j \quad (4)$$

From a generalized perspective of the reordering step we can distinguish between the DHA operator and the AHA operator. Note that if $w_j = 1/n$, for all a_i, we get the WA and if $\omega_j = 1/n$, for all a_i, we get the OWA operator. If $w_j = 1/n$, and $\omega_j = 1/n$, for all a_i, then, we get the arithmetic mean (AM).

When $\lambda = 0$, we get the HGA operator.

$$GHA(a_1, a_2\ldots, a_n) = \prod_{j=1}^{n} b_j^{\,w_j} \quad (5)$$

Note that it is possible to distinguish between descending (DHGA) and ascending (AHGA) orders. Note that if $w_j = 1/n$, for all a_i, we get the WGM and if $\omega_j = 1/n$, for all a_i, we get the OWG operator.

When $\lambda = -1$, we get the HHA operator.

$$GHA(a_1, a_2\ldots, a_n) = \frac{1}{\sum_{j=1}^{n} \frac{w_j}{b_j}} \quad (6)$$

In this case, we get the descending HHA (DHHA) operator and the ascending HHA (AHHA) operator.

When $\lambda = 2$, we get the HQA operator.

$$GHA(a_1, a_2\ldots, a_n) = \left(\sum_{j=1}^{n} w_j b_j^2 \right)^{1/2} \quad (7)$$

In this case, we also get the descending HQA (DHQA) operator and the ascending HQA (AHQA) operator. If $w_j = 1/n$, for all a_i, we get the WQM and if $\omega_j = 1/n$, for all a_i, we get the OWQA operator. If $w_j = 1/n$, and $\omega_j = 1/n$, for all a_i, then, we get the quadratic mean (QM).

Note that we could analyze other families by using different values of the parameter λ. Also note that it is possible to study these families individually.

5 QUASI-ARITHMETIC MEANS IN THE HA OPERATOR

Going a step further, it is possible to generalize the GHA operator by using quasi-arithmetic means in a similar way as it was done for the GOWA operator (Beliakov, 2005). The result is the Quasi-HA operator which is a hybrid version of the Quasi-OWA operator (Fodor et. al., 1995). It can be defined as follows.

Definition 4. A Quasi-HA operator of dimension n is a mapping QHA: $R^n \rightarrow R$ that has an associated weighting vector W of dimension n such that the sum of the weights is 1 and $w_j \in [0,1]$, then:

$$\textit{Quasi-HA}(a_1, \ldots, a_n) = g^{-1}\left(\sum_{j=1}^{n} w_j g\left(b_{(j)}\right)\right) \qquad (8)$$

where b_j is the jth largest of the $\hat{a}_i$ ($\hat{a}_i = n\omega_i a_i$, $i = 1,2,\ldots,n$), $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weighting vector of the a_i, with $\omega_i \in [0, 1]$ and the sum of the weights is 1.

As we can see, we replace b^λ with a general continuous strictly monotone function $g(b)$. In this case, the weights of the ascending and descending versions are also related by $w_j = w^*_{n-j+1}$, where w_j is the jth weight of the Quasi-DHA and w^*_{n-j+1} the jth weight of the Quasi-AHA operator.

Note that all the properties and particular cases commented in the GHA operator, are also included in this generalization (Merigó, 2007).

6 APPLICATION IN FINANCIAL DECISION MAKING

Now, we are going to develop an application of the new approach in a decision making problem. We will analyze an investment selection problem where an investor is looking for an optimal investment.

We will develop the analysis considering a wide range of particular cases of the GHA operator such as the arithmetic mean (AM), the WA, the OWA, the OWQA, the HA, the AHA, the HQA and the HGA. Note that we do not consider the hybrid maximum and the hybrid minimum because sometimes its results are inconsistent.

Assume an investor wants to invest some money in an enterprise in order to get high profits. Initially, he considers five possible alternatives.

In order to evaluate these investments, the investor uses a group of experts. This group of experts considers that the key factor is the economic environment of the economy. After detailed analysis, they consider five possible situations for the economic environment: S_1 = Very bad, S_2 = Bad, S_3 = Normal, S_4 = Good, S_5 = Very good. The expected results depending on the state of nature S_i and the alternative A_k are shown in Table 1.

Table 1: Payoff matrix.

	S_1	S_2	S_3	S_4	S_5
A_1	30	60	50	80	20
A_2	30	30	90	60	40
A_3	70	40	50	20	60
A_4	50	70	30	40	50
A_5	90	10	10	70	70

In this example, we assume the following weighting vector for all the cases of the WA and the OWA operator: W = (0.1, 0.1, 0.2, 0.3, 0.3).

With this information, it is possible to aggregate it in order to take a decision. First, we consider the results obtained with some basic aggregation operators. The results are shown in Table 2.

Table 2: Aggregated results 1.

	Max	Min	AM	WA	OWA
A_1	80	20	48	49	39
A_2	90	30	50	54	44
A_3	70	20	48	45	41
A_4	70	30	48	45	40
A_5	90	10	50	54	36

As we can see, the optimal investment is different depending on the operator used.

In the following, we will consider other particular cases of the GHA operator with more complexity. The results are shown in Table 3.

Table 3: Aggregated results 2.

	OWQ	HA	AHA	HQA	HGA
A_1	43.4	36.5	61.5	46.9	29.4
A_2	45.0	39	69	49.7	28.3
A_3	44.1	36	54	41.1	32.1
A_4	44.6	37	53	40.3	34.4
A_5	48.3	34.5	73.5	51.4	17.5

Again, we can see that the optimal investment is not the same for all the aggregations used. Note that other types of GHA operators may be used in the analysis such as the ones explained in Section 4.

A further interesting issue is to establish an ordering of the investments. This is very useful when the investor wants to consider more than one alternative. The results are shown in Table 4.

Table 4: Ordering of the investments.

	Ordering
Max	$A_5 \succ A_3 \succ A_4 \succ A_1 = A_2$
Min	$A_2 = A_4 \succ A_1 = A_3 \succ A_5$
AM	$A_2 = A_5 \succ A_1 = A_3 = A_4$
WA	$A_2 = A_5 \succ A_1 \succ A_3 = A_4$
OWA	$A_2 \succ A_3 \succ A_4 \succ A_1 \succ A_5$
OWQA	$A_5 \succ A_2 \succ A_4 \succ A_3 \succ A_1$
HA	$A_2 \succ A_4 \succ A_1 \succ A_3 \succ A_5$
AHA	$A_5 \succ A_2 \succ A_1 \succ A_3 \succ A_4$
HQA	$A_5 \succ A_2 \succ A_1 \succ A_3 \succ A_4$
HGA	$A_4 \succ A_3 \succ A_1 \succ A_2 \succ A_5$

As we can see, we get different orderings of the investments depending on the aggregation operator used.

7 CONCLUSIONS

We have introduced the generalized hybrid averaging (GHA) operator. It is a generalization of the hybrid averaging (HA) operator by using generalized means. We have seen that it is very useful when we want to consider subjective probabilities and the attitudinal character of the decision maker in the same problem. With this generalization we have found different special cases such as the hybrid geometric averaging (HGA), the hybrid quadratic averaging (HQA), the WA, the OWA operator, the OWG operator, etc. We have further generalized the GHA operator by using quasi-arithmetic means. Then, we have obtained the quasi-HA operator.

We have ended the paper with an application of the new approach in a decision making problem. In this case, we have focussed in a financial problem where we have seen the usefulness of the new approach in the selection of investments.

In future research, we expect to develop further extensions to the GHA operator by adding new characteristics in the problem such as the use of inducing variables.

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