OPTIMAL NONLINEAR IMAGE DENOISING METHODS IN HEAVY-TAILED NOISE ENVIRONMENTS

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Abstract: The statistics for the neighbor differences between the particular pixels and their neighbors are introduced. They are incorporated into the filter to enhance images contaminated by additive Gaussian and impulsive noise. The derived denoising method corresponds to the maximum likelihood estimator for the heavy-tailed Gaussian distribution. The error norm corresponding to our estimator from the robust statistics is equivalent to Huber's minimax norm. This estimator is also optimal in the respect of maximizing the efficacy under the above noise environment. It is mixed with the myriad filter to propose an amplitude-limited myriad filter. In order to reduce visually grainy output due to impulsive noise, Impulse-like signal detection is introduced so that it can be processed in different manner from the remaining pixels. Our approaches effectively remove both Gaussian and impulsive noise, not blurring edges severely.

1 INTRODUCTION

Noise introduced into images via image acquisition devices such as digital cameras can be adequately assumed to be additive zero-mean Gaussian distributed. Such impulsive noise as caused by transmission of images can be more approximated as α stable distribution. In general, the noise with zero-mean and independent properties can be easily removed by locally averaging pixel values. A mean filter is known to be a maximum likelihood estimator for additive Gaussian noise and is optimal in the sense of minimizing mean square error. This filter, however, tends to degrade the sharpness of the boundaries between regions of an image although it effectively removes noise inside the smooth regions. Basically linear filters can not overcome this problem. That is why nonlinear methods should be employed for this purpose. One of the simplest nonlinear filtering algorithms is the median-based filter. It is a maximum likelihood estimator for Laplacian distribution. It has a relatively good property of preserving fine details except for thin lines and corners. It is known to be robust to impulsive noise. Stack filter, weighted median and relaxed median are among its variations to improve the performance. Median-based methods basically select one of the samples in the input window. Thus, it is known that they can not reduce noise effectively. Motivated by the above limitations, several kinds of myriad filters have been proposed, which are known to be maximum likelihood estimator under Cauchy distribution (Gonzalez, Arce, 2001), (Zurbach, et al., 1996). Optimality of myriad filters are presented under α stable distributions (Gonzalez, Arce, 2001). (Hamza and Krim, 2001) proposed mean-relaxed median and mean-LogCauchy filters by combining a mean filter with a relaxed median or LogCauchy filter. They are maximum likelihood estimators under the assumption that the noise probability distribution is a linear combination of normal distribution and heavy-tailed distribution such as Laplacian or Cauchy distribution. Another popular methods are the anisotropic diffusion techniques into which a variety of research has been devoted since the work of (Perona and Malik, 1990). Recent researches have shown that nonlinear methods such as median filters and anisotropic diffusions can be reinterpreted using the theory of robust statistics (Huber, 1981). Robust-statistics-based denoising algorithms are developed, which deal with intensity discontinuities to adapt the analysis window size (Rabie, 2005). He chose a Lorenzian redescending estimator in which the influence function tends to zero with increasing distance.

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A large number of image denoising algorithms proposed so far are limited to the case of Gaussian noise or impulsive noise, not to both of them. The algorithms tuned for Gaussian noise or impulsive noise alone present serious performance degradation in case images are corrupted with both kinds of noise. To tackle the problem, an amplitude-limited sample average filter is proposed. It is also a maximum likelihood estimator in the density function which is Gaussian on $(-\delta, \delta)$, but Laplacian outside the region. Its idea is incorporated into the myriad filter to propose an amplitude-limited myriad filter. In order to reduce visually grainy output due to impulsive noise, Impulse-like signal detection is introduced so that it can be processed in different manner from the remaining pixels. Our approaches effectively remove both Gaussian and impulsive noise, not blurring edges severely.

After reviewing the problems of finding the best estimate of a model in terms of maximum likelihood estimate (MLE), given a set of data measurements, our estimators are interpreted based on the theory of robust estimation in both Gaussian and impulsive noise environment.

2 NOISE STATISTICS

In deriving our robust denoising filter, we employ an observed image model corrupted with additive Gaussian and impulsive noise

$$y_i = x_i + n_i, \quad i \in \mathbb{Z}^2 \tag{1}$$

where n_i is a zero-mean additive white Gaussian noise plus impulsive noise. n_i is uncorrelated to the image sequence x_i and y_i is the observed noise-contaminated image sequence. In this case, n_i can be assumed to have a density function whose tails are heavier than Gaussian. To ensure the unbiasedness of the maximum likelihood estimator, its density function is assumed to be symmetric. The density function of n_i is assumed to be Gaussian on $(-\delta, \delta)$, but Laplacian outside the region It has a shape of Gaussian distribution with heavier exponential tails given by

$$f(x) = \begin{cases} Ce^{-a\delta^2/2} e^{-a\delta(x-\delta)}, x > \delta \\ Ce^{-ax^2/2}, -\delta \le x \le \delta \\ Ce^{-a\delta^2/2} e^{a\delta(x+\delta)}, x < -\delta \end{cases} \tag{2}$$

where, of course, C should be chosen so that the density $f(x)$ has unit area by proper adjustment of a and δ. Its statistics can be modelled as symmetric α stable $(S\alpha S)$ distribution.

3 OUR PROPOSED FILTERS

3.1 Amplitude-Limited Sample Average Filter

Let us found out the MLE of the mean of a normal random variable with known variance from M independent observations. The density function for M independent observations is

$$p(\underline{x}/\mu) = \frac{1}{(2\pi)^{M/2} (\sigma^2)^{M/2}} e^{-\frac{1}{2}\sum_{i=1}^{M}(x_i-\mu)^2 / \sigma^2}. \tag{3}$$

The MLE of μ that maximizes the above density function is given by

$$\hat{\mu} = \frac{1}{M}\sum_{i=1}^{M} x_i = \arg\min_{\mu} \sum_{i=1}^{M}(x_i - \mu)^2. \tag{4}$$

The MLE is just the sample mean and $\hat{\mu}$ is known to be a minimum variance unbiased and consistent estimate. This means that the MLE for estimating the signal under the additive Gaussian model is a mean filter. It can be interpreted as optimum filter in the sense of mean-square errors.

Likewise, when the observations have a density of Laplacian instead of Gaussian, the density function for M independent observations is

$$p_L(x/\eta) = \frac{1}{(2)^{M/2}\sigma^M} e^{-\frac{\sqrt{2}}{\sigma}\sum_{i=1}^{M}|x_i - \eta|} \tag{5}$$

and the MLE of η that maximizes the above equation is given by

$$\hat{\eta} = \arg\min_{\eta} \sum_{i=1}^{M}|x_i - \eta| \tag{6}$$

Its MLE corresponds to the median filter which selects the sample located at the center after arranging the observations in the ascending order. Thus, combining the results given in Eq. (4) and (6) we obtain the MLE of θ for the density function given in Eq. (2).

$$\hat{\theta} = \arg\min_{\theta} \left\{ \sum_{|x_i| \le \delta} \left(x_i - \theta \right)^2 + \sum_{|x_i| > \delta} \left| x_i - \theta \right| \right\} \tag{7}$$

The corresponding filter can be easily implemented by

$$\hat{\theta}_{\delta} = \sum_{i=1}^{M} g\left(x_i\right) \tag{8}$$

where
$$g\left(x\right) = \begin{cases} a\delta, \text{..........} x > \delta \\ ax, . -\delta \le x \le \delta \\ -a\delta, \text{......} x < -\delta \end{cases} \tag{9}$$

We call this filter as an amplitude-limited sample average filter (ALSAF). The efficacy of the estimate can be found out as follows,

$$\xi = \frac{\left[\int_{-\infty}^{\infty} g'\left(y\right) f\left(y\right) dy \right]^2}{\int_{-\infty}^{\infty} g^2\left(y\right) f\left(y\right) dy} \tag{10}$$

In the above equation, $f\left(x\right)$, given in Eq.(2), represents the density function for each observation.

Since $g\left(x\right) = -\frac{f'(x)}{f(x)}$, Efficacy ξ has the maximum value. Thus, the ALSAF given above is the optimal estimate in terms of maximizing the efficacy under the above noise environment. The error norm corresponding to our estimator from the robust statistics is given by

$$\rho\left(x\right) = \begin{cases} \frac{ax^2}{2} \text{.......................} |x| \le \delta \\ a\delta |x| - \frac{a\delta^2}{2} \text{............} |x| > \delta \end{cases} \tag{11}$$

This is equivalent to Huber's minimax norm (Huber, 1981), (Black, et al., 1998). To apply our denoising filter, we need to choose the variables a and δ as given in Eq. (2) and Eq. (9), which depends on the statistics of the noisy images. The value of δ is inversely proportional to the amount of outliers such as impulsive noise. If the value of δ is equal to the standard deviation σ of the density function given in Eq. (2), the distribution will be similar to Gaussian, which means that the outliers rarely exist. Thus, δ should be less than σ (typically $\delta = 0.8\sigma$). The probability p_G that the noise is greater than δ is computed as

$$p_G = 2Ce^{-a\delta^2/2} \int_k^{\infty} e^{-a\delta(x-\delta)} dx = \frac{2C}{a\delta} e^{-a\delta^2/2} \tag{12}$$

And the probability p_L that the noise is less than δ is

$$p_L = C \frac{\sqrt{2\pi}}{\sqrt{a}} \left(\Phi\left(\sqrt{a}\delta\right) - \Phi\left(-\sqrt{a}\delta\right) \right) \tag{13}$$

where $\Phi\left(x\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy$

a is chosen empirically for each specific image such that $p_G = p_L$ to optimize the estimate. The ALSAF is iteratively applied to reduce any residual noise by estimating the variables a and δ from the statistics of the neighbor differences at each iteration. The algorithm stops when the residual error between the current and the next estimate falls below some threshold at each pixel, which is usually less than δ.
Recall the Perona-Malik (PM) anisotropic diffusion (Perona and Malik, 1990)

$$I_t = \vec{\nabla} \cdot \left\{ h\left(\left|\nabla I\right|\right) \nabla I \right\} \tag{14}$$

where $\vec{\nabla}$, ∇ denotes divergency and gradient, respectively. Since the robust estimation can be posed as:

$$\min_I \int_{\Omega} \rho\left(\left|\nabla I\right|\right) d\Omega \tag{15}$$

where Ω is the domain of the image. Eq. (15) can be solved using the gradient descent as follows:

$$I_t = \vec{\nabla} \cdot \left\{ \rho'\left(\left|\nabla I\right|\right) \frac{\nabla I}{\left|\nabla I\right|} \right\} \tag{16}$$

Comparing Eq. (14) with Eq. (16), we can obtain the relation

$$h\left(x\right) = \frac{\rho'\left(x\right)}{x} = \begin{cases} a \text{.................} |x| \le \delta \\ ak \frac{\operatorname{sgn}\left(x\right)}{x} \text{....} |x| > \delta \end{cases} \tag{17}$$

Thus, our denoising algorithm can be implemented using PM anisotropic diffusion by selecting the edge stopping function $h\left(x\right)$ given in Eq. (17) (Black, et al., 1998).

3.2 Amplitude-Limited Myriad Filter

Similarly, the myriad filter which is the MLE of location under heavier-tailed Cauchy distributed noise is defined as

$$\hat{\beta}_k = \arg\min_{\beta} \prod_{i=1}^{M} \left(k^2 + \left(x_i - \beta \right)^2 \right) \tag{18}$$

The behavior of the myriad filter is determined by the value of k , which is called the linearity parameter. Given a set of samples $x_1, x_2, \cdots, x_M$, the sample myriad $\hat{\beta}_k$ in Eq. (18) converges to sample mean $\hat{\mu}$ in Eq. (4), as $k \to \infty$ (Gonzalez, Arce, 2001). It is proposed in this paper that outliers which are samples outside the region $(-\delta, \delta)$, are limited, as shown in Eq. (9). That is, the sample myriad is computed as

$$\hat{\gamma}_k = \arg\min_{\beta} \prod_{i=1}^{M} \left(k^2 + \left(g\left(x_i \right) - \gamma \right)^2 \right) \tag{19}$$

where $g(\cdot)$ is as given in Eq. (9). This filter is named an amplitude-limited myriad filter (ALMF). Its sample myriad $\hat{\gamma}_k$ results in amplitude-limited sample average $\hat{\theta}_\delta$ depicted in Eq. (8), as $k \to \infty$ .This can be easily proved in the same way as the myriad filter converges to a mean filter as $k \to \infty$ as given in (Gonzalez, Arce, 2001).

3.3 Filtering Scheme

As mentioned above, if the given image pixel is known to belong to one of the smooth regions Gaussian noise can be reduced by a mean filter This filter, however, tends to degrade the sharpness of the boundaries between regions of an image if it belongs to the boundary regions. This problem can be reduced effectively by the ALSAF, which however, produces visually grainy output as the amount of impulsive noise increases. Thus, our proposed approach utilizes the statistics of the samples in the window. The parameter k in Eq. (19).is determined according to the presence of impulsive noise in the window.

3.3.1 Processing of Impulsive Noise

Deciding which pixels in an image are replaced with impulsive noise is not clearly defined yet. Especially in cases they are also corrupted with Gaussian noise, the problem will be very complicated. Fortunately, image pixel values does not vary severely from its surrounding pixels even in the boundary regions. Thus, each pixel isolated with its neighbors is detected as an impulse-like pixel.
In order to decide how impulse-like each pixel is, the pixels within the window are arranged in the ascending order for each pixel location, and it is decided whether the pixel is located at some predefined range as given in Eq. (20),

$$D_i = \begin{cases} 0, & x_i \in \left\{ \left[w_i \right]_l, \left[w_i \right]_u \right\} \\ 1, & otherwise \end{cases} \tag{20}$$

where $\left[w_i \right]_k$ is the k th-order statistics of the samples in the window of size $2N+1$, that is

$$\left[w_i \right]_1 \le \left[w_i \right]_2 \le \cdots \le \left[w_i \right]_{2N+1} \tag{21}$$

and l and u are such that $1 \le l \le N+1 \le u \le 2N+1$. If D_i corresponding to the pixel x_i equals 0, then the ALSAF or ALMF with a large value of k is applied to the samples in the window because it is more probable the pixel belongs to smooth regions. However, when $D_i = 1$, the pixel is regarded as impulse-like if the mean of absolute values of its neighbour differences (MAD), as given in Eq. (22) is above the predefined threshold,

$$MAD = \sum_{x \in \Omega} \left| x - y \right| \tag{22}$$

where y is the center pixel and Ω is the set of its neighbors. It is verified experimentally to be a good indicator of impulsive noise. Its idea is borrowed from (Garnett, et al., 2005). Fig. 1 and Fig. 2 depict mean MAD values on whole image pixels as functions of types of noise and its amount. Impulsive noise pixels have much larger mean MAD values than the uncorrupted pixels or the pixels corrupted with Gaussian noise. When impulsive noise exists at some pixel in Lena image, its mean MAD value is

127, which does not vary with the amount of Gaussian noise. In our method, the image pixels whose MAD values exceed 80 are classified as impulsive noise. The pixels decided to be impulse-like are separated to process with an ALMF, whose parameter k as given in Eq. (19) is small. In case there is no impulse within the window, k is set to a large value so that the ALMF may function as an ALSAF.

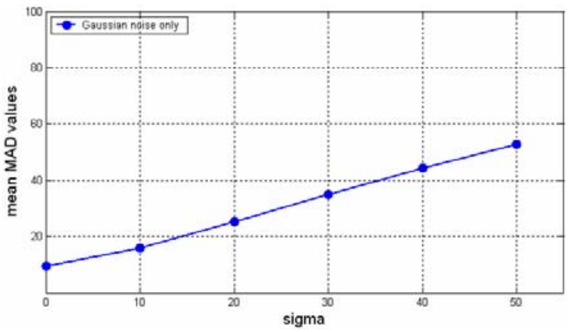

Figure 1: Mean MAD values as a function of standard deviation of Gaussian noise.

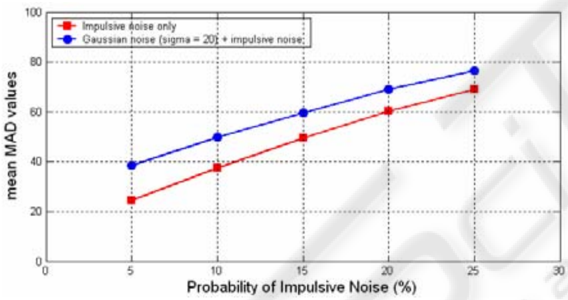

Figure 2: Mean MAD values as a function of probability of impulsive noise.

4 EXPERIMENTAL RESULTS

The widely used gray-scale Lena image is selected to test our proposed method. Impulsive noise as well as Gaussian noise are injected to the test image. In other words, the pixel corrupted with Gaussian noise is replaced randomly with impulse, which has the value of 0 ("black") or 255("white") with equal probability. Simulations are carried out for a wide range of noise density levels. The performance of our denoising filter is evaluated by way of mean-square-error (MSE) metric and peak signal-to-noise ratio (PSNR) given by

$$PSNR = 20\log_{10}\left(\frac{255}{\sigma_e}\right) \tag{23}$$

where σ_e is the standard deviation of the residual errors

$$\sigma_e = \sqrt{\frac{1}{|\Omega|}\sum_{i\in\Omega}\left(x_i - \hat{x}_i\right)^2} \tag{24}$$

In the above equation, $|\Omega|$ represents the number of pixels in the image.

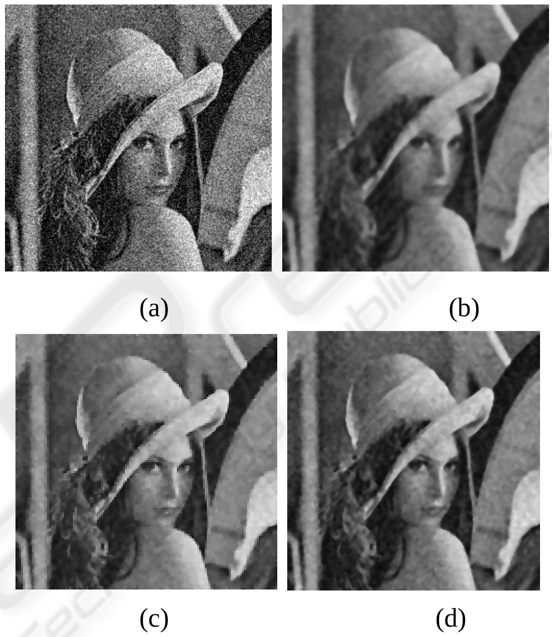

Figure 3: (a) Corrupted Lena image degraded by Gaussian noise of variance $\sigma_n^2 = 924$, with a measured $PSNR = 18.5dB$ (b) PM anisotropic diffused image after 10 iterations with $\sigma_n^2 = 153.3$ and $PSNR = 26.3dB$ (c) Output of the ALSAF after 10 iterations with $\sigma_n^2 = 137.6$ and $PSNR = 26.8dB$ (d) Output of the ALMF with $\sigma_n^2 = 155.2$ and $PSNR = 26.2dB$.

Fig. 3 shows the simulation results when gray scale image of size 256×256 is corrupted with additive Gaussian noise of variance $\sigma_n^2 = 924$ $(PSNR = 20dB)$. Obviously, both our methods suppress additive Gaussian noise without severely destroying the fine details compared with PM equation in spite of the fact that there are no significant differences in their PSNR values. Simulation results are depicted in Fig. 4 when the Lena image is corrupted with both Gaussian noise of variance $\sigma^2 = 900$ and 10% of impulsive noise $(PSNR = 20dB)$. Simulation results show that the ALSAF is not effective in removing impulsive noise, while the myriad filter can be extended to reduce both Gaussian and impulsive noise by limiting the amplitude of samples outside predefined range as given in Eq. (9). This ALMF tends to preserve the fine details while reducing both Gaussian and impulsive noise.

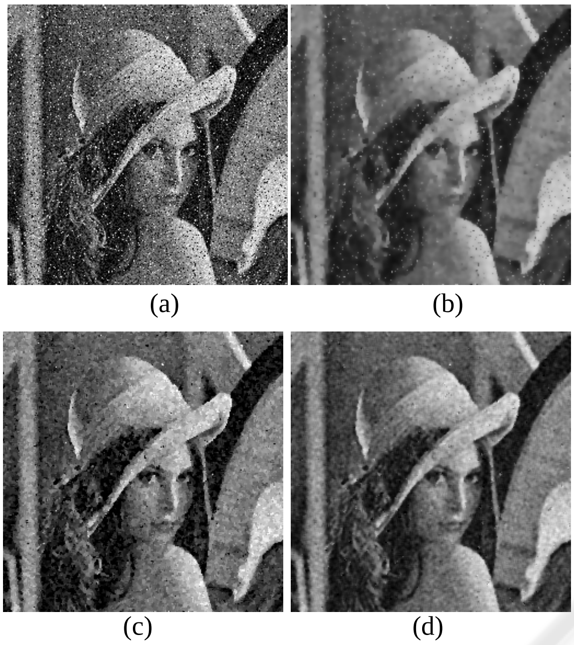

Figure 4: (a) Lena image corrupted with both Gaussian noise of $\sigma = 30$ and impulsive noise of $p = 10\%$, with a measured residual variance $\sigma_n^2 = 2059.3$ and $PSNR = 15.0dB$, (b) Output of the ALSAF after 10 iterations with $\sigma_n^2 = 359.6$ and $PSNR = 22.6dB$ (c) Output of myriad filter with $\sigma_n^2 = 557.9$ and $PSNR = 20.67dB$ (d) Output of ALMF with $\sigma_n^2 = 234.7$ and $PSNR = 24.42dB$.

5 CONCLUSIONS

Optimal nonlinear filter which maximizes the efficacy under mixed Gaussian noise environment is derived. This filter effectively can be implemented using PM anisotropic diffusion by selecting the appropriate edge stopping function. However, it produces visually grainy output as the amount of impulsive noise increases. Thus, impulse-like signal detection is introduced to process impulsive pixels differently from the remaining pixels. For this process, a myriad filter is selected, which is a maximum log-likelihood estimator of the location parameter for Cauchy density. The filter is known to outperform median-based filters in removing impulsive noise. By combining ALSAF which is a MLE in mixed Gaussian noise with a myriad filter, the resulting filter (ALMF) effectively removes both Gaussian and impulsive noise, preserving the fine details.

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