Partition-Form Cooperative Games in Two-Echelon Supply Chains

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Abstract: Competition and cooperation are inherent features of any multi-echelon supply chain. The interactions among the agents across the same echelon and that across various echelons influence the percolation of market demand across echelons. The agents may want to collaborate with others in pursuit of attracting higher demand and thereby improving their own revenue. We consider one supplier (at a higher echelon) and two manufacturers (at a lower echelon and facing the customers) and study the collaborations that are 'stable'; the main differentiator from the existing studies in supply chain literature is the consideration of the following crucial aspect – the revenue of any collaborative unit also depends upon the way the opponents collaborate. Such competitive scenarios can be modeled using what is known as partition form games.

Our study reveals that the grand coalition is not stable when the product is essential and the customers buy it from any of the manufacturers without a preference. The supplier prefers to collaborate with only one manufacturer, the one stronger in terms of market power; further, such collaboration is stable only when the stronger manufacturer is significantly stronger. Interestingly, no stable collaborative arrangements exist when the two manufacturers are nearly equal in market power.

1 INTRODUCTION

Supply chains are complex systems that involve multiple agents at multiple echelons. These agents compete and/or collaborate with each other to acquire the maximum possible market share at 'good' prices. The agents look for collaborative opportunities to provide better quality service, thereby attracting more customers, resulting in enhanced individual performance, while others compete with each other if they find it beneficial (e.g., in 2016, Walmart teamed with JD.com to compete with Amazon and Alibaba in China).

Handbooks in Operations Research, (Chen, 2003), discusses the importance of coordination on the effectiveness of the supply chain (SC). Cooperative game theory facilitates a systematic study of these interactions among the agents of SC (e.g., (Arshinder et al., 2011; Thun, 2005; Nagarajan and Sošić, 2008)).

We examine the interplay between cooperation and competition in a two-echelon SC, with two manufacturers at the lower echelon directly facing the customers, and a single supplier at the upper echelon. Customers choose to buy (or not buy) the product from one of the two manufacturers based on factors like the quoted price, the reputation of the entities involved, the importance of the product (essentialness), etc. The manufacturers compete with each other to attract 'good' amount of customer base at 'good' prices and rely on the supplier for the raw material. The supplier at the upper echelon quotes a per-unit price for raw materials to the manufacturers, and, the latter respond by either quoting a price of final product to the customers or by deciding not to operate; the choice of manufacturers also depends upon the production costs, demand response of the customers, etc.

This paper aims to find 'optimal' pricing and collaborative strategies of the agents using sophisticated cooperative game theoretic tools. Majority of these games (e.g., in (Li et al., 2023; Zheng et al., 2021)) focus on the stability of the grand coalition and further on scenarios where the worth of a coalition depends just upon its members. But many times, the grand coalition may not be stable, and further, the worth of the cooperating agents may depend upon the arrangement of agents outside the coalition. Such games are referred as partition form games, and a recent thesis, (Singhal, 2023), provides a comprehensive summary of these games (see also (Aumann and Dreze, 1974)).

In any real-world SC, the revenue or the worth generated (for example) by a supplier, when all the manufacturers collaborate (i.e., operate as a single

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unit) will obviously be different from that in a scenario where the manufacturers also compete among themselves. Thus partition-form based study is essential to capture the frictions in SC.

The first main contribution of this paper is to capture the above realistic aspects in an SC by modeling it as a partition form game and deriving the ingredients of the same – to the best of our knowledge, none of the papers in SC literature consider this. We further consider that the agents in any coalition operate together as a single unit by pooling the best resources from each partner; furthermore, the possibilities of vertical and/or horizontal cooperation are also explored.

The exhaustive partition-form game based study resulted in some interesting insights. When the product is essential, and when the customers are (almost) indifferent to the manufacturers, the grand coalition is not stable. It is actually the vertical cooperation between the supplier and one of the manufacturers that results in a stable configuration. More interestingly, only the collaboration with the stronger manufacturer (strong in terms of market power) is stable – no other attribute of the manufacturers makes a difference (when their reputation among the customers is almost the same); the weaker manufacturer operates alone and competes with the collaborating pair. Even more interestingly, no collaboration is stable when the manufacturers are of comparable market strengths.

When the supplier leads by quoting a price, there exists a Stackelberg equilibrium (SBE) at which the agents operate, in contrast, a scenario where all the agents make a simultaneous move results in a Nash equilibrium at which none of the agents operate. In fact, majority of the literature in SC seems to understand this at some level and considers the Stackelberg (SB) framework (e.g., (Li et al., 2023; Zheng et al., 2021)). The SB framework significantly favours the supplier – the supplier enjoys a huge fraction of the revenue generated, which becomes even higher with the competition at the lower echelon.

The model is described in Section 2, the partition form games are in Section 4, and the SC is analysed in Section 5. Some of the proofs are in Appendices, and others are in technical report (Wadhwa et al., 2024).

Literature Survey. There is a vast literature that studies the scope of SC coordination. Almost all the studies consider contract based cooperation (e.g., (Cachon, 2003) and subsequent papers). There are few strands of literature that study coalition formation ideas, where the agents are bound without any such enforcement, because they find it beneficial to do so. Important and relevant papers in this category are (Nagarajan and Sošić, 2009), (Zheng et al., 2021) and (Li et al., 2023) etc.

In (Zheng et al., 2021), authors study a two-echelon sustainable SC with two manufacturers and a single supplier; they neglect the partition-form aspects by defining the worth of a coalition to be the pessimal worth, the minimum (anticipated worth) that the said coalition can generate irrespective of the arrangement of the left-over agents. However, if a coalition (not currently operating) has to block/oppose an operating configuration (the set of operating coalitions and revenues/shares of all the agents of SC), the coalition should anticipate to derive a better revenue than the sum total revenue that its members are currently deriving. In other words, the anticipation is required only for estimating the worth of future (or blocking) coalition (as upfront it is not sure of the retaliatory actions of the others), and not for the worth currently derived (as considered in (Zheng et al., 2021)). We consider pessimal worth as the anticipated worth of blocking-coalition, while the (current) worth(s) in any operating configuration is derived by solving an appropriate game or optimization problem.

Another recent study in (Li et al., 2023) considers two assemblers (like manufacturers in our study) and many irreplaceable suppliers, where the second assembler only competes for customer base and has its own set of suppliers – hence this study is not directly comparable to ours. However, the study again neglects the partition form nature of the game – the worth of any coalition is defined just based on its size. As already argued, when one neglects the inherent partition form nature, the results could be misleading – it would be interesting to analyze the SC of (Li et al., 2023), after incorporating partition form aspects.

In (Nagarajan and Sošić, 2009), authors study coalitional stability considering partition-form aspects. However, as is mentioned in the same paper, they do not consider the worth of the coalition (based on partition), rather assume that all the players in the coalition to agree to quote a common (best) price. This (rather restricted) assumption facilitates in the derivation of the revenue generated by a single agent in any partition and thereby study the stability aspects. In our study, we derive the utility of any coalition depending upon the partition and then consider stability aspects based on the division of that worth and the anticipated utility of the 'opposing coalition'.

2 MODEL

We consider a two-echelon supply chain (SC), with two manufacturers at the lower echelon and a single supplier at the upper echelon. The customers purchase the final product from the manufacturers depending upon various factors (price and the essentialness of the product, reputation of the manufacturer, etc.); while manufacturers obtain the required raw materials from the supplier depending upon their own customer demand and the price quoted by the supplier, production cost, etc.

Any manufacturer can operate alone, or can collaborate with the other manufacturer, or with the supplier, or with both of them – when both the manufacturers operate together, they choose the best among them for any aspect (e.g., influence, reputation, production capacity), while the supplier and manufacturer pair quote one price directly to the customers.

We examine the impact of the interplay between cooperation and competition in the above SC using a cooperative game-theoretic framework; in particular, our research aims to explore the potential for horizontal (within the same echelon) and vertical cooperation (across echelons) in an SC. We now describe the ingredients of this study in detail.

2.1 Coalitions and Partitions

All the agents or a subset of them can operate together by forming coalitions. Basically, the agents within a coalition make joint decisions to generate a common revenue while facing competition from other coalitions or agents. One may have more than one coalition operating in the system. Any partition, say $\mathbb{P} = \{\mathbb{C}_1, \cdots, \mathbb{C}_k\}$, represents the operating arrangement of agents into distinct coalitions and satisfies the following:

$$\cup_m \mathbb{C}_m = \{M_1, M_2, S\}, \text{and } \mathbb{C}_m \cap \mathbb{C}_l = \emptyset \text{ if } m \neq l.$$

The goal of this paper is to study the interactions between these coalitions and predict the emergence of stable partition(s) (if any). Prior to this, we need to understand the criteria for declaring a partition stable. Even prior to this, we need to derive the revenues generated by various coalitions in each partition – we refer to these revenues as the worths of the coalitions, a term commonly used in the cooperative game theory literature (Singhal et al., 2021; Singhal, 2023; Aumann and Dreze, 1974). The stability concepts are discussed in Section 4, while the worths related to various partitions are derived in various sections. For now, we discuss important and interesting partitions and coalitions.

When two manufacturers operate together, we have a coalition $\mathbb{M} = \{M_1, M_2\}$, with horizontal cooperation (HC) at the lower echelon. When the supplier and a manufacturer operate together, we have a coalition with vertical cooperation (VC), e.g., $\mathbb{V}_i = \{M_i, S\}$. When any agent operates alone, we have a coalition with a single player, e.g., $\mathbb{M}_i = \{M_i\}$ or $\mathbb{S} = \{S\}$. When all the agents operate together as in a centralized SC, we have a grand coalition (GC), represented by $\mathbb{G} = \{S, M_1, M_2\}$.

The partition $\mathbb{P}_{\mathbb{G}} = \{\mathbb{G}\}$ where all the agents operate together is referred to as the GC partition. While we have an ALC partition $\mathbb{P}_{\mathbb{A}} = \{\mathbb{S}, \mathbb{M}_1, \mathbb{M}_2\}$, when all the agents operate alone. We also have VC (vertical cooperation) partition $\mathbb{P}_{\mathbb{V}i} = \{\mathbb{V}_i, \mathbb{M}_{-i}\}$ and HC (horizontal cooperation) partition $\mathbb{P}_{\mathbb{H}} = \{\mathbb{S}, \mathbb{M}\}$.

The worth, the revenue generated by any coalition must be *shared appropriately among its members and this payoff division also influences the stability aspects* (Aumann and Dreze, 1974; Singhal et al., 2021; Singhal, 2023). Further, departing from a majority of the literature (Li et al., 2023; Zheng et al., 2021), the worth of any coalition depends upon the operating partition leading to a partition form cooperative game (Singhal et al., 2021; Singhal, 2023); as already mentioned, the analysis of such games is significantly complicated, and the results obtained by omission of this dependency can be misleading.

In all, as a result of the choices made by various agents in the system, each agent derives some revenue/share. The agents are selfish and aim to maximize their individual revenue/share, which drives their choices, including their collaboration attempts; the paper precisely works in identifying the *'stable configurations' – the partitions and the corresponding payoff divisions.*

2.2 Market Segmentation

In an SC, the manufacturers satisfy customers' demands and rely upon suppliers for raw materials or intermediate products. Customers choose one manufacturer (or none) based on the price, reputation, loyalty, and other factors. The demand segmentation is also influenced by the essentialness of the product, which we capture using a parameter γ and a cross-linking factor ε that also captures the customers' affinity to switch loyalties. When the manufacturers do not operate together, the market is segmented between them based on their selling prices p_i and essentialness factors (γ, ε) as in equation (1) given below. This is inspired by the models commonly used in SC literature (see, e.g., (Zheng et al., 2021; Li et al., 2023). With $y^+ = \max\{0, y\}$, the demand derived by M_i equals:

$$D_{\mathbb{M}_i} = \left(\bar{d}_{\mathbb{M}_i} - \alpha_{\mathbb{M}_i} p_i + \varepsilon \alpha_{\mathbb{M}_{-i}} p_{-i}\right)^+, \quad (1)$$

with $\alpha_{\mathbb{M}_i} := \tilde{\alpha}_{\mathbb{M}_i}(1-\gamma)$, where,

- $\bar{d}_{\mathbb{M}_i}$ is the dedicated market size of M_i,
- $\alpha_{\mathbb{M}_i} p_i$ is the fraction of demand lost by M_i due to its price p_i, sensitized by parameter $\alpha_{\mathbb{M}_i}$,
- The essentialness factor γ dictates the sensitivity of price p_i on demand – for example, when $\gamma \approx 1$, the product is highly essential, and the customers are insensitive to price,
- $\varepsilon \alpha_{\mathbb{M}_{-i}} p_{-i}$ is the fraction of customer base of M_{-i} that rejected M_{-i} and shifted to M_i,
- The demand is positive as long as the term inside $(\cdot)^+$ is positive; else, the demand is zero.

The product is essential, either when $\gamma \approx 1$ or when $\varepsilon \approx 1$ and then almost all the customers buy the product (for these parameters, observe $D_{\mathbb{M}_1} + D_{\mathbb{M}_2} \approx \bar{d}_{\mathbb{M}_1} + \bar{d}_{\mathbb{M}_2}$, the total market size). When $\varepsilon \approx 1$, the customers buy the product from one or the other manufacturer (need not be loyal); otherwise, they prefer to buy from their own manufacturer (are loyal). Generally, the sum of demands of both manufacturers is strictly less than the total market size, and the gap depends upon the essentialness parameters (γ, ε).

HC Coalition. When both manufacturers operate together, as in $\mathbb{M}$ or $\mathbb{G}$, they can potentially attract both customer bases. Further, for manufacturing purposes, the coalition uses the methods of the manufacturer with the lowest manufacturing cost; thus, its per-unit manufacturing cost is $C_{\mathbb{M}} = \min_{M_i \in \mathbb{C}} C_{\mathbb{M}_i}$, where $C_{\mathbb{M}_i}$ is the per-unit manufacturing cost of the manufacturer M_i. As the best of the two capabilities are utilized, and as the customers are aware of it, we assume the reputation of the coalition equals that of the best. In all, we assume the demand function of coalition with horizontal cooperation to be:

$$\begin{aligned} D_{\mathbb{M}} &= \bar{d}_{\mathbb{M}} - \alpha_{\mathbb{M}} p, \text{ with } \alpha_{\mathbb{M}} := \min\{\alpha_{\mathbb{M}_1}, \alpha_{\mathbb{M}_2}\}, \\ \bar{d}_{\mathbb{M}} &= \bar{d}_{\mathbb{M}_1} + \bar{d}_{\mathbb{M}_2}. \end{aligned} \quad (2)$$

There is obviously no cross-linking (shift of customers from one manufacturer to the other), or basically, the customers have no choice. In some cases, this can be fatal to the system, as the customers can get discouraged by the unavailability of options. The product may not appear essential anymore, and the customers may find solace in other related products. We observe this phenomenon has significant influence on stability results of sections 5.1-5.2.

2.3 Costs, Actions and the Utilities

Any agent, supplier, manufacturer, or coalition has a fixed cost of operation. Therefore, if the agent/coalition does not generate sufficient profit, it incurs negative revenue and can choose not to operate. Let n_o represent the choice of not operating. The utility of any agent/coalition is 0 when it chooses n_o.

The supplier can decide not to operate, or can quote a price $q \in [0, \infty)$. Thus, the action set of supplier when operating alone is $\mathscr{A}_{\mathbb{S}} := \{n_o\} \cup [0, \infty)$, and its action $a_{\mathbb{S}} \in \mathscr{A}_{\mathbb{S}}$. Similarly, when manufacturer M_i decides to operate alone, it quotes a selling price $p_i \in [0, \infty)$. Thus, the action and the action set of manufacturer M_i is $a_{\mathbb{M}_i} \in \mathscr{A}_{\mathbb{M}_i} := \{n_o\} \cup [0, \infty)$.

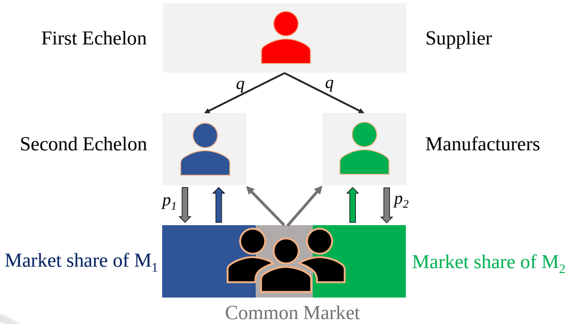

Figure 1: System model, when all agents operate alone.

In a VC coalition $\mathbb{V}_i = \{S, M_i\}$, the coalition sets a price q for supplying raw materials to the manufacturer outside the coalition, and sets a price p_i directly to customers by jointly producing the final product. In the grand coalition $\mathbb{G} = \{S, M_1, M_2\}$, which involves vertical and horizontal cooperation, the coalition directly quotes a price p for customers and makes a combined effort to produce the final product. The respective actions are represented by $\mathbf{a}_{\mathbb{V}}$ and $\mathbf{a}_{\mathbb{G}}$ and the action sets by $\mathscr{A}_{\mathbb{V}}$ and $\mathscr{A}_{\mathbb{G}}$ (defined as before).

2.3.1 Utilities in ALC Partition

We begin with describing the utilities of various agents when all of them operate alone, i.e., when the partition is $\mathbb{P}_{\mathbb{A}}$ (see Figure 1). Let $\mathbf{a} := (a_{\mathbb{S}}, \mathbf{a}_{\mathbb{M}})$ represent the actions of all the agents (the supplier, and both the manufacturers), where $\mathbf{a}_{\mathbb{M}} := (a_{\mathbb{M}_1}, a_{\mathbb{M}_2})$.

Manufacturers' Utility. The utility of manufacturer M_i is zero either if it chooses not to operate or if the supplier does not operate. Otherwise, utility is the total profit gained minus the operating cost, where the former is the product of the demand attracted $D_{\mathbb{M}_i}$ (1) and the profit gained per-unit:

$$U_{\mathbb{M}_i}(\mathbf{a}) = (D_{\mathbb{M}_i}(\mathbf{a}_{\mathbb{M}})(p_i - C_{\mathbb{M}_i} - q)\mathscr{F}_{\mathbb{S}} - O_{\mathbb{M}_i})\mathscr{F}_{\mathbb{M}_i}, \quad (3)$$

where $C_{\mathbb{M}_i}$ is the per-unit production cost incurred by M_i, $O_{\mathbb{M}_i}$ is the fixed operating/setup cost, $\mathscr{F}_{\mathbb{C}} = \mathbb{1}_{\{a_{\mathbb{C}} \neq n_o\}}$ represents the flag that coalition $\mathbb{C}$ operates, and q denotes the wholesale price quoted by supplier.

Suppliers' Utility. The demand for the supplier's raw materials (at higher echelon) percolates from the lower echelon (manufacturers), based on the choices of the manufacturers. This dictates the utility of the supplier, which is non-zero only if the supplier and at least one of the manufacturers operate. In all, the utility of the supplier S when it operates alone equals:

$$U_{\mathbb{S}}(\mathbf{a}) = \left(\left(\sum_{i=1}^{2} D_{\mathbb{M}_i}(\mathbf{a}_{\mathbb{M}}) \mathscr{F}_{\mathbb{M}_i} \right) (q - C_{\mathbb{S}}) - O_{\mathbb{S}} \right) \mathscr{F}_{\mathbb{S}}, \tag{4}$$

where $C_{\mathbb{S}}$ is the cost for procurement of a bundle of raw material required for producing one unit of product and $O_{\mathbb{S}}$ is the fixed operational cost of the supplier.

2.3.2 Utility in General Partition

In a general partition, the utility of a coalition is defined as the sum of the utilities of all the agents within the coalition. As in equation (2) and as described in the corresponding sub-section, the coalition utilizes the best agent for each feature. Also, any VC-based coalition directly quotes a price to the customers. Thus, for example, the action and utility of the grand coalition $\mathbb{G}$ are:

$$a_{\mathbb{G}} \in \{p \in [0,\infty)\} \cup \{n_o\}, \text{ and}$$
$$U_{\mathbb{G}}(a_{\mathbb{G}}) = \left((\bar{d}_{\mathbb{M}} - \alpha_{\mathbb{G}} p)(p - C_{\mathbb{G}}) - O_{\mathbb{G}}\right) \mathscr{F}_{\mathbb{G}}, \text{ where,}$$

- $O_{\mathbb{G}}$ is the combined operational cost, defined as $O_{\mathbb{G}} = \min\{O_{\mathbb{M}_1}, O_{\mathbb{M}_2}\} + O_{\mathbb{S}}$.
- $C_{\mathbb{G}}$ is the combined production cost, defined as $C_{\mathbb{G}} = \min\{C_{\mathbb{M}_1}, C_{\mathbb{M}_2}\} + C_{\mathbb{S}}$.
- $\alpha_{\mathbb{G}}$ is the price sensitivity of the grand coalition, defined as $\alpha_{\mathbb{G}} = \min\{\alpha_{\mathbb{M}_1}, \alpha_{\mathbb{M}_2}\}$.

It is important to note here that the agents in the grand coalition share the revenue generated ($U^*_{\mathbb{G}} = \sup_{a_{\mathbb{G}}} U_{\mathbb{G}}$), and there is no price per item to be paid between any subset of them. The definitions of utilities for other partitions follow similar logic and will be discussed in the respective sections.

We conclude this section by making an important assumption (inspired by commonly made choices in practical scenarios):

A.1 If any agent, either supplier or manufacturer, is indifferent between the action $a = n_o$ and an $a \neq n_o$, the agent prefers operating choices.

We begin with studying an SC with a single manufacturer and supplier, which provides the basis and benchmark for analysing the more generic SC (with two manufacturers) of section 5. The stability concepts of partition-form games are in section 4.

3 SINGLE MANUFACTURER SC

In a two-echelon SC with one supplier and one manufacturer, the agents either operate together to form GC partition $\{\mathbb{G}\}$ or operate alone to form ALC partition $\{\mathbb{S},\mathbb{M}\}$ (*observe that* $\mathbb{M} = \{M\}$, $\mathbb{G} = \{S,M\}$ *are coalitions with one manufacturer in this section, while the same respectively represent* $\mathbb{M} = \{M_1, M_2\}$ *and* $\mathbb{G} = \{S, M_1, M_2\}$ *in the rest of the paper*). To completely understand the coalitional stability aspects of such a system, it is sufficient to analyze these two partitions. Note that there is no competition among the agents in the same echelon in this case. In the absence of this competition, the market demand (1) for the manufacturer simplifies to (as in (2)):

$$D_{\mathbb{M}}(a_{\mathbb{M}}) = \left(\bar{d}_{\mathbb{M}} - \alpha_{\mathbb{M}} p\right). \tag{5}$$

Recall the demand decreases as the price increases, but the drop is reduced as the product becomes more and more essential (when $\gamma \approx 1$) and $\alpha_{\mathbb{M}} = \tilde{\alpha}_{\mathbb{M}}(1-\gamma)$. We assume the following:

A.2 The total market size $\bar{d}_{\mathbb{M}} > \alpha_{\mathbb{M}} C_{\mathbb{G}} + 2\sqrt{\alpha_{\mathbb{M}} O}$, with $O := \max\{2O_{\mathbb{S}}, O_{\mathbb{G}}, 4O_{\mathbb{M}}\}$.

Basically, the available market size has to be above a certain threshold so that the profit from the attracted demand surpasses the operating and manufacturing costs – this ensures it is optimal for the agents to operate (see for e.g., Lemma 4 of the Appendix). If it is optimal for the agents not to operate, then there is nothing left to analyse.

3.1 ALC Partition

In this partition, both agents operate independently and aim to maximize their respective utilities. We consider a Stackelberg game framework, where the supplier first quotes a price q per bundle of raw material to the manufacturer. The manufacturer then quotes a price $a_{\mathbb{M}} = p$ (per unit of product) to the customers; the customers, in turn, respond by generating a demand $D_{\mathbb{M}}(a_{\mathbb{M}})$, as in (5). Thus, the Stackelberg game between the supplier and the manufacturer is given by (with $\mathbf{a} = (a_{\mathbb{S}}, a_{\mathbb{M}})$, see (3)-(4)):

$$U_{\mathbb{S}}(\mathbf{a}) = (D_{\mathbb{M}}(a_{\mathbb{M}}) \mathscr{F}_{\mathbb{M}} (q - C_{\mathbb{S}}) - O_{\mathbb{S}}) \mathscr{F}_{\mathbb{S}}, \tag{6}$$
$$U_{\mathbb{M}}(\mathbf{a}) = (D_{\mathbb{M}}(a_{\mathbb{M}})(p - q - C_{\mathbb{M}}) \mathscr{F}_{\mathbb{S}} - O_{\mathbb{M}}) \mathscr{F}_{\mathbb{M}}. \tag{7}$$

We now derive the Stackelberg equilibrium (SBE) of the above game.

Theorem 1. *Assume* **A**.*1 and* **A**.*2. There exists an SBE under which both the agents operate, which is given by* $a^*_{\mathbb{M}} = p^*$ *and* $a^*_{\mathbb{S}} = q^*$, *where*

$$p^* := \frac{3\bar{d}_{\mathbb{M}} + \alpha_{\mathbb{M}}(C_{\mathbb{S}} + C_{\mathbb{M}})}{4\alpha_{\mathbb{M}}}, \; q^* := \frac{\bar{d}_{\mathbb{M}} + \alpha_{\mathbb{M}}(C_{\mathbb{S}} - C_{\mathbb{M}})}{2\alpha_{\mathbb{M}}}.$$

Further, the utilities at this SBE are given by

$$(U^*_{\mathbb{M}}, U^*_{\mathbb{S}}) = (\phi - O_{\mathbb{M}}, 2\phi - O_{\mathbb{S}}), \quad (8)$$

$$\textit{where } \phi := \frac{\left(\bar{d}_{\mathbb{M}} - \alpha_{\mathbb{M}}(C_{\mathbb{S}} + C_{\mathbb{M}})\right)^2}{16\alpha_{\mathbb{M}}}.$$

Proof. The utility of manufacturer (7) resembles that in Lemma 4 and thus for any q, the optimizer of the manufacturer is operating (not equal to n_o) only when $\bar{d}_{\mathbb{M}} \geq \alpha_{\mathbb{M}}(C_{\mathbb{M}} + q) + 2\sqrt{\alpha_{\mathbb{M}} O_{\mathbb{M}}}$ and then is given by $p^*(q) = \bar{d}_{\mathbb{M}}/2\alpha_{\mathbb{M}} + (C_{\mathbb{M}}+q)/2$.

If one neglects the operating conditions, then after substituting $p^*(q)$ in (6) we have:

$$U_{\mathbb{S}}(q) = U_{\mathbb{S}}(q, p^*(q)) = \frac{(\bar{d}_{\mathbb{M}} - \alpha_{\mathbb{M}}(C_{\mathbb{M}} + q))(q - C_{\mathbb{S}}) - 2O_{\mathbb{S}}}{2}$$

which is again similar to that in Lemma 4, and then by the same lemma, the optimal q would have been $q^* = (\bar{d}_{\mathbb{M}} + \alpha_{\mathbb{M}}(C_{\mathbb{S}} - C_{\mathbb{M}}))/2\alpha_{\mathbb{M}}$ – this is true when q^* and $p^* := p^*(q^*) = (3\bar{d}_{\mathbb{M}} + \alpha_{\mathbb{M}}(C_{\mathbb{S}} + C_{\mathbb{M}}))/4\alpha_{\mathbb{M}}$ both satisfy the required operating conditions (i.e., respective Δs are ≥ 0) – these are satisfied, as by **A.2**:

$$\alpha_{\mathbb{M}}(q^* + C_{\mathbb{M}}) = \frac{\alpha_{\mathbb{M}} C_{\mathbb{G}} + \bar{d}_{\mathbb{M}}}{2} < \bar{d}_{\mathbb{M}} - 2\sqrt{\alpha_{\mathbb{M}} O_{\mathbb{M}}}, \text{ and}$$

$$(\bar{d} - \alpha_{\mathbb{M}} C_{\mathbb{M}}) - \alpha_{\mathbb{M}} C_{\mathbb{S}} - 2\sqrt{2O_{\mathbb{S}}} > 0.$$

By substituting $(q^*, p^*(q^*))$ in (6) and (7), we derive the optimal utilities. □

3.2 GC Partition

Both the agents operate together, and the system directly faces the customers and quotes a common price p. The per-unit cost $C_{\mathbb{G}} = C_{\mathbb{S}} + C_{\mathbb{M}}$ of the system includes the procurement cost (of the raw materials) and the production cost (see Section 2.3). Furthermore, the system also has a fixed operating cost $O_{\mathbb{G}} = O_{\mathbb{S}} + O_{\mathbb{M}}$, when it operates. Thus, the overall utility of the system is:

$$U_{\mathbb{G}} = \left(D_{\mathbb{M}}(a_{\mathbb{M}})\left(p - C_{\mathbb{G}}\right) - O_{\mathbb{G}}\right)\mathscr{F}_{\mathbb{G}}. \quad (9)$$

The utility of any coalition is defined as the optimal utility that it can derive. Therefore, we have the following simple optimization problem for deriving the utility of the GC:

$$\sup_{\mathbf{a}_{\mathbb{G}} \in \mathscr{A}_{\mathbb{G}}} \left(D_{\mathbb{M}}(\mathbf{a}_{\mathbb{M}})(p - C_{\mathbb{G}}) - O_{\mathbb{G}}\right)\mathscr{F}_{\mathbb{G}}. \quad (10)$$

This problem can be solved using basic (derivative-based) methods, and the solution is as follows:

Theorem 2. *Assume* **A***.1 and* **A***.2. There exists an optimizer at which the system operates, and the corresponding optimal price is* $a^*_{\mathbb{G}} = p^*_{\mathbb{G}}$, *where*

$$p^*_{\mathbb{G}} = \frac{\bar{d}_{\mathbb{M}}}{2\alpha_{\mathbb{M}}} + \frac{C_{\mathbb{G}}}{2}, \quad (11)$$

and the corresponding optimal utility is given by:

$$U^*_{\mathbb{G}} = \frac{\left(\bar{d}_{\mathbb{M}} - \alpha_{\mathbb{M}} C_{\mathbb{G}}\right)^2}{4\alpha_{\mathbb{M}}} - O_{\mathbb{G}}. \quad (12)$$

Proof. Again the utility function of GC (10), resembles that in Lemma 4. Furthermore, by assumption **A.2**, $\Delta > 0$ for the GC, which implies that the GC will operate. Thus the proof follows by Lemma 4. □

Remarks. By Theorems 1 and 2 (observe here $C_{\mathbb{G}} = C_{\mathbb{S}} + C_{\mathbb{M}}$ as in subsection 2.3.2),

$$U^*_{\mathbb{G}} - (U^*_{\mathbb{M}} + U^*_{\mathbb{S}}) = \phi. \quad (13)$$

Thus, the agents derive higher utility in GC than the combined utility that they derive when they operate alone (i.e., ALC). This means that both agents can benefit by forming a coalition, as long as they agree to share the extra profit $(U^*_{\mathbb{G}} - (U^*_{\mathbb{M}} + U^*_{\mathbb{S}}))$ in a way that benefits both of them. Consider a configuration $(GC, (x_{\mathbb{M}}, x_{\mathbb{S}}))$, where x_i represents the payoff allocation of agent i and which satisfies:

$$x_{\mathbb{M}} + x_{\mathbb{S}} = U^*_{\mathbb{G}}, \;\; x_{\mathbb{M}} > U^*_{\mathbb{M}}, \;\; \text{and } x_{\mathbb{S}} > U^*_{\mathbb{S}}. \quad (14)$$

When the profits are shared as above, none of the agents prefer to operate alone. Now consider a configuration $(GC, (x_{\mathbb{M}}, x_{\mathbb{S}}))$ that does not satisfy (14). When $x_{\mathbb{M}} + x_{\mathbb{S}} < U^*_{\mathbb{G}}$, the generated revenue $U^*_{\mathbb{G}}$ is not completely shared; if share $x_{\mathbb{M}} < U^*_{\mathbb{M}}$, the manufacturer would prefer to operate alone, as it would then derive $U^*_{\mathbb{M}}$; similarly if $x_{\mathbb{S}} < U^*_{\mathbb{S}}$, the supplier would prefer to operate alone. Thus such configurations are 'opposed' and hence are not stable. Before we study an SC with two manufacturers, let us formally discuss the notions of stability in partition form games.

4 PARTITION FORM GAMES

A partition form game is described using the tuple $\langle N, (w^{\mathbb{P}}_{\mathbb{C}})\rangle$ where N is the set of players and $w^{\mathbb{P}}_{\mathbb{C}}$ is the worth of the coalition $\mathbb{C}$ under partition $\mathbb{P}$ and is defined only when $\mathbb{C} \in \mathbb{P}$. As mentioned in the introduction, here the worth $w^{\mathbb{P}}_{\mathbb{C}}$ also depends upon the partition $\mathbb{P}$ – basically $w^{\mathbb{P}}_{\mathbb{C}}$ need not equal $w^{\mathbb{P}'}_{\mathbb{C}}$ for two different partitions $\mathbb{P} \neq \mathbb{P}'$ both of which contain $\mathbb{C}$.

Given a partition $\mathbb{P}$ and the worths $\{w^{\mathbb{P}}_{\mathbb{C}}\}$ of each coalition in $\mathbb{P}$, the next question is about a 'pay-off' vector which defines the allocation to each agent in N. A pay-off vector $\mathbf{x} = (x_1, \cdots, x_n)$ is defined to be consistent with respect to partition $\mathbb{P}$ if (see (Aumann and Dreze, 1974; Singhal et al., 2021; Singhal, 2023)):

$$\sum_{i \in \mathbb{C}_j} x_i = w^{\mathbb{P}}_{\mathbb{C}_j} \text{ for all } \mathbb{C}_j \in \mathbb{P}. \quad (15)$$

The pair $(\mathbb{P}, \mathbf{x})$ is defined to be a configuration if the latter is consistent with the former.

The quest now is to study a 'solution' of the partition form game. The 'solution' in this context describes the configurations that are stable; in other words, it identifies the partitions and their companion consistent payoff vectors that can emerge or operate stably without being 'opposed'.

To study the stability aspects, one first needs to understand if a certain coalition which is not a part of the partition can 'block' (or oppose) the given configuration – such a blocking is possible if the coalition has an 'anticipation' of the value it can achieve (irrespective of all scenarios that can result after coalition blocks) and if the anticipated value is bigger than what the members of the coalition are deriving in the current configuration. Basically, if there exists at least one division of this anticipated value among the members of the blocking coalition that renders all the members to achieve more than that in the given payoff vector, then the coalition has a tendency to oppose the current configuration. The above concepts are made precise in the following definitions ((Singhal et al., 2021; Singhal, 2023)):

Definition 1 (Blocking of a configuration by a coalition)**.** *A configuration, the tuple of partition and the consistent payoff vector,* $(\mathbb{P}, \mathbf{x})$*, is blocked by a coalition* $\mathbb{C} \notin \mathbb{P}$*, under the pessimal anticipation rule, if the coalition derives better than that in the original configuration irrespective of the arrangement of opponent players, i.e., if the pessimal anticipated utility*

$$w_{\mathbb{C}}^{pa} := \min_{\mathbb{P}': c \in \mathbb{P}'} w_{\mathbb{C}}^{\mathbb{P}'} > \sum_{i \in \mathbb{C}} x_i. \quad (16)$$

Definition 2 (Stability)**.** *A configuration* $(\mathbb{P}, \mathbf{x})$ *is said to be stable if there exists no coalition* $\mathbb{C} \notin \mathbb{P}$ *that blocks it. A partition* $\mathbb{P}$ *can be said to be stable if there exists at least one configuration* $(\mathbb{P}, \mathbf{x})$ *involving* $\mathbb{P}$ *which is stable.*

We now apply the above stability concepts to the single manufacturer case study of the previous section. In this case, $N = \{S, M\}$ and the only possible partitions are $\mathbb{P}_{\mathbb{G}} = \{N\}$ and $\mathbb{P}_{\mathbb{A}} = \{\{S\}, \{M\}\}$.

Clearly, the worth of grand coalition $w_{\mathbb{G}}^{\mathbb{P}_{\mathbb{G}}} = U_{\mathbb{G}}^*$ given in (11), and that of manufacturer and supplier, while operating alone, are respectively given by $w_{\mathbb{M}}^{\mathbb{P}_{\mathbb{A}}} = U_{\mathbb{M}}^*$ and $w_{\mathbb{S}}^{\mathbb{P}_{\mathbb{A}}} = U_{\mathbb{S}}^*$ of Theorem 1. These complete the definition of the partition form game. Also observe that any pay-off vector $\mathbf{x} = (x_{\mathbb{M}}, x_{\mathbb{S}})$ is consistent with GC partition if and only if $x_{\mathbb{M}} + x_{\mathbb{S}} = U_{\mathbb{G}}^* = w_{\mathbb{G}}^{\mathbb{P}_{\mathbb{G}}}$. On the other hand, the only payoff vector consistent with the ALC partition is $\mathbf{x} = (w_{\mathbb{M}}^{\mathbb{P}_{\mathbb{A}}}, w_{\mathbb{S}}^{\mathbb{P}_{\mathbb{A}}})$. The Theorems 1-2 immediately imply the following stability result:

Lemma 1. *In the single manufacturer SC: i) ALC partition* $\mathbb{P}_{\mathbb{A}}$ *is blocked by grand coalition; and ii) The GC-core, the set of consistent pay-off vectors that form stable configurations with* $\mathbb{P}_{\mathbb{G}}$ *(see* (8)*, is:*

$$\{\mathbf{x} : x_{\mathbb{S}} > 2\phi - O_{\mathbb{S}},\ x_{\mathbb{M}} > \phi - O_{\mathbb{M}} \text{ and } x_{\mathbb{S}} + x_{\mathbb{M}} = 4\phi - O_{\mathbb{G}}\}.$$

Proof. From (16), the pessimal anticipated utilities with $|N| = 2$, clearly equal $w_{\mathbb{C}}^{pa} = w_{\mathbb{C}}^{\mathbb{P}_{\mathbb{G}}}$ for any $\mathbb{C}$. Thus by Theorems 1-2, $w_{\mathbb{G}}^{pa} = w_{\mathbb{G}}^{\mathbb{P}_{\mathbb{G}}} > w_{\mathbb{M}}^{\mathbb{P}_{\mathbb{A}}} + w_{\mathbb{S}}^{\mathbb{P}_{\mathbb{A}}}$ and hence part (i); part (ii) follows by direct verification. □

Remarks. From (3)- (4), if the supplier and the manufacturer participate in a strategic form game (i.e., when they make choices simultaneously), the resultant Nash Equilibrium (NE) is (n_o, n_o) – the best response of the manufacturer is n_o for any $a_{\mathbb{S}} = q > C_{\mathbb{M}}$ or when $a_{\mathbb{S}} = n_o$, while that of the supplier is n_o when $a_{\mathbb{M}} = n_o$ and equals infinity when $a_{\mathbb{M}} \neq n_o$. Thus if both the agents compete at the same level, the SC would not operate and both of them derive 0 revenue.

On the other hand, when the supplier leads the market as in the SB game, by Theorem 1 the system operates resulting in positive revenues for both the agents. They derive even better utilities by operating together and hence GC is stable as shown in Lemma 1. However the supplier gets a much better share; again from Lemma 1, the share of supplier $x_{\mathbb{S}}$ is at least $2\phi - O_{\mathbb{S}}$ while that of the manufacturer is at most $2\phi - O_{\mathbb{M}}$. These observations motivate us to analyse a more generic SC with competition at the lower echelon. The aim in particular is to understand the stable configurations, the profit shares, etc., in the presence of lower echelon competition.

5 TWO MANUFACTURER SC

In this section, we explore the case of two echelon SC consisting of a supplier and two manufacturers. As explained in Section 2.2 and equation (1), the fraction of demand captured by each manufacturer is :

$$D_{\mathbb{M}_i}(\mathbf{a}_{\mathbb{M}}) = (\bar{d}_{\mathbb{M}_i} - \alpha_{\mathbb{M}_i} p_i + \varepsilon \alpha_{\mathbb{M}_{-i}} p_{-i}) \text{ for all } i. \quad (17)$$

If there is no cross-linking (i.e., if $\varepsilon = 0$), the demand functions get decoupled, and each of them resemble to that of the single manufacturer SC (see (5)).

One needs to derive the worths $\{w_{\mathbb{C}}^{\mathbb{P}}\}$ for all possible coalitions $\mathbb{C}$ and partitions $\mathbb{P}$ to study the stability aspects. The worth $w_{\mathbb{C}}^{\mathbb{P}}$ can be defined as the 'best' utility (the maximum sum utility) that the members of $\mathbb{C}$ can derive, while facing the competition from agents outside the coalition arranged as in partition $\mathbb{P}$.

The competition between various coalitions is captured via a Stackelberg game (as in Subsection 3.1), when at least one manufacturer is not collaborating with the supplier – the partitions of this type are, ALC partition $\mathbb{P}_{\mathbb{A}} = \{\mathbb{S}, \mathbb{M}_1, \mathbb{M}_2\}$, HC partition $\mathbb{P}_{\mathbb{H}} = \{\mathbb{S}, \mathbb{M}\}$ and the VC partition $\mathbb{P}_{\mathbb{V}i} = \{\mathbb{V}_i, \mathbb{M}_{-i}\}$. In all these cases, the leader is the coalition $\mathbb{C}_L$ with the supplier. The coalitions with only manufactures form the followers - the followers respond optimally for any given action $\mathbf{a}_L$ (the quoted prices or n_o) of the leader. The solution of the followers is either an optimizer (when all manufacturers form a coalition) or an NE. Let $\mathbf{a}^*_{\mathbb{M}}(\mathbf{a}_L)$ represent this solution in either case. The leader coalition is aware of this optimal choice, i.e., $\mathbf{a}^*_{\mathbb{M}}(\mathbf{a}_L)$ for every $\mathbf{a}_L$ is a common knowledge. Thus, the optimal choice of the leader is,

$$\mathbf{a}^*_L \in \arg\max_{\mathbf{a}_L} \sum_{j \in \mathbb{C}_L} U_j\left(\mathbf{a}_L, \mathbf{a}^*_{\mathbb{M}}(\mathbf{a}_L)\right)\right),$$

and then $(\mathbf{a}^*_L, \mathbf{a}^*_{\mathbb{M}}(\mathbf{a}^*_L))$ represents the SBE. We then define the worth of the leader coalition by:

$$w^{\mathbb{P}}_{\mathbb{C}_L} = \sum_{j \in \mathbb{C}_L} U_j\left(\mathbf{a}^*_L, \mathbf{a}^*_{\mathbb{M}}(\mathbf{a}^*_L)\right).$$

The worth of the rest of the coalitions of $\mathbb{P}$ can be defined similarly using the SBE $(\mathbf{a}^*_L, \mathbf{a}^*_{\mathbb{M}}(\mathbf{a}^*_L))$.

We are just left with the GC partition $\mathbb{P}_{\mathbb{G}} = \{\mathbb{G}\}$, which can be analysed exactly as in Subsection 3.2 and is considered in the immediate next – we once again assume 'operating-conditions' assumption **A**.2 (with terms like $C_{\mathbb{G}}$ etc., accordingly changed); *without loss of generality, we consider $\bar{d}_{\mathbb{M}_1} \geq \bar{d}_{\mathbb{M}_2}$.*

5.1 GC Partition

In GC partition $\mathbb{P}_{\mathbb{G}}$, the two manufacturers and the supplier operate together as explained in Subsection 2.3.2. The optimization problem is similar to that in (10), hence we have the following with proof exactly as in that of Theorem 2:

Corollary 1. *Assume **A.1** and **A.2**. The worth of $\mathbb{P}_{\mathbb{G}}$ defined using the optimizer is given by*

$$w^{\mathbb{P}_{\mathbb{G}}}_{\mathbb{G}} = U^*_{\mathbb{G}} = \frac{\left(\bar{d}_{\mathbb{M}} - \alpha_{\mathbb{M}} C_{\mathbb{G}}\right)^2}{4\alpha_{\mathbb{M}}}. \qquad \square$$

5.2 HC Partition

We now consider the HC partition $\mathbb{P}_{\mathbb{H}}$, where both the manufacturers $\mathbb{M} = \{M_1, M_2\}$ operate together. The coalition of manufacturers $\mathbb{M}$ quotes a selling price p to the customers, and the leader (supplier) S quotes a price q to $\mathbb{M}$. Recall any of them may decide not to operate (choose n_o). The SBE $\mathbf{a}^* := (\mathbf{a}^*_{\mathbb{M}}, a^*_{\mathbb{S}})$ of the Stackelberg game satisfies the following as before:

$$\mathbf{a}^*_{\mathbb{M}} = \mathbf{a}^*_{\mathbb{M}}(a^*_{\mathbb{S}}), \text{ and } a_{\mathbb{S}^*} \in \arg\max_{a_{\mathbb{S}} \in \mathscr{A}_{\mathbb{S}}} U_{\mathbb{S}}(a_{\mathbb{S}}, \mathbf{a}^*_{\mathbb{M}}(a_{\mathbb{S}})),$$

where the utilities and the optimizers are given by:

$$\begin{aligned} U_{\mathbb{S}}(\mathbf{a}) &:= \left(D_{\mathbb{M}}(\mathbf{a}_{\mathbb{M}})(q - C_{\mathbb{S}})\mathscr{F}_{\mathbb{M}} - O_{\mathbb{S}}\right)\mathscr{F}_{\mathbb{S}} \\ \mathbf{a}^*_{\mathbb{M}}(a_{\mathbb{S}}) &:= \arg\max_{a_{\mathbb{M}} \in \mathscr{A}_{\mathbb{M}}} U_{\mathbb{M}}(a_{\mathbb{M}}, a_{\mathbb{S}}) \text{ with} \\ U_{\mathbb{M}}(\mathbf{a}) &:= \left(D_{\mathbb{M}}(\mathbf{a}_{\mathbb{M}})(p - C_{\mathbb{M}} - q)\mathscr{F}_{\mathbb{S}} - O_{\mathbb{M}}\right)\mathscr{F}_{\mathbb{M}}, \end{aligned}$$

with $D_{\mathbb{M}}(a_{\mathbb{M}})$ defined in (2). This game is similar to that considered Subsection 3.1, and hence the following result using the proof of Theorem 1.

Corollary 2. *Assume **A.1** and **A.2**, the worths of the agents in $\mathbb{P}_{\mathbb{H}}$ (defined using operating SBE) equal:*

$$\{w^{\mathbb{P}_{\mathbb{H}}}_{\mathbb{S}}, w^{\mathbb{P}_{\mathbb{H}}}_{\mathbb{M}}\} = \{2\phi - O_{\mathbb{S}}, \phi - O_{\mathbb{M}}\},$$

where $\phi = \frac{\left(\bar{d}_{\mathbb{M}} - \alpha_{\mathbb{M}} C_{\mathbb{G}}\right)^2}{16\alpha_{\mathbb{M}}}$. $\square$

Using Corollaries 1-2, as in Lemma 1, it is easy to conclude that the HC partition is blocked by grand coalition $\mathbb{G}$ and hence is not stable.

5.3 Worth-Limits

For further analysis, one needs to study the ALC and VC partitions. However the expressions for these two partitions are significantly complex and hence we begin with a specific yet an important asymptotic case study in this conference paper – while the complete generality would be considered in future. We consider an *asymptotic regime near $(\varepsilon, \gamma) \approx (1, 1)$; as mentioned previously, here the customers are willing to switch the loyalties towards their manufacturers and hence we call such a regime as Essential and Substitutable-Manufacturer (ESM) regime.* We also consider manufacturers of equal reputation, *i.e., with* $\tilde{\alpha}_{\mathbb{M}_1} = \tilde{\alpha}_{\mathbb{M}_2}$. Towards obtaining the asymptotic study we consider the following procedure.

ESM Regime: For any partition-coalition $(\mathbb{P}, \mathbb{C})$, consider the function $(\gamma, \varepsilon) \mapsto (1-\gamma)(1-\varepsilon)w^{\mathbb{P}}_{\mathbb{C}}$. From all the expressions derived in this paper, i.e., for all $(\mathbb{P}, \mathbb{C})$, these functions are continuous. Hence the following limits exist (for each $(\mathbb{P}, \mathbb{C})$) and can be rewritten as below:

$$\begin{aligned} f^{\mathbb{P}}_{\mathbb{C}} &:= \lim_{(\gamma,\varepsilon) \to (1,1)} (1-\gamma)(1-\varepsilon) w^{\mathbb{P}}_{\mathbb{C}} \\ &= \lim_{\varepsilon \to 1} \lim_{\gamma \to 1} (1-\gamma)(1-\varepsilon) w^{\mathbb{P}}_{\mathbb{C}}. \end{aligned} \tag{18}$$

We refer to the above as worth-limits, with slight abuse of notation. Similarly define $\{f^{pa}_{\mathbb{C}}\}$ using anticipated worths $\{w^{pa}_{\mathbb{C}}\}$. We will also require the limits of the following derivatives

$$f_{\mathbb{C}}^{(1),\mathbb{P}} := \lim_{\varepsilon\to 1} \frac{d\tilde{w}_{\mathbb{C}}^{\mathbb{P}}}{d\varepsilon} \text{ with } \tilde{w}_{\mathbb{C}}^{\mathbb{P}} := (1-\varepsilon)\lim_{\gamma\to 1}(1-\gamma)w_{\mathbb{C}}^{\mathbb{P}}, \quad (19)$$

and also that of the anticipated worths, $\{f_{\mathbb{C}}^{(1),pa}\}$. *The idea is to derive the stability results by comparing the worth-limits instead of the actual worths* $\{w_{\mathbb{C}}^{\mathbb{P}}\}$, *and further using the derivative limits* (19) *when the worth-limits are equal.* We claim that such stability results are applicable for all (ε,γ) in a neighbourhood of $(1,1)$ because of the following reasons and procedure.

From (16) a configuration $(\mathbb{P},\mathbf{x})$ is stable if the following set of inequalities are satisfied:

$$\sum_{i\in\mathbb{C}} x_i \geq w_{\mathbb{C}}^{pa} \text{ for all } \mathbb{C} \notin \mathbb{P}. \quad (20)$$

(we identify only the configurations that satisfy the above with strict inequality, a more complete study is again a part of the future work, and the reasons for this omission is evident in the immediate next).

If the inequalities in (20) are satisfied in a strict manner by some vector $\mathbf{y}$ and for some partition $\mathbb{P}$ using limits $\{f_{\mathbb{C}}^{pa}\}$ in place of $w_{\mathbb{C}}^{pa}$, then by continuity there exists $\bar{\gamma}$ and $\bar{\varepsilon}$ such that the above inequalities (finitely many) are satisfied for all $\gamma > \bar{\gamma}$ and $\varepsilon > \bar{\varepsilon}$ — this implies that for all those (γ,ε), the configuration

$$(\mathbb{P},\beta\mathbf{y}), \text{ with, } \beta := \frac{1}{(1-\varepsilon)(1-\gamma)},$$

is stable; thus one can obtain stability results near ESM regime using the worth-limits $\{f_{\mathbb{C}}^{\mathbb{P}}, f_{\mathbb{C}}^{pa}\}$ (when strict inequalities are considered in (20)).

During blocking by mergers, i.e., say when blocking coalition $\mathbb{C} = \mathbb{C}_1 \cup \mathbb{C}_2$, then recall by consistency in (15), the inequality (20) modifies to

$$\sum_{i\in\mathbb{C}} x_i = w_{\mathbb{C}_1}^{P} + w_{\mathbb{C}_2}^{\mathbb{P}} \geq w_{\mathbb{C}}^{pa}. \quad (21)$$

And if now the worth-limits of both right and left hand sides are equal, then the comparison in neighbourhood is possible only by considering the derivatives. This is because for such limits, by Taylors series expansion, near $\varepsilon \approx 1$ we have (see (19)):

$$\begin{aligned} &\tilde{w}_{\mathbb{C}_1}^{P} + \tilde{w}_{\mathbb{C}_2}^{\mathbb{P}} - \tilde{w}_{\mathbb{C}}^{pa} \quad (22)\\ &= (\varepsilon-1)\left.\frac{d\left(\tilde{w}_{\mathbb{C}_1}^{P} + \tilde{w}_{\mathbb{C}_2}^{\mathbb{P}} - \tilde{w}_{\mathbb{C}}^{pa}\right)}{d\varepsilon}\right|_{\varepsilon\to 1} + o((1-\varepsilon)^2)\\ &= (\varepsilon-1)\left(f_{\mathbb{C}_1}^{(1),\mathbb{P}} + f_{\mathbb{C}_2}^{(1),\mathbb{P}} - f_{\mathbb{C}}^{(1),pa}\right) + o((1-\varepsilon)^2), \end{aligned}$$

where the limits $\{f_{\mathbb{C}}^{(1),pa}, f_{\mathbb{C}}^{(1),\mathbb{P}}\}$ are defined in (19). Thus the required stability results can be established if now the derivative limits satisfy the required inequalities – and then there exists an $\bar{\varepsilon} < 1$ such that the stability results are true in a neighbourhood as below:

$$\{(\varepsilon,\gamma) : \varepsilon \geq \bar{\varepsilon} \text{ and } \gamma \geq \bar{\gamma}_{\varepsilon}\}, \quad (23)$$

where $\bar{\gamma}_{\varepsilon} < 1$ is a lower bound depending upon ε.

Further to ensure that the coalitions under consideration are operating, one would require conditions like that in **A**.2. However these conditions are trivially satisfied in the limits $\gamma \to 1$, once $\bar{d}_{\mathbb{M}_i} > 0$ for all i; furthermore, the conditions will also be satisfied in the neighbourhood of $(1,1)$ (if required by shrinking the neighbourhood further) due to similar reasons.

5.4 VC Partition

Recall in the partition with vertical cooperation, $\mathbb{P}_{\mathbb{V}i}$ the supplier collaborates with one of the manufacturers M_i and competes with the other. The Stackelberg game is between the coalition $\mathbb{V}_i$ as leader and the manufacturer M_{-i} as follower. The manufacturer M_{-i} (when it operates) obtains raw material from $\mathbb{V}_i$, quotes p_{-i} and the demand $D_{\mathbb{M}_{-i}}$ attracted by M_{-i} also contributes towards the revenue of $\mathbb{V}_i$; the VC coalition $\mathbb{V}_i$ also derives utility due to its own demand $D_{\mathbb{M}_i}$ (recall here a direct price p_i is quoted to the customers). Thus the utilities of the two coalitions are:

$$\begin{aligned} U_{\mathbb{V}_i} = \big[&D_{\mathbb{M}_i}(\mathbf{a}_{\mathbb{M}})(p_i - C_{\mathbb{M}_i} - C_{\mathbb{S}}) + \mathscr{F}_{\mathbb{M}_{-i}} D_{\mathbb{M}_{-i}}(\mathbf{a}_{\mathbb{M}})(q - C_{\mathbb{S}})\\ &- O_{\mathbb{S}} - O_{\mathbb{M}_i}\big]\mathscr{F}_{\mathbb{V}_i}, \quad (24)\\ U_{\mathbb{M}_{-i}} = \big(&D_{\mathbb{M}_{-i}}(\mathbf{a}_{\mathbb{M}})(p_{-i} - q - C_{\mathbb{M}_{-i}})\mathscr{F}_{\mathbb{V}_i} - O_{\mathbb{M}_{-i}}\big)\mathscr{F}_{\mathbb{M}_{-i}}. \quad (25) \end{aligned}$$

The SBE $(\mathbf{a}_{\mathbb{V}_i}^*, a_{\mathbb{M}_{-i}}^*)$ (when exists) satisfies:

$$a_{\mathbb{M}_{-i}}^* = a_{\mathbb{M}_{-i}}^*(\mathbf{a}_{\mathbb{V}_i}^*),\ \ a_{\mathbb{M}_{-i}}^*(\mathbf{a}_{\mathbb{V}_i}) := \arg\max_{a_{\mathbb{M}_{-i}}} U_{\mathbb{M}_{-i}}(a_{\mathbb{M}_{-i}}, \mathbf{a}_{\mathbb{V}}), \text{ and}$$
$$\mathbf{a}_{\mathbb{V}_i}^* \in \arg\max_{\mathbf{a}_{\mathbb{V}}\in\mathscr{A}_{\mathbb{V}_i}} U_{\mathbb{V}_i}(\mathbf{a}_{\mathbb{V}}, a_{\mathbb{M}_{-i}}^*(\mathbf{a}_{\mathbb{V}})),$$

and defines the worths, $w_{\mathbb{V}_i}^{\mathbb{P}_{\mathbb{V}}} = U_{\mathbb{V}_i}(\mathbf{a}^*)$ and $w_{\mathbb{M}_{-i}}^{\mathbb{P}_{\mathbb{V}}} = U_{\mathbb{M}_{-i}}(\mathbf{a}^*)$. As already mentioned, it is complicated to analyze this game theoretically, we instead obtain the ESM limits in the following:

Lemma 2. *Assume* $\alpha_{\mathbb{M}_1} = \alpha_{\mathbb{M}_2} = \alpha$. *The worth-limits* $(f_{\mathbb{V}_i}^{\mathbb{P}_{\mathbb{V}i}}, f_{\mathbb{M}_{-i}}^{\mathbb{P}_{\mathbb{V}i}})$ *and the derivative limits* $(f_{\mathbb{V}_i}^{(1),\mathbb{P}_{\mathbb{V}i}}, f_{\mathbb{M}_{-i}}^{(1),\mathbb{P}_{\mathbb{V}i}})$ *for ESM regime are respectively in Tables 1a and 1b.*

Proof. Refer to (Wadhwa et al., 2024) for the proof. □

5.5 ALC Partition

Recall the partition, $\mathbb{P}_{\mathbb{A}} = \{\mathbb{S}, \mathbb{M}_1, \mathbb{M}_2\}$, where all the agents operate alone and compete with each other. In this partition, we have a SB game, where supplier $\mathbb{S}$ is the leader, quoting its price via the action $a_{\mathbb{S}}$; the competing manufacturers $\{\mathbb{M}_1, \mathbb{M}_2\}$ are followers in the lower echelon, who respond to $a_{\mathbb{S}}$ via a non-cooperative strategic form game (inner game between

Table 1: ESM regime near $(\gamma, \varepsilon) \approx (1,1)$.

$f_{\mathbb{G}}^{\mathbb{P}_{\mathbb{G}}} = 0$	$\left(f_{\mathbb{S}}^{\mathbb{P}_{\mathbb{H}}}, f_{\mathbb{M}}^{\mathbb{P}_{\mathbb{H}}}\right) = (0,0)$
$\left(f_{\mathbb{V}_i}^{\mathbb{P}_{\mathbb{V}i}}, f_{\mathbb{M}_{-i}}^{\mathbb{P}_{\mathbb{V}i}}\right) = \left(\frac{\bar{d}_{\mathbb{M}}^2}{8\tilde{\alpha}}, 0\right)$	$\left(f_{\mathbb{S}}^{\mathbb{P}_{\mathbb{A}}}, f_{\mathbb{M}_1}^{\mathbb{P}_{\mathbb{A}}}, f_{\mathbb{M}_2}^{\mathbb{P}_{\mathbb{A}}}\right) = \left(\frac{\bar{d}_{\mathbb{M}}^2}{8\tilde{\alpha}}, 0, 0\right)$

(a) Worth-limits

$f_{\mathbb{V}_i}^{(1),\mathbb{P}_{\mathbb{V}i}} = \frac{2\bar{d}_{\mathbb{M}_i}\bar{d}_{\mathbb{M}_{-i}} + \bar{d}_{\mathbb{M}_{-i}}^2 - \bar{d}_{\mathbb{M}_i}^2}{16\tilde{\alpha}}, \ f_{\mathbb{M}_{-i}}^{(1),\mathbb{P}_{\mathbb{V}i}} = -\frac{\bar{d}_{\mathbb{M}_{-i}}^2}{16\tilde{\alpha}}$
$f_{\mathbb{M}_i}^{(1),\mathbb{P}_{\mathbb{A}}} = \frac{-\left(5\bar{d}_{\mathbb{M}_i} + \bar{d}_{\mathbb{M}_{-i}}\right)^2}{144\tilde{\alpha}} \ \forall i, \ f_{\mathbb{S}}^{(1),\mathbb{P}_{\mathbb{A}}} = \frac{\bar{d}_{\mathbb{M}}^2}{8\tilde{\alpha}}$

(b) Derivative-limits

the manudacturers), parametrized by fixed action of supplier $a_{\mathbb{S}}$. As a leader of SB game, the supplier is aware of the Nash Equilibrium (NE) $\mathbf{a}_{\mathbb{M}}^*(a_{\mathbb{S}})$ of the inner game $\mathbf{a}_{\mathbb{M}}^*(a_{\mathbb{S}})$ for every $a_{\mathbb{S}}$. The utility of each of these agents are given in (3)-(4). The utility of the supplier $\mathbb{S}$ and a manufacturer $\mathbb{M}_i$ is ,

$$U_{\mathbb{S}}(a_{\mathbb{S}}) = \left(\left(\sum_{i=1}^{2} D_{\mathbb{M}_i}(\mathbf{a}_{\mathbb{M}}^*(a_{\mathbb{S}}))\mathscr{F}_{\mathbb{M}_i}\right)(q - C_{\mathbb{S}}) - O_{\mathbb{S}}\right)\mathscr{F}_{\mathbb{S}}, \tag{26}$$

$$U_{\mathbb{M}_i}(\mathbf{a}_{\mathbb{M}}; a_{\mathbb{S}}) = \left(\left(D_{\mathbb{M}_i}(\mathbf{a}_{\mathbb{M}}(a_{\mathbb{S}}))(p_i - C_{\mathbb{M}_i} - q)\mathscr{F}_{\mathbb{S}} - O_{\mathbb{M}_i}\right)\mathscr{F}_{\mathbb{M}}. \tag{27}$$

The equilibrium of the SB game satisfies

$$a_{\mathbb{S}}^* = \arg\max_{a_{\mathbb{S}}} U_{\mathbb{S}}\left(a_{\mathbb{S}}, \mathbf{a}_{\mathbb{M}}^*(a_{\mathbb{S}})\right)),$$
$$\mathbf{a}_{\mathbb{M}_i}^* = \arg\max_{a_{\mathbb{M}_i}} U_{\mathbb{M}_i}(a_{\mathbb{M}_i}, a_{\mathbb{M}_{-i}}; a_{\mathbb{S}}^*) \text{ for all } i.$$

The worth of supplier $\mathbb{S}$ is, $w_{\mathbb{S}}^{\mathbb{P}_{\mathbb{A}}} = U_{\mathbb{S}}(a_{\mathbb{S}}^*)$, and the worth of manufacturer $\mathbb{M}_i$ is, $w_{\mathbb{M}_i}^{\mathbb{P}_{\mathbb{A}}} = U_{\mathbb{M}_i}(\mathbf{a}_{\mathbb{M}}^*)$. As already mentioned, it is complicated to analyze this game theoretically, we instead obtain the ESM limits in the following:

Lemma 3. *Assume $\alpha_{\mathbb{M}_1} = \alpha_{\mathbb{M}_2} = \alpha$. The worth-limits $(f_{\mathbb{S}}^{\mathbb{P}_{\mathbb{A}}}, f_{\mathbb{M}_i}^{\mathbb{P}_{\mathbb{A}}}, f_{\mathbb{M}_{-i}}^{\mathbb{P}_{\mathbb{A}}})$ and the derivative limits $(f_{\mathbb{S}}^{(1),\mathbb{P}_{\mathbb{A}}}$, $f_{\mathbb{M}_i}^{(1),\mathbb{P}_{\mathbb{A}}}$ $f_{\mathbb{M}_{-i}}^{(1),\mathbb{P}_{\mathbb{A}}})$ for ESM regime are respectively in Tables 1a and 1b.*

Proof. Refer to (Wadhwa et al., 2024) for the proof. □

6 STABILITY RESULTS: ESM REGIME

We have derived the worths $\{w_{\mathbb{C}}^{\mathbb{P}}\}$ and the worth-limits $\{f_{\mathbb{C}}^{\mathbb{P}}\}$ in the previous section and now aim to identify the partitions and configurations that are stable in ESM regime. We have worth-limits only for VC and ALC partitions, and thus for the comparison purposes (see (16), (20) and (21)), compute the same for GC and HC partitions. By Corollaries 1-2, $f_{\mathbb{G}}^{\mathbb{P}_{\mathbb{G}}} = f_{\mathbb{M}}^{\mathbb{P}_{\mathbb{H}}} = f_{\mathbb{S}}^{\mathbb{P}_{\mathbb{H}}} = 0$; these are also tabulated in Table 1.

From Table 1, the immediate result is that, the GC and HC partitions (irrespective of the pay-off vectors) are both blocked by coalition $\mathbb{V}_i$. Thus none among these two partitions are stable. As discussed at the end of Subsection 2.2, the customers may no longer feel the product is essential in the absence of choices, and this may be the reason for non-stability of the GC and HC partitions.[1]

We now identify the stable configurations. Towards this, the pessimal anticipatory worth-limits (see equation 16), using Table 1, are:

$$\begin{aligned} f_{\mathbb{S}}^{pa} &= \min\{f_{\mathbb{S}}^{\mathbb{P}_{\mathbb{H}}}, f_{\mathbb{S}}^{\mathbb{P}_{\mathbb{A}}}\} = 0 \\ f_{\mathbb{M}}^{pa} &= f_{\mathbb{M}}^{\mathbb{P}_{\mathbb{H}}} = 0 \end{aligned} \tag{28}$$

$$\begin{aligned} f_{\mathbb{M}_i}^{pa} &= \min\{f_{\mathbb{M}_i}^{\mathbb{P}_{\mathbb{V}-i}}, f_{\mathbb{M}_i}^{\mathbb{P}_{\mathbb{A}}}\} = 0, \ f_{\mathbb{G}}^{pa} = 0, \text{ and} \\ f_{\mathbb{V}_i}^{pa} &= f_{\mathbb{V}_i}^{\mathbb{P}_{\mathbb{V}i}} = \frac{\bar{d}_{\mathbb{M}}^2}{8\tilde{\alpha}}. \end{aligned} \tag{29}$$

In the following we prove that: i) the partitions $\mathbb{P}_{\mathbb{A}}$ and $\mathbb{P}_{\mathbb{V}2}$ are not stable, while $\mathbb{P}_{\mathbb{V}1}$ is stable when

$$\bar{d}_{\mathbb{M}_1} > (\sqrt{2}+1)\bar{d}_{\mathbb{M}_2}; \tag{30}$$

and ii) none of the partitions are stable when the above inequality (30) is negated strictly. Towards this we establish few (strict) inequalities at ESM limit for each configuration – the set of inequalities are strict and demonstrate either that the configuration is stable or that a coalition blocks it. Then the said configuration remains stable (or is blocked by the said coalition) for all (γ, ε) around $(1,1)$ and in a set as in (23) of subsection 5.3. We begin with ALC partition.

ALC Partition Is Not Stable. Note that there is only one scaled configuration at limit involving ALC

[1]Observe when a worth-limit is non-zero,the corresponding worth/optimal-revenue increases to infinity as $(\varepsilon, \gamma) \to (1,1)$ and this is true only for worths related to VC and ALC partitions in ESM regime; by Corollaries 1- 2 the worths for GC and HC partitions also increase to infinity with $\gamma \to 1$, but not with $\varepsilon \to 1$ and hence the corresponding worth-limits are zero.

$(\mathbb{P}_{\mathbb{A}}, \mathbf{y})$ with $y_{\mathbb{S}} = f_{\mathbb{S}}^{\mathbb{P}_{\mathbb{A}}}$, $y_{\mathbb{M}_1} = f_{\mathbb{M}_1}^{\mathbb{P}_{\mathbb{A}}}$ and $y_{\mathbb{M}_2} = f_{\mathbb{M}_2}^{\mathbb{P}_{\mathbb{A}}}$, because of consistency. The coalitions that can possibly block ALC partition are the merger coalitions such as $\mathbb{G}$, $\mathbb{M}$, $\mathbb{V}_i$. Consider a merger $\mathbb{V}_i$ which gets the same utility as $y_{\mathbb{S}} + y_{\mathbb{M}_i}$ of ALC at limit. Thus we compare using the derivative limits of Table 1b ,

$$f_{\mathbb{S}}^{(1),\mathbb{P}_{\mathbb{A}}} + f_{\mathbb{M}_i}^{(1),\mathbb{P}_{\mathbb{A}}} = -\frac{\bar{d}_{\mathbb{M}}^2}{8\bar{\alpha}} + \frac{\left(5\bar{d}_{\mathbb{M}_i} + \bar{d}_{\mathbb{M}_{-i}}\right)^2}{144\bar{\alpha}} < \frac{\bar{d}_{\mathbb{M}_i}^2 - 2\bar{d}_{\mathbb{M}_i}\bar{d}_{\mathbb{M}_{-i}} - \bar{d}_{\mathbb{M}_{-i}}^2}{16\bar{\alpha}} = f_{\mathbb{V}_i}^{(1),\mathbb{P}_{\mathbb{V}}}, \quad (31)$$

and hence the merger $\mathbb{V}_i$ blocks $\mathbb{P}_{\mathbb{A}}$ (see (22) and observe $(\varepsilon - 1)$ is negative) and this is true with a strict inequality at limit. Therefore, $\mathbb{P}_{\mathbb{A}}$ is not a stable partition in the ESM regime.

Continuing in this manner we obtain the following result (the remaining proof is in the Appendix):

Theorem 3. [ESM regime] *Consider any* $\bar{\alpha}, \bar{d}_{\mathbb{M}_1}, \bar{d}_{\mathbb{M}_2}, \{C_{\mathbb{C}}\}$ *and* $\{O_{\mathbb{C}}\}$ *with* $\bar{d}_{\mathbb{M}_1} \geq \bar{d}_{\mathbb{M}_2}$. *There exists a* $\bar{\varepsilon} < 1$ *such that for every* $\varepsilon \in (\bar{\varepsilon}, 1)$, *there exists a* $\bar{\gamma}_{\varepsilon} < 1$ *and for any system with above parameters and with* $(\varepsilon, \gamma) \in \{\varepsilon \geq \bar{\varepsilon}, \gamma \geq \bar{\gamma}_{\varepsilon}\}$, *the following are true:*

i) When $\bar{d}_{\mathbb{M}_2} \leq \bar{d}_{\mathbb{M}_1} < (\sqrt{2}+1)\bar{d}_{\mathbb{M}_2}$, *none of the partitions are stable.*

ii) When $\bar{d}_{\mathbb{M}_1} > (\sqrt{2}+1)\bar{d}_{\mathbb{M}_2}$, *then only* $\mathbb{P}_{\mathbb{V}1}$ *partition is stable. Further the configuration* $(\mathbb{P}_{\mathbb{V}1}, \mathbf{x})$ *is stable if*

$$x_{\mathbb{M}_1} \in \left(w_{\mathbb{M}}^{pa} - w_{\mathbb{M}_2}^{\mathbb{P}_{\mathbb{V}1}},\ w_{\mathbb{V}_1}^{\mathbb{P}_{\mathbb{V}1}} - w_{\mathbb{V}_2}^{\mathbb{P}_{\mathbb{V}2}} + w_{\mathbb{M}_2}^{\mathbb{P}_{\mathbb{V}1}} \right), \quad (32)$$

and the above interval is non-empty. □

Remarks. When there is a huge disparity between the two manufacturers in terms of the dedicated market demands, the SC has stable configurations. The supplier prefers to collaborate with stronger manufacturer ($\mathbb{P}_{\mathbb{V}1}$ is stable) and the weaker manufacturer has no choice (no partner finds it beneficial to oppose $\mathbb{P}_{\mathbb{V}1}$ by collaborating with weaker M_2).

However the share of the manufacturer that collaborates with supplier is negligible in comparison to that of the supplier (the upper bound on scaled payoff $(1-\varepsilon)(1-\gamma)x_{\mathbb{M}_1}$ to 0 as seen from (32) and Table 1, while the lower bound on that of the supplier converges to $\bar{d}_{\mathbb{M}}^2/8\bar{\alpha}$); in fact the scaled share of the non-collaborating manufacturer M_2 also converges to 0 (from Table 1 $f_{\mathbb{M}_2}^{\mathbb{P}_{\mathbb{V}1}} = 0$). Thus in the essential and substitutable manufacturer regime, the supplier has even higher advantage than that in the single manufacturer SC – the competition at the lower echelon added with substitutability significantly improved the benefits and the position of the supplier.

Another interesting aspect is that the supplier prefers to collaborate with stronger manufacturer ($\mathbb{P}_{\mathbb{V}1}$ is stable, but not $\mathbb{P}_{\mathbb{V}2}$) – in the prelimit, the supplier derives better coalitional utility when it collaborates with stronger M_1.

On the other hand, when the manufacturers are of comparable strengths, as in part i of Theorem 3, the SC has no stable configuration. The system probably keeps switching configurations (any operating configuration is blocked by one or the other coalition).

7 CONCLUSIONS

The main takeaway of this work is that it establishes the possible stability of vertical mergers (the collaboration between the supplier and a manufacturer) and instability of centralised supply chain (all the members of SC); this is true for the SC that supplies essential goods and that caters to not-so loyal customers. This is in contrast to the current literature which usually establishes the stability of centralized SC. This contrast is the consequence of the realistic consideration of the partition-form aspects – where the worth of a coalition depends upon the arrangement of the agents outside the coalition. When the manufacturers are significantly different in terms of market power, the vertical cooperation (or merger) between the supplier and the stronger manufacturer is stable, while the weaker manufacturer is left-out to compete with the collaborating pair. Surprisingly, no collaboration is stable when the manufacturers are of comparable market strengths.

In reality, the worth of any coalition depends upon the arrangement of opponents in the market space; for example, the revenue generated by the supplier with the two manufacturers operating together is different from that when the two manufacturers operate independently. This realistic aspect is captured by studying the SC using partition-form games, which paved way to the above mentioned contrasting results.

Further, the competition at the lower echelon significantly favors the higher level supplier – the supplier enjoys a huge fraction of the revenue generated, while the manufacturers draw a negligible fraction irrespective of their market powers and irrespective of whether they collaborate with the supplier or not.

This research has opened up many new questions – which configurations are stable in an SC that supplies luxury goods (non-essential) or in a SC with loyal customers? More interesting questions are about the stable configurations with competition at both the echelons along with vertical competition.

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APPENDIX

In this appendix, we consider the following optimization problem and derive its solution:

$$U(a) = \left((\bar{d}-\alpha p)^{+}(p-c) - O_c\right)\mathbb{1}_{\{a \neq n_o\}}. \qquad (33)$$

Lemma 4. *Define* $\Delta = \bar{d} - \alpha c - 2\sqrt{\alpha O_c}$. *The maximizer and the maximum value of* (33) *is given by:*

$$a^* = p^* \mathbb{1}_{\{\Delta>0\}} + n_o \mathbb{1}_{\{\Delta<0\}}, \text{ where } p^* = \frac{\bar{d}+c\alpha}{2\alpha},$$
$$U(a^*) = \left(\frac{(\bar{d}-\alpha c)^2}{4\alpha} - O_c\right)\mathbb{1}_{\{\Delta>0\}}.$$

When $\Delta = 0$, *we have two optimizers,* p^* *and* n_o.

Proof. Towards solving (33), we first consider optimizing the interior objective, more precisely, $w(p) = (\bar{d}-\alpha p)(p-c)$ only w.r.t. p. The solution to this optimization problem (using derivative techniques) is,

$$p^* = \frac{\bar{d}}{2\alpha} + \frac{c}{2}, \text{ and } w^* = w(p^*) = \frac{(\bar{d}-\alpha c)^2}{4\alpha}. \qquad (34)$$

Returning to the original problem (33), if $\Delta > 0$, then,

$$\alpha p^* = \frac{\bar{d}}{2} + \frac{\alpha c}{2} < \bar{d} - \sqrt{\alpha O_c} < \bar{d},$$

and hence, $(\bar{d}-\alpha p^*)^+ = \bar{d} - \alpha p^*$, $w^* - O_c > 0$. Thus, p^* is also the maximizer of (33) with $U^* = w^* - O_c$. If $\Delta < 0$ then n_o is the optimizer, as $U^* = U(n_o) = 0 > w^* - O_c$. The last sentence now follows trivially. □

Proof continued, Theorem 3. Without loss of generality, consider stability of $\mathbb{P}_{\mathbb{V}1}$. To ensure $(\mathbb{P}_{\mathbb{V}1}, \mathbf{x})$ is stable, it should not be blocked by HC coalition $\mathbb{M}$, as well as VC coalition $\mathbb{V}_2$. In other words, we require,

$$w_{\mathbb{M}}^{pa} - w_{\mathbb{M}_2}^{\mathbb{P}_{\mathbb{V}1}} \leq x_{\mathbb{M}_1} \leq w_{\mathbb{V}_1}^{\mathbb{P}_{\mathbb{V}1}} - w_{\mathbb{V}_2}^{\mathbb{P}_{\mathbb{V}2}} + w_{\mathbb{M}_2}^{\mathbb{P}_{\mathbb{V}1}}, \qquad (35)$$

and this is because for any pay-off vector consistent with $\mathbb{P}_{\mathbb{V}1}$, we have $x_{\mathbb{M}_2} = w_{\mathbb{M}_2}^{\mathbb{P}_{\mathbb{V}1}}$ and $x_{\mathbb{M}_1} + x_{\mathbb{S}} = w_{\mathbb{V}_1}^{\mathbb{P}_{\mathbb{V}1}}$ and $w_{\mathbb{V}_2}^{pa} = w_{\mathbb{V}_2}^{\mathbb{P}_{\mathbb{V}2}}$. Towards this, consider the limit[2]

$$\begin{aligned}
&\frac{\tilde{w}_{\mathbb{V}_1}^{\mathbb{P}_{\mathbb{V}1}} - \tilde{w}_{\mathbb{V}_2}^{\mathbb{P}_{\mathbb{V}2}} + \tilde{w}_{\mathbb{M}_2}^{\mathbb{P}_{\mathbb{V}1}} - (\tilde{w}_{\mathbb{M}}^{pa} - \tilde{w}_{\mathbb{M}_2}^{\mathbb{P}_{\mathbb{V}1}})}{1-\varepsilon} \\
&= \left(\frac{-2\bar{d}_{\mathbb{M}_1}\bar{d}_{\mathbb{M}_2} + \bar{d}_{\mathbb{M}_1}^2 - \bar{d}_{\mathbb{M}_2}^2}{16\tilde{\alpha}}\right) - \left(\frac{-2\bar{d}_{\mathbb{M}_1}\bar{d}_{\mathbb{M}_2} + \bar{d}_{\mathbb{M}_2}^2 - \bar{d}_{\mathbb{M}_1}^2}{16\tilde{\alpha}}\right) \\
&\qquad + o((1-\varepsilon)) + 2\frac{\bar{d}_{\mathbb{M}_2}^2}{16\tilde{\alpha}} - \frac{\bar{d}_{\mathbb{M}}^2}{16\tilde{\alpha}} \\
&= \frac{\bar{d}_{\mathbb{M}_1}^2 - \bar{d}_{\mathbb{M}_2}^2}{8\tilde{\alpha}} + \frac{\bar{d}_{\mathbb{M}_2}^2}{8\tilde{\alpha}} - \frac{\bar{d}_{\mathbb{M}}^2}{16\tilde{\alpha}} + o((1-\varepsilon)) \\
&= \frac{\bar{d}_{\mathbb{M}_1}^2 - \bar{d}_{\mathbb{M}_2}^2 - 2\bar{d}_{\mathbb{M}_1}\bar{d}_{\mathbb{M}_2}}{16\tilde{\alpha}} + o((1-\varepsilon)) \qquad (36)
\end{aligned}$$

We get the above as $\tilde{w}_{\mathbb{M}}^{pa} = \tilde{w}_{\mathbb{M}}^{\mathbb{P}_{\mathbb{H}}}$ and then refer to Table 1b and equation (22). Similarly, to ensure the configuration $(\mathbb{P}_{\mathbb{V}1}, \mathbf{x})$ is not blocked by singletons $\mathbb{S}$ and $\mathbb{M}_2$, we require

$$w_{\mathbb{M}_1}^{pa} \leq x_{\mathbb{M}_1} \leq w_{\mathbb{V}_1}^{\mathbb{P}_{\mathbb{V}1}} - w_{\mathbb{S}}^{pa}. \qquad (37)$$

Finally, for stability against blocking by GC, we require

$$w_{\mathbb{V}_1}^{\mathbb{P}_{\mathbb{V}1}} + w_{\mathbb{M}_2}^{\mathbb{P}_{\mathbb{V}1}} \geq w_{\mathbb{G}}^{\mathbb{P}_{\mathbb{G}}}. \qquad (38)$$

In view of the above three required inequalities (35), (37) and (38), we require (if possible) an

[2] It is easy to observe that the derivative limit in case of HC from Corollary 2 is equal to $f_{\mathbb{M}}^{1,\mathbb{P}_{\mathbb{H}}} = -\bar{d}_{\mathbb{M}}^2/16\tilde{\alpha}$.

$\bar{\varepsilon} < 1$ such that the following inequalities are satisfied for each $\varepsilon \geq \bar{\varepsilon}$:

$$\begin{aligned} \tilde{w}_{\mathbb{V}_1}^{\mathbb{P}_{\mathbb{V}1}} - \tilde{w}_{\mathbb{V}_2}^{\mathbb{P}_{\mathbb{V}2}} + \tilde{w}_{\mathbb{M}_2}^{\mathbb{P}_{\mathbb{V}1}} - (\tilde{w}_{\mathbb{M}}^{pa} - \tilde{w}_{\mathbb{M}_2}^{\mathbb{P}_{\mathbb{V}1}}) &> 0, \\ \tilde{w}_{\mathbb{V}_1}^{\mathbb{P}_{\mathbb{V}1}} + \tilde{w}_{\mathbb{M}_2}^{\mathbb{P}_{\mathbb{V}1}} - \tilde{w}_{\mathbb{G}}^{\mathbb{P}_{\mathbb{G}}} &> 0, \text{ and} \\ \tilde{w}_{\mathbb{V}_1}^{\mathbb{P}_{\mathbb{V}1}} - \tilde{w}_{\mathbb{S}}^{pa} - \tilde{w}_{\mathbb{M}_1}^{pa} &> 0. \end{aligned} \quad (39)$$

If the above inequalities are satisfied, then one can choose by continuity a $\bar{\gamma}_\varepsilon < 1$ for each $\varepsilon \geq \bar{\varepsilon}$ such that the strict inequalities in (39) are now satisfied with $\{w_{\mathbb{C}}^{\mathbb{P}}\}$ in place of $\{\tilde{w}_{\mathbb{C}}^{\mathbb{P}}\}$, for all $\gamma \geq \bar{\gamma}_\varepsilon$ and for each $\varepsilon \geq \bar{\varepsilon}$. Clearly, for such (ε, γ), the configuration $(\mathbb{P}_{\mathbb{V}1}, \mathbf{x})$ is stable when:

$$x_{\mathbb{M}_1} \in \left(w_{\mathbb{M}}^{pa} - w_{\mathbb{M}_2}^{\mathbb{P}_{\mathbb{V}1}},\ w_{\mathbb{V}_1}^{\mathbb{P}_{\mathbb{V}1}} - w_{\mathbb{V}_2}^{\mathbb{P}_{\mathbb{V}2}} + w_{\mathbb{M}_2}^{\mathbb{P}_{\mathbb{V}1}} \right), \quad (40)$$

and the above interval is non-empty.

There exists an $\bar{\varepsilon} < 1$ such that the last two inequalities of (39) are satisfied (see Table 1 and (Wadhwa et al., 2024) for the proof of the values in the table, which follows in similar lines as (31)).

Thus from (36), all three strict inequalities of (39) are definitely satisfied (if required for a larger $\bar{\varepsilon}$), when (30) is satisfied.

Instability. On the other hand, say (30) is negated with strict inequality. Then from (36) there exists an $\bar{\varepsilon} < 1$ such that for all $\varepsilon \geq \bar{\varepsilon}$ the following is satisfied:

$$\tilde{w}_{\mathbb{V}_1}^{\mathbb{P}_{\mathbb{V}1}} - \tilde{w}_{\mathbb{V}_2}^{\mathbb{P}_{\mathbb{V}2}} + \tilde{w}_{\mathbb{M}_2}^{\mathbb{P}_{\mathbb{V}1}} < (\tilde{w}_{\mathbb{M}}^{pa} - \tilde{w}_{\mathbb{M}_2}^{\mathbb{P}_{\mathbb{V}1}})$$

As before there exists $\bar{\gamma}_\varepsilon$ for each $\varepsilon \geq \bar{\varepsilon}$, and then for any $\gamma \geq \bar{\gamma}_\varepsilon$ and ε we have:

$$w_{\mathbb{V}_1}^{\mathbb{P}_{\mathbb{V}1}} - w_{\mathbb{V}_2}^{\mathbb{P}_{\mathbb{V}2}} + w_{\mathbb{M}_2}^{\mathbb{P}_{\mathbb{V}1}} < (w_{\mathbb{M}}^{pa} - w_{\mathbb{M}_2}^{\mathbb{P}_{\mathbb{V}1}}).$$

For all such (ε, γ) there exists no pay-off division, to be more precise, no $x_{\mathbb{M}_1}$ such that (35) is satisfied. Thus $\mathbb{P}_{\mathbb{V}1}$ is blocked for any configuration either by $\mathbb{V}_2$ or by HC.

Now assume without loss of generality, $\bar{d}_{\mathbb{M}_1} \geq \bar{d}_{\mathbb{M}_2}$. Then (30) can never be satisfied when roles of $\mathbb{M}_2$ and $\mathbb{M}_1$ interchanged and thus the partition $\mathbb{P}_{\mathbb{V}2}$ is never stable in ESM regime. However, as proved above, $\mathbb{P}_{\mathbb{V}1}$ is stable with payoff vectors additionally satisfying (40) when (30) is satisfied and unstable when (30) is negated with strict inequality.

Thus we have proved the theorem. □