Multi-Server Queue, with Heterogeneous Service Valuations Induced by Travel Costs

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Abstract: This work presents a variation of Naor's strategic observable model (Naor, 1969) for a loss system M/G/2/2, with a heterogeneous service valuations induced by the location of customers in relation to two servers, A, located at the origin, and B, located at M. Customers incur a "travel cost" which depends linearly on the distance of the customer from the server. Arrival of customers is assumed to be Poisson with a rate that is the integral of a nonnegative intensity function. We find the Nash equilibrium threshold strategy of the customers, and formulate the conditions that determine the optimal social welfare strategy. For the symmetric case (i.e., both servers have the same parameters and the intensity function is symmetric), we find the socially optimal strategies; Interestingly, we find that when only one server is idle, then under social optimality, the server also serves far away consumers, consumers whom he would not serve if he was a single server (i.e., in M/M/1/1).

1 INTRODUCTION

Customers of a service system often have heterogeneous service valuations, and this heterogeneity may be caused by various reasons. In this paper, we study a model with two servers, each located at a different site, therefore a consumer (in general) incures different "travel costs" when arriving at each service site. In such circumstances, customers need to decide whether to arrive for service, and if so, to what service point to arrive. A realistic example may be that of a network of public schools, hospitals, etc., from which an individual needs to choose. Of course "location" may refer to a geographic location or it may serve as a metaphoric way expressing different preferences on the ideal type of service.

The performance of service systems with strategic customers has attracted much attention in recent years (see, for example, Hassin & Haviv, 2003; Hassin, 2016). Naor (1969), was the first to introduce a queueing model that describes customer rational decisions. The model considers an FCFS M/M/1 system with homogeneous customers, a fixed reward associated with service completion, and linear waiting costs. The Nash equilibrium solution in Naor's model is simple since there exists a dominant pure threshold strategy n_e, such that an arriving customer joins the queue if and only if the observed queue upon arrival is shorter than n_e. This strategy maximizes the individual's expected welfare regardless of the strategies adopted by the others. The socially-optimal behavior is also characterized by a pure threshold strategy n^*, such that $n^* \leq n_e$.

Naor assumes that customers are homogeneous with respect to service valuation, and much of the literature on observable queues (i.e., assuming customers know the queue length before joining it) follow this assumption. Some exceptions are described in Section 2.5 of Hassin & Haviv (2003). For example, Larsen (1998) assumes that the service value is a continuous random variable and proves that the profits and social welfare are unimodal functions of the price. For the case of a loss system (where customers join *iff* the server is idle), Larsen proves that the profit-maximizing fee exceeds the socially optimal fee. Miller and Buckman (1987) consider an M/M/s/s loss system with heterogeneous service values and characterize the socially optimal fee.

Some authors investigated the *price of anarchy* (PoA) in various service systems (see, for example, Koutsoupias & Papadimitriou, 1999; Mavronicolas & Spirakis, 2001; Hassin, 2016). The *price of an-*

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archy measures the inefficiency of selfish behavior. It is defined as the ratio of the social welfare under optimum to the Social welfare in equilibrium. In Gilboa-Freedman, Hassin and Kerner (2014), the PoA in Naor's model is shown to have an odd behavior. It increases sharply (from 1.5 to 2) as the arrival rate comes close to the service rate and becomes unbounded exactly when the arrival rate is greater than the service rate, which is odd since the system is always stable.

Most relevant to our work is the work of Hassin, Nowik and Shaki (2018), in which heterogeneity in service valuation is introduced through a Hotelling-type model where customers reside in a "linear city" and incur "transportation costs" from their locations to the location of the server. Similar models have been investigated (e.g. D'aspremont & Jaskold, 1979; Dobson & Stavrulaki, 2007; Economides, 1986; Gallay, Olivier and Max-Olivier Hongler, 2008; Hotelling, 1929; Kwasnica & Euthemia, 2008; Pangburn & Stavrulaki, 2008; Ray & Jewkes, 2004; §6.7 and §7.5 in Hassin, 2016) but they all assume a constant density (possibly restricted to an interval). In contrast, Hassin, Nowik and Shaki allow non-uniform distributions of customer locations, and the potential arrival of customers with distances less than x from the service facility is assumed to be distributed according to Poisson with rate $\lambda(x) = \int_0^x h(y)dy < \infty$, where $h(y)$ is a nonnegative "intensity" function of the distance y. The definition of $\lambda(x)$ by an integral is natural since the customers *accumulate* from location 0, to location x. The intensity function and (linear) travel costs jointly generate the distribution of customer service valuations. A simple example is a two-dimensional city, in which the arrival of customers is uniform. In this case the intensity function can be defined as $h(x) = 2\pi x$, and so the arrival of customers with distances less than x is assumed to be a Poisson process with rate $\lambda(x) = \int_0^x 2\pi y dy = \pi x^2$. In a loss system M/G/1/1, Hassin, Nowik and Shaki (2018) define the threshold Nash equilibrium strategy x_e and the socially-optimal threshold strategy x^*. They investigate the dependence of the PoA on the parameter x_e and the intensity function h. They develop an explicit formula to calculate $\lim_{x_e \to \infty} PoA(h, x_e)$ when it exists.

As in Gilboa-Freedman, Hassin and Kerner (2014), the number 2 arrises repeatadely in several results of Hassin, Nowik and Shaki (2018), relating to the limit of PoA when x_e, goes to infinity. For instance, if h converges to a positive constant then PoA converges to 2; if h increases (decreases) then the limit of PoA is at least (at most) 2. In a system with a queue they prove that PoA may be unbounded already in the simplest case of uniform arrival.

The goal of this work is to extend Hassin, Nowik and Shaki's model to the case of two servers (instead of a single server), where server A is located at the origin and server B is located at a point denoted as M. If the servers' points are distsnce from each other then the system is just a combination of two single-server systems. It becomes more interesting when the servers are closer, creating a dilemma for some consumers regarding what service point to arrive at.

The value of information sharing between service providers lies in its capacity to enhance coordination, optimize resource allocation, and improve overall system efficiency. When service providers have common knowledge about each other's status, they can collaborate more effectively, leading to a better distribution of workloads and resources. This coordination often results in increased efficiency, reduced response times, and improved service quality. The ability to access real-time information about the status of other providers allows for more informed decision-making, enabling adaptive strategies that respond dynamically to changing conditions. Ultimately, the value of such information is reflected in its power to streamline operations, enhance service delivery, and contribute to a more resilient and responsive system. Think for example of Air Traffic Control Towers; In a situation where two air traffic control towers manage adjacent airspaces and are aware of each other's workload, they can coordinate and optimize the allocation of incoming flights. If one tower is busy, the other can efficiently handle additional aircraft to maintain smoother air traffic operations. Another example is of a hospital with two emergency rooms, if each ER is aware of the patient load and occupancy status of the other, medical staff can coordinate patient assignments. Deo and Gurvich (2011) consider a routing problem motivated by the diversion of ambulances to neighboring hospitals. These examples illustrate situations where the level of information sharing between service providers can significantly impact their ability to optimize resource allocation and overall system efficiency.

2 THE M/G/$1/1$ MODEL: NOTATIONS AND FUNDEMENTAL RESULTS

In the model of one server, for all $x \geq 0$, customers with distances less than x, arrive to the system according to a Poisson process with rate $\lambda(x) = \int_0^x h(y)dy$, where $h(y)$ is an intensity function. The service distribution is general with average rate μ and the benefit from a service is R. There is a waiting cost c_w per unit time while in the system and a traveling cost of c_t per unit distance. The optimal (individual) strategy of a customer located at a distance x from the origin, is to enter service if the server is idle and the utility is positive, namely: $R \geq \frac{c_w}{\mu} + c_t x$. This implies that under individual optimization, a consumer located at a distance x from the origin, is to enter service iff the server is idle and

$$x \leq \frac{R - c_w/\mu}{c_t}.$$

Denote $v = \frac{R - c_w/\mu}{c_t}$. (Note that the optimal individual threshold strategy is denoted in Hassin, Nowik and Shaki (2018), as x_e, and in this paper as v.). Hence there is a unique individual optimal strategy (i.e., Nash equilibrium): $v = \frac{R - c_w/\mu}{c_t}$. Under this strategy, a customer located at a distance x, enters service iff the server is idle and $x \leq v$.

The utility of a customer entering service from location x is: $R - c_w/\mu - c_t x = c_t(v - x)$. The balance equation for the probability $\pi_0(x)$, of an idle server satisfies:

$$\pi_0(x)\lambda(x) = (1 - \pi_0(x))\mu.$$

This implies that:

$$\pi_0(x) = \frac{1}{1 + \rho(x)} = \frac{1}{1 + \frac{1}{\mu}\int_0^x h(y)dy},$$

where $\rho(x) = \lambda(x)/\mu$.

The expected social benefit per unit of time associated with threshold x satisfies

$$S_1(x) = c_t \int_0^x (v - y)\, h(y)\pi_0(x)dy = \frac{c_t \int_0^x (v - y)h(y)dy}{1 + \frac{1}{\mu}\int_0^x h(y)dy}. \quad (1)$$

Let x^* be the threshold strategy that maximizes social welfare. Under this strategy, only consumers with distances less than x^*, will enter the system. It is shown in Hassin, Nowik and Shaki (2018), that;

$$x^* < v,$$

and that given v, the optimal threshold strategy x^* is unique and satisfies,

$$v = \frac{1}{\mu}\int_0^{x^*} (x^* - y)h(y)dy + x^*, \quad (2)$$

(see Proposition 3.1 in Hassin, Nowik and Shaki (2018)).

3 THE M/G/$2/2$ MODEL

3.1 Model Description

We consider two servers A and B on the interval $[0, M]$. A is located at the origin, and B is located at a point M. The model makes the following assumptions:

1. All customers reside on the interval $[0, M]$.
2. The arrivals to the servers follow a Poisson process with rates defined according to a given 'intensity function' h, defined over the interval $[0, M]$. For any x, if consumers from interval $[0, x]$, turn to server A, then the arrival rate from that interval is $\lambda_A(x) = \int_0^x h(y)dy$. Similarily, for any x, if consumers from interval $[x, M]$, turn to server B, then the arrival rate from that interval is $\lambda_B(x) = \int_x^M h(y)dy$.
3. The intensity function h may be any nonnegative function for which $\int_0^1 h(y)dy$ is finite for all $x \geq 0$.
4. Customers know their distance from each of the two servers.
5. The status of the servers is observable.
6. Customers are risk neutral, maximizing expected net benefit.
7. The service distribution of servers A and B, is exponentially with rate μ_A, and μ_B, respectively. The system is a loss system.
8. The benefit from a completed service is R.
9. The waiting cost is c_w per unit time (while in the system).
10. The traveling cost to servers A and B are c_t^A and c_t^B, respectively, per unit distance, and traveling is instantaneous.
11. All processes are mutually independent.
12. The decision of the customer is whether to enter to service and if so then which of the servers to turn to.

The states of the system are denoted with (i, j), $i, j \in \{0, 1\}$, where $i = 0$ means that server A is free, and $i = 1$ means that Server A is busy. The same for j and server B.

For State $(0,0)$, let x_{A00}, and x_{B00}, be the arrival thresholds, for servers A and B respectively. Namely, if both servers are idle, then consumers with locations closer to the origin than x_{A00}, (i.e., with locations x, s.t., $x \leq x_{A00}$), turn to server A, and similarily, consumers with locations closer to M than x_{B00}, (i.e., with locations x, s.t., $x \geq x_{B00}$), turn to server B. For State $(0,1)$, let x_{A01} be the arrival threshold, from which we allow consumers to arrive to Server A, when Server B is busy, and similarly for state $(1,0)$, let x_{B10} be the arrival threshold, from which we allow consumers to arrive to Server B, when Server A is busy.

Every strategy is described by 4-dimensional vector $\overrightarrow{x} = (x_{A00}, x_{A01}, x_{B00}, x_{B10})$.

For the strategy to be well defined, it is necessary that,

$$x_{A00} \leq x_{B00}, \tag{3}$$

since if $x_{B00} < x_{A00}$, then when both servers are idle, consumers that are located between x_{B00} and x_{A00}, should turn to server A according to x_{A00}, but according to x_{B00} they should turn to server B.

Given $\overrightarrow{x}$, denote $\pi_{ij}(\overrightarrow{x})$, $i,j \in \{0,1\}$, as the probability that the system is in state (i,j), $i,j \in \{0,1\}$.

In the following sections, we consider individual and socially optimal strategies.

4 INDIVIDUAL OPTIMIZATION

Assume for the moment that only one server is idle (i.e., state $(0,1)$ or state $(1,0)$). Then the optimal strategy of a customer located at a distance x from server A (at a distance $M-x$ from server B), is to arrive to server A (server B) if server A (server B) is idle and $R \geq \frac{c_w}{\mu_A} + c_t^A x$ ($R \geq \frac{c_w}{\mu_B} - c_t^B(M - x_e^B)$). In other words, the threshold strategies are:

$$v_A = \frac{R - c_w/\mu_A}{c_t^A}, \text{ and } v_B = M - \frac{R - c_w/\mu_B}{c_t^B}. \tag{4}$$

For state $(0,0)$, we need to relate separately to two cases; $v_A \leq v_B$, (case 1), and $v_A > v_B$, (case 2).

4.1 Case 1. $v_A \leq v_B$

In this case, which is illustrated by Figure 1, every customer between the origin and v_A will turn to server A, if he is idle, and every customer between v_B and M will turn to server B, if he is idle. The customers between v_A and v_B will not turn to any server (as their utility is negative when turning to either server). In fact, since the intervals $[0, v_A]$ and intervals $[v_B, M]$ are disjoint, our system is equivalent to two independent service systems. Therefore, the individual optimal strategy (i.e., Nash equilibrium) is:

$$\overrightarrow{x}_E = (x_{A00}, x_{A01}, x_{B00}, x_{B10}) = (v_A, v_A, v_B, v_B).$$

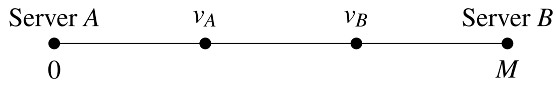

Figure 1: $v_A \leq v_B$.

4.2 Case 2. $v_A > v_B$

In this case, which is illustrated by Figure 2, if only server A is idle, consumers located between the origin and v_A will turn to server A. If only server B is idle consumers located between v_B and M will turn to server B. However, if both servers are idle, consumers located between the origin and v_B will turn to server A, and customers located between v_A and M will turn to server B. But customers located in the interval $[v_B, v_A]$ may potentially go to either server A or B, (since the utility by going to either server, is positive). Thus, the optimizing individual strategy would be turnning to the server which yields the greatest utility for the consumer.

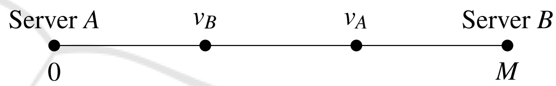

Figure 2: $v_B < v_A$.

If the customer turns to A, his benefit will be $R - \frac{c_w}{\mu_A} + c_t^A x = c_t^A(v_A - x)$, whereas if he turns to B, his benefit will be $R - \frac{c_w}{\mu_B} - c_t^B(M - x) = c_t^B(x - v_B)$. Consequently, if

$$c_t^B(x - v_B) < c_t^A(v_A - x), \tag{5}$$

the customer will turn to A. The above is equivalent to

$$x < \frac{(c_t^A v_A + c_t^B v_B)}{(c_t^A + c_t^B)}. \tag{6}$$

Substituting v_A and v_B, from (4), in (6), we get

$$x < \frac{c_t^B M + c_w\left(\frac{1}{\mu_B} - \frac{1}{\mu_A}\right)}{(c_t^A + c_t^B)}.$$

Denote $v_T = \frac{c_t^B M + c_w\left(\frac{1}{\mu_B} - \frac{1}{\mu_A}\right)}{(c_t^A + c_t^B)}$, then a customer located at x, and observes that the two servers are idle, will turn to A if his location x, satisfies $x < v_T$, and otherwise, will turn to B. Hence, the individual optimal strategy in this case is,

$$\overrightarrow{x}_E = (x_{A00}, x_{A01}, x_{B00}, x_{B10}) = (v_T, v_A, v_T, v_B).$$

Note that in the special case, in which $c_t^A = c_t^B$, and $\mu_A = \mu_B$, then the individual optimal strategy is:

$$\overrightarrow{x}_E = (\frac{M}{2}, v_A, \frac{M}{2}, M - v_A). \tag{7}$$

5 SOCIALLY OPTIMAL STRATEGY

Under social optimality, the mutual influences of actions chosen by the players must be taken into consideration. These influences are not trivial. For example, given $\overrightarrow{x} = (x_{A00}, x_{A01}, x_{B00}, x_{B10})$, if we increase x_{A00}, to include consumers that are further away from server A but still have positive utility (namely their location x, satisfies $x_{A00} < x < v_A$), then, on one hand it may increase the social welfare function since consumers that are further away from server A will now get service. But on the otherhand, we may lose some of the closer, (thus more valuable), consumers that may find the server busy more often than before and this may reduce the social welfare function.

Recall that the arrivals to server A from $[0,x]$ follow a Poisson process with rate of $\lambda_A(x) = \int_0^x h(y)dy$, and the arrivals to server B from $[x,M]$ follow a Poisson process with rate of $\lambda_B(x) = \int_x^M h(y)dy$, where h is the intensity function. Also recall that $\pi_{ij}(\overrightarrow{x})$, is the probability that the system is in state (i,j) when $\overrightarrow{x} = (x_{A00}, x_{A01}, x_{B00}, x_{B10})$. Given $\overrightarrow{x}$, the transition diagram is presented in Figure 3.

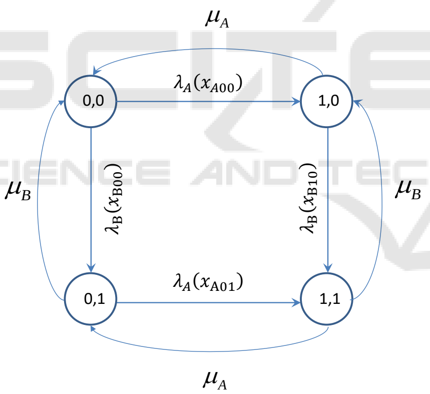

Figure 3: Transition Diagram.

In order to find the steady-state probabilities, we need to solve the following balance equation system:

$$\begin{aligned}
&1)\ \ \mu_B\pi_{01} + \mu_A\pi_{10} = [\lambda_A(x_{A00}) + \lambda_B(x_{B00})]\pi_{00}\\
&2)\ \ \mu_B\pi_{11} + \lambda_A(x_{A00})\pi_{00} = [\lambda_B(x_{B10}) + \mu_A]\pi_{10}\\
&3)\ \ \lambda_B(x_{B10})\pi_{10} + \lambda_A(x_{A01})\pi_{01} = [\mu_A + \mu_B]\pi_{11}\\
&4)\ \ \pi_{00} + \pi_{10} + \pi_{01} + \pi_{11} = 1.
\end{aligned} \tag{8}$$

Let $S(\overrightarrow{x}) = S(x_{A00}, x_{A01}, x_{B00}, x_{B10})$ denote the social welfare function. We have

$$\begin{aligned}
S(\overrightarrow{x}) = \pi_{00}(\overrightarrow{x}) &\int_0^{x_{A00}} c_t^A(v_A - y)h(y)dy +\\
\pi_{01}(\overrightarrow{x}) &\int_0^{x_{A01}} c_t^A(v_A - y)h(y)dy +\\
\pi_{00}(\overrightarrow{x}) &\int_{x_{B00}}^{M} c_t^B(y - v_B)h(y)dy +\\
\pi_{10}(\overrightarrow{x}) &\int_{x_{B10}}^{M} c_t^B(y - v_B)h(y)dy.
\end{aligned} \tag{9}$$

We wish to find: $\overrightarrow{x}^* = (x^*_{A00}, x^*_{A01}, x^*_{B00}, x^*_{B10})$ that maximizes the social Welfare function S. Recall first the model with a single server (see Section 2). According to (2), if server A was a single server, located at the origin, then the optimal threshold strategy x^*_A is unique and satisfies,

$$v_A = \frac{1}{\mu_A}\int_0^{x^*_A}(x^*_A - y)h(y)dy + x^*_A. \tag{10}$$

Similarly, if server B was a single server, located at M, then the optimal threshold strategy x^*_B is unique and satisfies,

$$v_B = x^*_B - \frac{1}{\mu_B}\int_{x^*_B}^{M}(y - x^*_B)h(y)dy. \tag{11}$$

The values of x^*_A and x^*_B depend on the parameters of the model and on the intensity function h. Under some conditions, $x^*_A \le x^*_B$, (Case A), and under other conditions, $x^*_A > x^*_B$, (Case B).

As we show in the sequal, x^*_A and x^*_B, although originated in the single server mode, nevertheless play a significant role in the model with two servers.

For simplicity, we assume from now on that servers have the same capacity. Additionally, We normalize all other parameters according to this common capacity hence

$$\mu_A = \mu_B = 1.$$

Lemma 5.1. *For all* $0 < x < v$,

- *If* $x < x^*_A$, *then* $\int_0^x (v-y)h(y)dy < (\lambda_A(x)+1)(v-x)$
- *If* $x = x^*_A$, *then* $\int_0^x (v-y)h(y)dy = (\lambda_A(x)+1)(v-x)$
- *If* $x > x^*_A$, *then* $\int_0^x (v-y)h(y)dy > (\lambda_A(x)+1)(v-x)$

Proof. Note that,

$$\int_0^x (v-y)h(y)dy = \int_0^x (v-x+x-y)h(y)dy = (v-x)\lambda_A(x) + \int_0^x (x-y)h(y)dy. \tag{12}$$

It follows that,

$$\int_0^x (v-y)h(y)dy =$$

$$(v-x)\lambda_A(x) - x + \left[x + \int_0^x (x-y)h(y)dy\right]. \quad (13)$$

By (10), x_A^* satisfies,

$$v = x_A^* + \int_0^{x_A^*} (x_A^* - y)h(y)dy. \quad (14)$$

Thus, substituting $x = x_A^*$ in (13), we get:

$$\int_0^{x_A^*} (v-y)h(y)dy = (v - x_A^*)\lambda_A(x_A^*) - x_A^* + v$$
$$= (\lambda_A(x_A^*)+1)(v - x_A^*), \quad (15)$$

proving the second statement of the lemma. Note that $x + \int_0^x (x-y)h(y)dy$, appearing in the square brackets at the right hand side of (13) is increasing in x, hence it follows from (14) that:

- For $x < x_A^*$,

$$\int_0^x (v-y)h(y)dy < (v-x)\lambda_A(x) - x + v$$
$$= (\lambda_A(x)+1)(v-x), \quad (16)$$

and,
- For $x > x_A^*$,

$$\int_0^x (v-y)h(y)dy < (v-x)\lambda_A(x) - x + v$$
$$= (\lambda_A(x)+1)(v-x), \quad (17)$$

proving the first and last statements of the lemma. □

A similar lemma, regarding server B is:

Lemma 5.2. *For all $v < x < M$,*

- *If $x < x_B^*$, then* $\int_x^M (y-v)h(y)dy > (\lambda_B(x)+1)(x-v)$.
- *If $x = x_B^*$, then* $\int_x^M (y-v)h(y)dy = (\lambda_B(x)+1)(x-v)$.
- *If $x > x_B^*$, then* $\int_x^M (y-v)h(y)dy < (\lambda_B(x)+1)(x-v)$.

5.1 Socially Optimal Strategies in the Symmetric Case

In this section we assume that the model is completely symmetric with regards to the two servers. We will show that in case A (namely when $x_A^* \le x_B^*$), consumers with distances that are less than x_A^* from server A, turn to server A in any case (wheather server B is avilable or not). Similarly, consumers with distances that are less than x_A^* from server B (i.e., their location is $M - x_A^*$ and beyond) turn to server B. In Case B (namely when $x_A^* > x_B^*$), we will show that if both servers are available, then server A serves consumers with locations between 0 and $M/2$, and server B serves consumers from that point on (until the end of the interval $[0,M]$). Interestingly, if only one server is available, say server A, then she serves consumers with distances that are beyond x_A^*, which was the service-threshold when server A was the only server in a single-server system.

We assume that,

- A1. $h(M-y) = h(y)$, $\forall 0 \le y \le M$.
- A2. $c_t^A = c_t^B$, thus travel cost is simply c_t.

The following 4 properties follow,

- P1. $v_B = M - v_A$. We denote $v = v_A$.
- P2. $x_B^* = M - x_A^*$.
- P3. $x_{B10}^* = M - x_{A01}^*$.
- P4. $x_{B00}^* = M - x_{A00}^*$.

Recall that in all cases (including the general non-symmetric case), $x_{A00} \le x_{B00}$, (see (3)). This together with P4 above gives in the symmetric case

$$x_{A00}^* \le \frac{M}{2}. \quad (18)$$

Lemma 5.3. *In the symmetric case, for all $0 \le x \le M$,*

$$\lambda_B(M-x) = \lambda_A(x).$$

Proof. By A1,

$$\lambda_B(M-x) = \int_{M-x}^M h(u)du =$$
$$\int_0^x h(M-u)du = \int_0^x h(u)du = \lambda_A(x). \quad (19)$$

□

Denote $a = 1 + \lambda_A(x_{A00})(2 + \lambda_A(x_{A01}))$.

Proposition 5.4. *In the symmetric case,*

$$\pi_{00} = \frac{1}{a}, \ \ \pi_{01} = \pi_{01} = \frac{\lambda_A(x_{A00})}{a},$$

and,

$$\pi_{11} = \frac{\lambda_A(x_{A00})\lambda_A(x_{A01})}{a}.$$

The proof follows from Lemma 5.3, Assumptions A1-A3 and Properties P1-P4 above.

Proposition 5.5. *In the symmetric case,*

$$S(\overrightarrow{x}) = \frac{2c_t}{a}$$
$$\left(\int_0^{x_{A00}} (v-y)h(y)dy \ + \ \lambda_A(x_{A00} \int_0^{x_{A01}} (v-y)h(y)dy\right) \quad (20)$$

Proof. By Proposition 5.4, the assumptions and properties, we obtain by (9) that:

$$\begin{aligned}
&S(\overrightarrow{x}) = \\
&\frac{1}{a}\int_0^{x_{A00}} c_t(v_A-y)h(y)dy+ \\
&\frac{\lambda_A(x_{A00})}{a}\int_0^{x_{A01}} c_t(v_A-y)h(y)dy+ \\
&+\frac{1}{a}\int_{M-x_{A00}}^{M} c_t(y-M+v_A)h(y)dy+ \\
&\frac{\lambda_A(x_{A00})}{a}\int_{M-x_{A01}}^{M} c_t(y-M+v_A)h(y)dy.
\end{aligned} \tag{21}$$

Now, by A1 and by substituting $t = M - y$ in $\int_{M-x_{A00}}^{M} c_t(y-M+v_A)h(y)dy$, we obtain

$$\int_{M-x_{A00}}^{M} c_t(y-M+v_A)h(y)dy = \int_0^{x_{A00}} c_t(v_A-t)h(t)dt. \tag{22}$$

Similarly,

$$\int_{M-x_{A01}}^{M} c_t(y-M+v_A)h(y)dy = \int_0^{x_{A01}} c_t(v_A-t)h(t)dt. \tag{23}$$

Substituting (22) and (23) in (21) gives

$$\begin{aligned}
&S(\overrightarrow{x}) = \\
&\frac{1}{a}\int_0^{x_{A00}} c_t(v_A-y)h(y)dy+ \\
&\frac{\lambda_A(x_{A00})}{a}\int_0^{x_{A01}} c_t(v_A-y)h(y)dy+ \\
&\frac{1}{a}\int_0^{x_{A00}} c_t(v_A-y)h(y)dy+ \\
&\frac{\lambda_A(x_{A00})}{a}\int_0^{x_{A01}} c_t(v_A-y)h(y)dy = \frac{2c_t}{a} \\
&\left(\int_0^{x_{A00}}(v_A-y)h(y)dy + \lambda_A(x_{A00})\int_0^{x_{A01}}(v_A-y)h(y)dy\right).
\end{aligned} \tag{24}$$

□

As Propositions 5.4 and 5.5 show, because of the symmetry, $S(\overrightarrow{x})$ can be presented as a function of the parameters of server A only, namely; x_{A00} and x_{A01}. The values of x_{B00} and x_{B10} are then derived according to properies P1-P4 above. Hence in this section we abbreviate the notations so that

$$v = v_A,\ \ x_{00} = x_{A00},\ \ x_{01} = x_{A01},$$
$$\lambda_{00} = \lambda_A(x_{A00}),\ \ \lambda_{01} = \lambda_A(x_{A01}).$$

In the new notations, we obtain from Proposition 5.5:

$$S(x_{00},x_{01}) = 2c_t\left(\frac{1}{a}\int_0^{x_{00}}(v-y)h(y)dy + \frac{\lambda_{00}}{a}\int_0^{x_{01}}(v-y)h(y)dy\right). \tag{25}$$

We wish to find (x_{00}^*, x_{01}^*), that maximizes $S(x_{00},x_{01})$. $(x_{00},x_{01})^*$ must satisfy that both derivatives of S, (with respect to x_{00}, and x_{01}) equal zero.

Recall that x_A^* is the socially optimal strategy in the case of a single server A (see Section 2). We wish to prove first that (x_A^*, x_A^*) is the unique maximum point of $S(x_{00},x_{01})$.

Proposition 5.6. *Given* $x_{00} > 0$, *the* $\tilde{x}_{01} = \tilde{x}_{01}(x_{00})$, *that satisfies*

$$\frac{\partial}{\partial x_{01}}S(x_{00},x_{01}) = 0,$$

is the unique local maximum point of $S(x_{00},\bullet)$. *If* $x_{00} \neq x_A^*$, *then* $\tilde{x}_{01}(x_{00}) > x_A^*$.

Proof. Note that the derivatives of π_{00} and π_{01} with respect to x_{01} are,

$$\pi'_{00} = -\frac{\lambda_{00}h(x_{01})}{(1+\lambda_{00}(2+l_{01}))^2} = -\frac{\lambda_{00}h(x_{01})}{a^2}, \tag{26}$$

and,

$$\pi'_{01} = -\frac{\lambda_{00}^2 h(x_{01})}{(1+\lambda_{00}(2+\lambda_{01}))^2} = -\frac{\lambda_{00}^2 h(x_{01})}{a^2}. \tag{27}$$

It follows from (25), (26) and (27) that

$$\begin{aligned}
&\frac{\partial}{\partial x_{01}}S(x_{00},x_{01}) = \frac{\lambda_{00}h(x_{01})}{a^2} \\
&\{-\int_0^{x_{00}}(v-y)h(y)dy + (v-x_{01})(1+\lambda_{00}(2+\lambda_{01}))- \\
&\lambda_{00}\int_0^{x_{01}}(v-y)h(y)dy\}
\end{aligned} \tag{28}$$

Thus, for a given x_{00}, the $\tilde{x}_{01} = \tilde{x}_{01}(x_{00})$ for which $\frac{\partial}{\partial x_{01}}S(x_{00},x_{01}) = 0$, solves:

$$-\int_0^{x_{00}}(v-y)h(y)dy + (v-x_{01})(1+\lambda_{00}(2+\lambda_{01}))- \lambda_{00}\int_0^{x_{01}}(v-y)h(y)dy = 0, \tag{29}$$

where $\lambda_{01} = \int_0^{x_{01}} h(y)dy$.

Now, if we differentiate the left hand side of (29) with respect to x_{01}, we arrive at

$$-\lambda_{00}(v-x_{01})h(x_{01}) - (1+\lambda_{00}(2+\lambda_{01}))+ \lambda_{00}(v-x_{01})h(x_{01}) = -a < 0. \tag{30}$$

Hence the left hand side of (29) is decreasing in x_{01}. It follows that,

$$\frac{\partial}{\partial x_{01}}S(x_{00},x_{01}) > 0,\quad \forall x_{01} < \tilde{x}_{01},$$
$$\frac{\partial}{\partial x_{01}}S(x_{00},x_{01}) = 0,\quad \textit{for } x_{01} = \tilde{x}_{01},$$

and

$$\frac{\partial}{\partial x_{01}} S(x_{00}, x_{01}) < 0, \quad \forall x_{01} > \tilde{x}_{01}.$$

Thus, $\tilde{x}_{01}$ is indeed the unique local maximum of $S(x_{00}, \bullet)$.

Denote $\lambda_A^* = \lambda_A(x_A^*) = \int_0^{x_A^*} h(y)dy$. For any given x_{00}, (29) presents the condition for $\frac{\partial}{\partial x_{01}} S(x_{00}, x_{01}) = 0$. Subtituting $x_{01} = x_A^*$ in the left hand side of (29) gives

$$\begin{aligned}
&-\int_0^{x_{00}} (v-y)h(y)dy - \\
&\lambda_{00} \int_0^{x_A^*} (v-y)h(y)dy + (v-x_A^*)(1+\lambda_{00}(2+\lambda_A^*)) \\
&= -\int_0^{x_{00}} (v-x_A^*+x_A^*-y)h(y)dy - \\
&\lambda_{00} \int_0^{x_A^*} (v-y)h(y)dy + (v-x_A^*)(1+\lambda_{00}(2+\lambda_A^*)) = \\
&-(v-x_A^*)\lambda_{00} - \int_0^{x_{00}} (x_A^*-y)h(y)dy - \\
&\lambda_{00} \int_0^{x_A^*} (v-y)h(y)dy + \\
&(v-x_A^*)(1+2\lambda_{00}+\lambda_{00}\lambda_A^*).
\end{aligned} \tag{31}$$

By (12),

$$\int_0^{x_A^*} (v-y)h(y)dy = (v-x_A^*)\lambda_A^* + \int_0^{x_A^*} (x_A^*-y)h(y)dy.$$

Substituting this in (31) gives

$$\begin{aligned}
&-(v-x_A^*)\lambda_{00} - \int_0^{x_{00}} (x_A^*-y)h(y)dy - \\
&\lambda_{00} \left((v-x_A^*)\lambda_A^* + \int_0^{x_A^*} (x_A^*-y)h(y)dy \right) \\
&+(v-x_A^*)(1+2\lambda_{00}+\lambda_{00}\lambda_A^*) = (v-x_A^*)(-\lambda_{00}-\lambda_{00}\lambda_A^*+ \\
&1+2\lambda_{00}+\lambda_{00}\lambda_A^*) + (-\lambda_{00}-1+1)\int_0^{x_A^*} (x_A^*-y)h(y)dy - \\
&\int_0^{x_{00}} (x_A^*-y)h(y)dy =
\end{aligned} \tag{32}$$

$$\begin{aligned}
&= (\lambda_{00}+1)\left[v - \left(x_A^* + \int_0^{x_A^*} (x_A^*-y)h(y)dy \right) \right] + \\
&\int_{x_{00}}^{x_A^*} (x_A^*-y)h(y)dy.
\end{aligned}$$

Note that by (10), the expression in the square brakets above equals 0, hence we get

$$\int_{x_{00}}^{x_A^*} (x_A^*-y)h(y)dy, \tag{33}$$

which is positive for all $x_{00} \neq x_A^*$, as now explained: For $x_A^* \geq x_{00}$ this is obvious. For $x_A^* < x_{00}$, the left hand side of (33) equals

$$-\int_{x_A^*}^{x_{00}} (x_A^*-y)h(y)dy = \int_{x_A^*}^{x_{00}} (y-x_A^*)h(y)dy > 0.$$

So we proved that for $x_{00} \neq x_A^*$, the left hand side of (29), is positive for $x_{01} = x_A^*$. Now, in comparission to that, if we substitute $x_{01} = \tilde{x}_{01}(x_{00})$ in the left hand side of (29), then by definition of $\tilde{x}_{01}(x_{00})$, this equals 0. Earlier (see (30)) we showed that the left hand side of (29) is decreasing in x_{01}. Hence it follows that $\tilde{x}_{01}(x_{00}) > x_A^*$. □

Similarly,

Proposition 5.7. *Given $x_{01} > 0$, the $\tilde{x}_{00} = \tilde{x}_{00}(x_{01})$, that satisfies*

$$\frac{\partial}{\partial x_{00}} S(x_{00}, x_{01}) = 0,$$

is unique and it is the local maximum point of $S(\bullet, x_{01})$. If $x_{01} \neq x_A^*$, then $\tilde{x}_{00}(x_{01}) < x_A^*$.

The proof of Proposition 5.7 is very similar to the proof of Proposition 5.6. and involves proving that for a given x_{01}, the $\tilde{x}_{00} = \tilde{x}_{00}(x_{01})$ for which $\frac{\partial}{\partial x_{00}} S(x_{00}, x_{01}) = 0$, solves:

$$\begin{aligned}
&-(2+\lambda_{01})\int_0^{x_{00}} (v-y)h(y)dy + \\
&\int_0^{x_{01}} (v-y)h(y)dy + (v-x_{00})a = 0.
\end{aligned} \tag{34}$$

Theorem 5.8. *The point (x_A^*, x_A^*) is the unique global maximum point of $S(x_{00}, x_{01})$, and it satisfies that*

$$S(x_A^*, x_A^*) = 2c_t(v-x_A^*).$$

Proof. Substituting $x_{00} = x_{01} = x_A^*$ in (20) and utilizing Lemma 5.1 indeed gives $2c_t(v-x_A^*)$. In order to prove that (x_A^*, x_A^*) is the unique global maximum point of $S(x_{00}, x_{01})$, we first, prove that (x_A^*, x_A^*) is the only point in which both derivatives of $S(x_{00}, x_{01})$, (with respect to x_{00} and with respect to x_{01}), equal 0. A point in which both derivatives equal 0, must satisfy both (29) and (57).

Recall that, Equation (29) is:

$$\begin{aligned}
&-\int_0^{x_{00}} (v-y)h(y)dy + (v-x_{01})(1+\lambda_{00}(2+\lambda_{01})) - \\
&\lambda_{00}\int_0^{x_{01}} (v-y)h(y)dy = 0,
\end{aligned} \tag{35}$$

and Equation (57) is:

$$\begin{aligned}
&-(2+\lambda_{01})\int_0^{x_{00}} (v-y)h(y)dy + \\
&\int_0^{x_{01}} (v-y)h(y)dy + (v-x_{00})a = 0.
\end{aligned} \tag{36}$$

From (29) we have:

$$\int_0^{x_{00}} (v-y)h(y)dy = (v-x_{01})a - \lambda_{00}\int_0^{x_{01}} (v-y)h(y)dy.$$

Subtituting this in (57) gives:

$$-(2+\lambda_{01})\left[(v-x_{01})a-\lambda_{00}\int_0^{x_{01}}(v-y)h(y)dy\right]+ \int_0^{x_{01}}(v-y)h(y)dy+(v-x_{00})a=0. \quad (37)$$

This implies that:

$$a\Big(v-x_{00}-(2+\lambda_{01})(v-x_{01})\Big)+ \Big(1+\lambda_{00}(2+\lambda_{01})\Big)\int_0^{x_{01}}(v-y)h(y)dy=0, \quad (38)$$

which is equivalent to:

$$a\Big(v-x_{00}-(2+\lambda_{01})(v-x_{01})\Big)+ a\int_0^{x_{01}}(v-y)h(y)dy=0. \quad (39)$$

From this we arrive at:

$$x_{01}-x_{00}+ \left[\int_0^{x_{01}}(v-y)h(y)dy-(1+\lambda_{01})(v-x_{01})\right]=0. \quad (40)$$

We first show that x_{00} must equal x_A^* : If on the contrary, $x_{00}\neq x_A^*$, then we proved in Proposition 5.6 that $\tilde{x}_{01}(x_{00})>x_A^*$. But in that case, by Lemma 5.1, the expression in the square brackets above is positive. Additionally, $x_{01}-x_{00}>0$, (since, $x_{00}\leq M/2<x_A^*<\tilde{x}_{01}(x_{00})$). Thus the left hand side of (40) is strictly positive and thus cannot satisfy (40). Hence,

$$x_{00}=x_A^*. \quad (41)$$

Now, if $\tilde{x}_{01}\neq x_A^*$, then by Proposition 5.7, $x_{00}<x_A^*$, which contradicts (41), thus,

$$\tilde{x}_{01}(x_A^*)=x_A^*. \quad (42)$$

Hence, we proved that (x_A^*,x_A^*) is the only point in which both derivatives of S equal 0. To prove that (x_A^*,x_A^*) is also the unique *global* maximum point, we need to check the value of S at the borders, namely at the boundaries of the rectangular $[0,M/2]\times[0,M]$: If $x_{00}=0$, then $\lambda_{00}=0$. Hence by Proposition 5.4,

$$\pi_{01}=\pi_{01}=\frac{\lambda_{00}}{a}=0.$$

Hence from the the balance equation for state $(1,1)$ (see equation 3 in the balance equation system appearing in (8)), we get:

$$0=[\mu_A+\mu_B]\pi_{11},$$

which implies that $\pi_{11}=0$, hence $\pi_{00}=1$. Thus the service-system is always empty and thus $S(0,x_{01})=0$, which is smaller than $S(x_A^*,x_A^*)=2c_t(v-x_A^*)$.

If $x_{01}=0$, then (25) gives

$$S(x_{00},0)=2c_t\left(\frac{1}{1+2\lambda_{00}}\int_0^{x_{00}}(v-y)h(y)dy\right),$$

which implies:

$$S(x_{00},0)<2c_t\left(\frac{1}{1+\lambda_{00}}\int_0^{x_{00}}(v-y)h(y)dy\right). \quad (43)$$

Recall that in M/G/1/1, (see Section 2), the probability that the system is empty when the service-threshold is x_{00}, (and $\mu=1$), is

$$\pi_0(x_{00})=\frac{1}{1+\lambda_{00}}.$$

In M/G/1/1, we denoted the social welfare function, when the service-threshold is x, as $S_1(x)$. By (1):

$$S_1(x)=c_t\int_0^x(v-y)\,h(y)\pi_0(x)dy. \quad (44)$$

Thus it follows from (43) that:

$$S(x_{00},0)<2c_t\int_0^{x_{00}}(v-y)h(y)\pi_0(x_{00})dy=$$
$$2S_1(x_{00})<2S_1(x_A^*)=2c_t\int_0^{x_A^*}(v-y)h(y)\pi_0(x_A^*)dy$$
$$=2c_t\int_0^{x_A^*}(v-y)h(y)\frac{1}{1+\lambda_A^*}dy=$$
$$\frac{2c_t}{1+\lambda_A^*}\int_0^{x_A^*}(v-y)h(y)dy. \quad (45)$$

By Lemma 5.1, the right hand side of (45) equals $2c_t(v-x_A^*)$, which by Theorem 5.8 equals $S(x_A^*,x_A^*)$. Hence,

$$S(x_{00},0)<S(x_A^*,x_A^*).$$

For the cases of $x_{00}=M/2$, or $x_{01}=M$, we can use the symmetry of the model with regards to the servers, and thus, for example, $x_{00}=M$, implies that for server B, $x_{B00}=M$, as well (see (3)) which yields $\lambda_{B00}=0$, and from that point to continue like in the case where $\lambda_{00}=0$. □

Corollary 5.9. *Given* x_{00},

$$S(x_{00},\tilde{x}_{01}(x_{00}))=2c_t(v-\tilde{x}_{01}(x_{00})).$$

Proof. By (25) we have:

$$S(x_{00},\tilde{x}_{01}(x_{00}))=\frac{2c_t}{a} \left(\int_0^{x_{00}}(v-y)h(y)dy\ +\lambda_{00}\int_0^{\tilde{x}_{01}(x_{00})}(v-y)h(y)dy\right). \quad (46)$$

Additionally, $\tilde{x}_{01}(x_{00})$ satisfies (29), implying that:

$$\lambda_{00}\int_0^{\tilde{x}_{01}(x_{00})}(v-y)h(y)dy = -\int_0^{x_{00}}(v-y)h(y)dy + a\big(v-\tilde{x}_{01}(x_{00})\big). \quad (47)$$

Subtituting (47) in (46), gives:

$$S(x_{00},\tilde{x}_{01}(x_{00})) = \frac{2c_t}{a}\Big[\int_0^{x_{00}}(v-y)h(y)dy - \int_0^{x_{00}}(v-y)h(y)dy + a(v-\tilde{x}_{01}(x_{00}))\Big] = 2c_t(v-\tilde{x}_{01}(x_{00})). \quad (48)$$

□

Recall that Case A is defined by $x^*_A \le x^*_B$, and Case B is defined by $x^*_A > x^*_B$. Theorem 5.10 ahead claims that under social optimality, in Case A, each server serves consumers that are x^*_A or less, away from the servic point, and that in Case B: When both servers are idle, then each server serves consumers that are $M/2$ or less, away from the service point. In case only one server is available, then she serves consumers with distance that exceeds x^*_A, where the service- threshold is determined according to (49) ahead.

Formally, going back to our earlier notations; recall that x_{00} was an abbreviation for x_{A00} the service-threshold of server A when both servers are idle, and x_{B00} was the service-threshold of server B when both servers are idle. Similarly, x_{A01} (x_{B10}) was the service threshold of A, (B) when only he was available.

Theorem 5.10. *In the symmetric model, the strategy* $\overrightarrow{x}^* = (x^*_{A00}, x^*_{A01}, x^*_{B00}, x^*_{B10})$ *that maximizes social welfare* $S(\overrightarrow{x})$, *is the following*

- ***Case A, namely when*** $x^*_A \le x^*_B$, *then:* $(x^*_{A00}, x^*_{A01}, x^*_{B00}, x^*_{B10}) = (x^*_A, x^*_A, M-x^*_A, M-x^*_A)$, *and* $S = 2c_t(v-x^*_A)$.
- ***Case B, namely when*** $x^*_A > x^*_B$,
  1. *If both servers are available then* $(x^*_{A00}, x^*_{B00}) = (M/2, M/2)$, *and* $S = 2c_t(v-M/2)$.
  2. *If only server A is available then,* $x^*_{A01} = \tilde{x}_{01}(M/2)$, *and if only server B is available then,* $x^*_{B10} = M-\tilde{x}_{01}(M/2)$. *In both cases,* $S = 2c_t(v-\tilde{x}_{01}(M/2))$, *where* $\tilde{x}_{01}(M/2)$, *is the unique solution for:*

$$\int_0^{\frac{M}{2}}(v-y)h(y)dy - (v-x_{01})\Big(1+\lambda_A(M/2)(2+\lambda_{01})\Big) + \lambda_A(M/2)\int_0^{x_{01}}(v-y)h(y)dy = 0. \quad (49)$$

Proof. Because of the symmetry between servers A and B, we have by properties P3 and P4, that $x^*_{B10} = M-x^*_{A01}$, and $x^*_{B00} = M-x^*_{A00}$, hence we only need to prove the theorem for server A. Recall that in the symmetric case x_{A00} must satisfy

$$x^*_{A00} \le \frac{M}{2},$$

(see (18)). Now, by Theorem 5.8, $(x^*_{00}, x^*_{01}) = (x^*_A, x^*_A)$ is the unique global maximum of the social welfare function S, hence whenever this solution is possible (i.e., $x^*_A \le M/2$), then this will be the strategy of server A. This proves the statement of the theorem regarding case A, (since in that case, $x^*_A \le M/2$.).

In Case B, $x^*_A > M/2$, hence x^*_A is not an option for x^*_{00} which must satisfy $x^*_{A00} \le \frac{M}{2}$.

Recall that (29) defines $\tilde{x}_{01}(x_{00})$ for a given x_{00}, where (29) is

$$-\int_0^{x_{00}}(v-y)h(y)dy + (v-x_{01})(1+\lambda_{00}(2+\lambda_{01})) - \lambda_{00}\int_0^{x_{01}}(v-y)h(y)dy = 0. \quad (50)$$

Define the left hand side of (29), as a function $f(x_{00}, x_{01})$. By definition of $\tilde{x}_{01}(x_{00})$,

$$f(x_{00}, \tilde{x}_{01}(x_{00})) = 0. \quad (51)$$

We showed, (see (30)), that $f(x_{00}, x_{01})$ is decreasing in x_{01}.

Below we show that $f(x_{00}, x_{01})$ is also decreasing in x_{00} for all $x_{00} \ne x^*_A$. This implies that $\tilde{x}_{01}(x_{00})$ is decreasing as a function of x_{00}, since if we increase x_{00} to $x_{00}+\varepsilon$ then since $f(x_{00}, x_{01})$ is decreasing in x_{00}, then $f(x_{00}+\varepsilon, \tilde{x}_{01}(x_{00})) < f(x_{00}, \tilde{x}_{01}(x_{00})) = 0$, and since f is also decreasing in x_{01}, then $f(x_{00}+\varepsilon, \tilde{x}_{01}(x_{00}+\varepsilon)) = 0$ implies that $\tilde{x}_{01}(x_{00}+\varepsilon) < \tilde{x}_{01}(x_{00})$.

To see that indeed $f(x_{00}, x_{01})$ is decreasing in x_{00}, for all $x_{00} \ne x^*_A$, note that the derivative of f with respect to x_{00}, is:

$$-(v-x_{00}) + (v-x_{01})(2+\lambda_{01})h(x_{00}) - h(x_{00})\int_0^{x_{01}}(v-y)h(y)dy, \quad (52)$$

which equals:

$$h(x_{00})\Big[(1+\lambda_{01})(v-x_{01}) - \int_0^{x_{01}}(v-y)h(y)dy\Big] - x_{01}h(x_{00}). \quad (53)$$

By Lemma 5.1, the expression in the square brackets is negative, (since by Proposition 5.6, if $x_{00} \ne x^*_A$, then $\tilde{x}_{01}(x_{00}) > x^*_A$).

Now, by Corollary 5.9, $S(x_{00}, \tilde{x}_{01}(x_{00})) = 2c_t(v-\tilde{x}_{01}(x_{00}))$, which is maximized for the smallest

$\tilde{x}_{01}(x_{00})$. Since we showed that $\tilde{x}_{01}(x_{00})$ is decreasing as a function of x_{00}, then $2c_t(v - \tilde{x}_{01}(x_{00}))$ is maximized for the largest x_{00} possible, which is $M/2$, (see (18)). Thus $x_{00} = M/2$ and $x_{01} = \tilde{x}_{01}(M/2)$.

The value of S in case A and case B1 in Theorem 5.10, follow from (25). The value of S in case B2 follow from Corollary 5.9. □

6 CONCLUSIONS

In this study we establish that if the model is symmetric with regards to the servers, then under social optimality, when the service points are distant from each other, each server behaves as he would if he was the sole server. But when the service points are within close proximity, then when only one server is idle, social optimality dictates that the available server also caters to distant customers, a behavior it would not exhibit if it were the sole server in the service system (i.e., in M/M/1/1). These results apply when both servers are informed about each other's status (idle/busy). While sharing information about server status is beneficial, there are situations where it is not feasible. Take, for instance, two ride-sharing drivers operating in the same area without real-time knowledge of each other's current ride status. In this scenario, each driver independently accepts ride requests without knowing whether the other is currently occupied. This lack of information may lead to suboptimal resource allocation as both drivers might end up serving nearby locations simultaneously, potentially reducing overall efficiency. For future research, comparing between the two models; the model where servers are informed about each other's status to the model where servers are ignorant of each other's status would be intriguing. How significant is this information? If there is a substantial difference between the outcomes, it may warrant consideration for intervention by authorities.

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APPENDIX

Proof of Proposition 5.7

Proof. Note that the derivatives of π_{00} and π_{01} with respect to x_{00} are,

$$\pi'_{00} = \left(\frac{1}{a}\right)' = \left(\frac{1}{1+\lambda_{00}(2+\lambda_{01})}\right)' = -\frac{(2+\lambda_{01})h(x_{00})}{(1+\lambda_{00}(2+\lambda_{01}))^2} = -\frac{(2+\lambda_{01})h(x_{00})}{a^2}, \quad (54)$$

and,

$$\pi'_{01} = \left(\frac{\lambda_{00}}{a}\right)' = \left(\frac{\lambda_{00}}{1+\lambda_{00}(2+\lambda_{01})}\right)' = \frac{h(x_{00})(1+\lambda_{00}(2+\lambda_{01}))-\lambda_{00}h(x_{00})(2+\lambda_{01})}{(1+\lambda_{00}(2+\lambda_{01}))^2}$$

$$= \frac{h(x_{00})(1+2\lambda_{00}+\lambda_{00}\lambda_{01}-2\lambda_{00}-\lambda_{00}\lambda_{01})}{(1+\lambda_{00}(2+\lambda_{01}))^2} = \frac{h(x_{00})}{(1+\lambda_{00}(2+\lambda_{01}))^2} = \frac{h(x_{00})}{a^2}. \quad (55)$$

It follows from (54), (55) and (25), that

$$\frac{\partial}{\partial x_{00}}S(x_{00},x_{01}) = 2c_t\frac{h(x_{00})}{a^2} \left[-(2+\lambda_{01})\int_0^{x_{00}}(v-y)h(y)dy + (v-x_{00})a + \int_0^{x_{01}}(v-y)h(y)dy\right]. \quad (56)$$

Thus, for a given x_{01}, the $\tilde{x}_{00} = \tilde{x}_{00}(x_{01})$ for which

$$\frac{\partial}{\partial x_{00}}S(x_{00},x_{01}) = 0,$$

solves:

$$-(2+\lambda_{01})\int_0^{x_{00}}(v-y)h(y)dy + \int_0^{x_{01}}(v-y)h(y)dy + (v-x_{00})a = 0, \quad (57)$$

where $\lambda_{00} = \int_0^{x_{00}} h(y)dy$, and $a = 1+\lambda_{00}(2+\lambda_{01})$.

Now, if we differentiate the left hand side of (57) with respect to x_{00}, we arrive at

$$-(2+\lambda_{01})(v-x_{00})h(x_{00}) - (1+\lambda_{00}(2+\lambda_{01})) + (2+\lambda_{01})(v-x_{00})h(x_{00}) = -a < 0. \quad (58)$$

It follows that,

$$\frac{\partial}{\partial x_{00}}S(x_{00},x_{01}) > 0, \quad \forall x_{00} < x^*_{00},$$

$$\frac{\partial}{\partial x_{00}}S(x_{00},x_{01}) = 0, \quad for\ x_{00} = x^*_{00},$$

and

$$\frac{\partial}{\partial x_{00}}S(x_{00},x_{01}) < 0, \quad \forall x_{00} > x^*_{00}.$$

We now prove that if $x_{01} \neq x^*_A$, then $x^*_{00}(x_{01}) < x^*_A$.

Recall that $\lambda^*_A = \lambda_A(x^*_A) = \int_0^{x^*_A} h(y)dy$. For any given x_{01}, (57) presents the condition for $\frac{\partial}{\partial x_{00}}S(x_{00},x_{01}) = 0$. Subtituting $x_{00} = x^*_A$ in the left hand side of (57) gives

$$-(2+\lambda_{01})\int_0^{x^*_A}(v-y)h(y)dy + \int_0^{x_{01}}(v-y)h(y)dy + (v-x^*_A)(1+\lambda^*_A(2+\lambda_{01})). \quad (59)$$

By Lemma 5.1:

$$\begin{aligned}
&-(2+\lambda_{01})\int_0^{x^*_A}(v-y)h(y)dy + \int_0^{x_{01}}(v-y)h(y)dy + (v-x^*_A)(1+\lambda^*_A(2+\lambda_{01})) = \\
&-(2+\lambda_{01})\left[(v-x^*_A)\lambda^*_A + \int_0^{x^*_A}(x^*_A-y)h(y)dy\right] + \\
&\int_0^{x_{01}}(v-x^*_A+x^*_A-y)h(y)dy + (v-x^*_A)(1+\lambda^*_A(2+\lambda_{01})) \\
&= -(2+\lambda_{01})\left[(v-x^*_A)\lambda^*_A + \int_0^{x^*_A}(x^*_A-y)h(y)dy\right] + \\
&(v-x^*_A)\lambda_{01} + \int_0^{x_{01}}(x^*_A-y)h(y)dy + \\
&(v-x^*_A)(1+\lambda^*_A(2+\lambda_{01})) = \\
&(v-x^*_A)(-2\lambda^*_A-\lambda_{01}\lambda^*_A+\lambda_{01}+1+2\lambda^*_A+\lambda_{01}\lambda^*_A) \\
&-(1+\lambda_{01}+1)\int_0^{x^*_A}(x^*_A-y)h(y)dy + \\
&\int_0^{x_{01}}(x^*_A-y)h(y)dy = \\
&(1+\lambda_{01})\left[v-\left(x^*_A+\int_0^{x^*_A}(x^*_A-y)h(y)dy\right)\right] - \\
&\int_{x_{01}}^{x^*_A}(x^*_A-y)h(y)dy.
\end{aligned} \quad (60)$$

Note that by (10), the expression in the square brakets above equals 0, hence we get that

$$-\int_{x_{01}}^{x^*_A}(x^*_A-y)h(y)dy,$$

which is negative for all $x_{00} \neq x^*_A$.

Now, in comparission to that, if we substitute $x_{00} = \tilde{x}_{00}(x_{01})$ in the left hand side of(57), then by definition of $\tilde{x}_{00}(x_{01})$, this equals 0. Earlier (see (58)) we showed that the left hand side of (57) is decreasing in x_{00}. Hence it follows that $\tilde{x}_{00}(x_{01}) < x^*_A$.

□

Proof of Lemma 5.2

Proof. Note that,

$$\int_x^M (y-v)h(y)dy = (x-v)\lambda_B(x) + x - \left[x - \int_x^M (y-x)h(y)dy\right]. \quad (61)$$

It follows from (61) and (11) that for $x = x_B^*$,

$$\int_{x_B^*}^M (y-v)h(y)dy = (x_B^* - v)\lambda_B^* + x_B^* - v = (x_B^* - v)(\lambda_B^* + 1), \quad (62)$$

proving the second statement of the lemma. Note that $x - \int_x^M (y-x)h(y)dy$, appearing in the square brackets at the right hand side of (61) is increasing in x, hence it follows from (62) that:

- If $x < x_B^*$, then $\int_x^M (y-v)h(y)dy > (x-v)\lambda_B(x) + x - v = (\lambda_B(x)+1)(x-v)$,
- If $x > x_B^*$, then $\int_x^M (y-v)h(y)dy < (x-v)\lambda_B(x) + x - v = (\lambda_B(x)+1)(x-v)$,

proving the first and last statements of the lemma. □