Compressing UNSAT CDCL Trees with Caching

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Abstract: We aim at providing users of SAT solvers with small, easily understandable proofs of unsatisfiability. Caching techniques have been proposed to identify redundant subproofs and reduce the size of some UNSAT proof trees. Branches are pruned when they correspond to subformulas that were proved unsatisfiable earlier in the tree. A caching mechanism based on subgraph isomorphism was proposed as postprocessing step both in the DPLL and CDCL architectures but the technique could only be integrated during the search on the DPLL architecture. This paper presents how to integrate such caching mechanism during the search for the CDCL case and presents a generalized caching mechanism for that architecture.

1 INTRODUCTION

SAT solvers are currently used to solve a wide range of combinatorial problems (Biere et al., 2021). They can be trusted since they can provide either a model when a solution exists or a certificate of unsatisfiability (Wetzler et al., 2014) that can be verified by an independent checker. Both kinds of certificates allow to check the answer provided by the SAT solver, but it is usually not an explanation for the user of the SAT solver. If there are more than one solution, why providing that one? If there are no solution, why is it the case? In the former case, it will be difficult to answer since the model found depends heavily on the implementation of the SAT solver used and has no logical explanation. In the latter case, it is possible to find a minimal subset of unsatisfiable clauses (MUS) (Ignatiev et al., 2015). However, there is no guarantee that a MUS is smaller than the complete formula, and a certificate can have an exponential number of steps.

In this work, we only consider unsatisfiable formulas and our goal is to significantly compress the search tree of a CDCL solver in order to obtain a proof that is small enough to be given as explanation to the user. To do so, we focus on finding recurring unsatisfiable patterns during the search because of their potentially huge impact on the tree size, and also because they can be ”explained” individually and independently to the user. We use a cache based on subgraph isomorphism detection to recognize these patterns. If the current subformula was already explored and proved unsatisfiable, the current branch can be pruned and a conflict clause must be provided to the CDCL solver. That work has been initiated in (Blomme et al., 2023), with a complete integration in the DPLL architecture but only as a postprocessing step in the CDCL architecture. This paper extends that work with a complete integration in a CDCL solver and a generalized caching mechanism dedicated to that architecture.

This paper is organized as follows. In Section 2, we define some basic notions and notations. In Section 3, we summarize the previous work on this problem by (Blomme et al., 2023). In Section 4, we propose an integration of a cache for unsatisfiable formulas into a CDCL solver. Then, we present a more general caching definition in Section 5. Some experimental results are presented in Section 6 and we discuss the generation of reduced prrofs in Section 7. Finally, we conclude and present some future works.

2 PRELIMINARIES

A Boolean *variable* v is either true or false. A *literal* is a variable v or its negation $\neg v$. A *clause* is a disjunction of literals and a formula in *Conjunctive Normal Form* (CNF) is a conjunction of clauses. An *assignment* maps the set of variables to the truth values 0 (for *false*) or 1 (for *true*). A clause is satisfied by an

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assignment if it contains at least one literal l which is assigned true. A formula is satisfied by an assignment if and only if all its clauses are satisfied. Deciding if there exists an assignment that satisfies a given formula in CNF is known as the *satisfiability problem* (SAT), which is NP-complete (Cook, 1971). The formula is *SAT* if it is possible to find such an assignment and it is *UNSAT* otherwise. Given an assignment I, $F_{|I}$ denotes the formula simplified by I: satisfied clauses are removed from the formula and falsified literals are removed from the remaining clauses. A *unit clause* is a clause C which contains only one non falsified literal l, therefore l must be assigned true. In this case, C is the *reason* for the assignment of l and will be denoted $r(l)$. Applying this operation until there remains no unit clause is called *unit propagation*. Extending an assignment with a literal assignment without reason is called *a decision*. A function $DL(l)$ provides the level of l, which is the number of decisions taken before l. SAT solvers are programs that solve the satisfiability problem. In the 90's, complete SAT solvers were based on the Davis Putnam Logemann Loveland (DPLL) architecture (Davis and Putnam, 1960)(Davis et al., 1962). In 2001, a new architecture called Conflict Driven Clause Learning (CDCL) (Silva and Sakallah, 1999; Moskewicz et al., 2001; Eén and Sörensson, 2003) appeared and made SAT solvers commodity software oracles for solving NP-Complete problems (Biere et al., 2021). SAT solvers explore a search tree, in which a path from the root to the leaves is a partial assignment, and leaves correspond to falsified clauses (a so-called conflict) when the formula is unsatisfiable. While DPLL solvers explore a binary tree by branching on variables, CDCL solvers use conflict analysis and clause learning to drive the search (Marques-Silva et al., 2021).

3 CACHING FOR COMPRESSING

3.1 Principle

In (Blomme et al., 2023), it was noticed that some unsatisfiable problems have a specific structure that can be used to reduce the search tree of a SAT solver. A widely known academic problem with such property is the Pigeon Hole Principle (*PHP*), famous for being hard for solvers and for featuring lots of symmetries (Haken, 1985). The problem is to assign $n+1$ pigeons to n holes with the constraints that a pigeon has to be associated with one hole and a hole cannot contain more than one pigeon. This problem is denoted PHP_n and is defined by variables $x_{i,k}$, which state that pigeon i is assigned hole k. The first constraint is encoded by using a clause of size n for each pigeon: $C_{1,n} = \bigwedge_{1 \le i \le n+1} (x_{i,1} \vee \cdots \vee x_{i,n})$. For the second constraint, we add the mutual exclusions between two pigeons and for a specific hole: $C_{2,n} = \bigwedge_{1 \le i < j \le n+1} \bigwedge_{1 \le k \le n} (\neg x_{i,k} \vee \neg x_{j,k})$. It can be noticed that $PHP_{n|x_{1,n}}$ is an instance of PHP_{n-1}. This occurs when exploring the n ways to place the first pigeon. Once the first PHP_{n-1} subproblem has been explored, it can be recognized as soon as any of the other $x_{1,k}$ is assigned *true* (this occurs $n-1$ times). This step can be repeated recursively until the problem PHP_2 is encountered. The latter only needs two branches to be fully explored. Figure 1 shows an imbricated *PHP* subproblems of size 3 inside a *PHP* problem of size 4. Figure 2 shows a reduced search tree when PHP subproblems are recognized (propagations have been omitted).

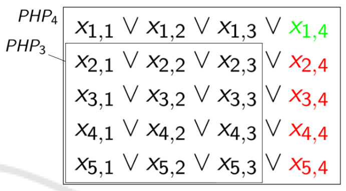

PHP_4
PHP_3
$x_{1,1} \vee x_{1,2} \vee x_{1,3} \vee x_{1,4}$
$x_{2,1} \vee x_{2,2} \vee x_{2,3} \vee x_{2,4}$
$x_{3,1} \vee x_{3,2} \vee x_{3,3} \vee x_{3,4}$
$x_{4,1} \vee x_{4,2} \vee x_{4,3} \vee x_{4,4}$
$x_{5,1} \vee x_{5,2} \vee x_{5,3} \vee x_{5,4}$

Figure 1: Pigeon Hole Principle problem of size 4.

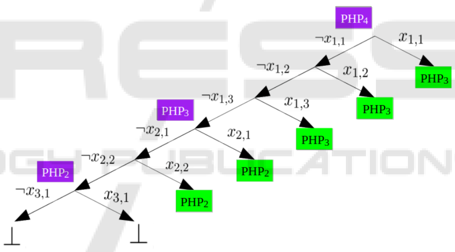


Figure 2: Expected single branch when solving PHP_4.

3.2 Subformula Recognition

The different PHP problems in the previous examples are not based on the same variables. As such, the recognition mechanism needs to support the notion of *equality modulo a renaming of the literals*. There is also a specificity related to dealing with UNSAT formulas: a formula is UNSAT if it contains an UNSAT subformula. So it is also important to detect *formulas included in a target formula*.

Inclusion and renaming can be implemented by solving an NP-complete *Subgraph Isomorphism* problem. For an UNSAT formula E and a current formula $F_{|I}$, one wants to determine if there exists a consistent renaming σ of literals (i.e. a permutation of literals such that if $\sigma(l) = l'$ then $\sigma(\neg l) = \neg l'$) such that $\sigma(E) \subseteq F_{|I}$. As such, (Blomme et al., 2023) proposed to implement a caching mechanism for UNSAT formulas based on subgraph isomorphism detection.

3.3 Sources of Inconsistency

The caching mechanism presented in (Blomme et al., 2023) is based on the notion of *sources*.

The *sources* of an unsatisfiable subformula $F_{|I}$ are the *initial* clauses of F used by the solver to prove the unsatisfiability of $F_{|I}$. This set will be denoted $S(F,I)$. These sources are easily obtained in the solver by gathering recursively the reason of each propagation leading to the conflicts. This process is in essence the same as conflict analysis in a CDCL solver, except that no resolution step is performed. *One important point is that the sources may only contain clauses of the initial formula.* In a CDCL solver, if a learned clause appears in the sources, it is replaced by the set of initial clauses that generated it. Formally, sources are defined as follows in (Blomme et al., 2023).

Definition 1. *We first define the source of a clause $S(C)$. When C is an initial clause, $S(C) = \{C\}$ When L is a learned clause, $S(L)$ is the set of initial clauses of F that appear in the derivation of L by resolution.*

Let $F_{|I}$ be an unsatisfiable subformula and let $\{I_1,\ldots I_m\}$ be the set of branches developed by the solver to prove this inconsistency. Each $F_{|I_j}$ contains a conflict C_j. We define $S_0(F,I_j) = \{C_j\}$ and $S_{i+1}(F,I_j) = S_i(F,I_j) \cup \{S(r(l)) | l \in c \wedge c \in S_i(F,I_j) \wedge DL(l) \geq DL(I_j)\}$. This sequence has a least fixed-point denoted $S(F,I_j)$. At last, the sources $S(F,I)$ of $F_{|I}$ are defined as $S(F,I) = \cup_j S(F,I_j)$.

The cache of a CDCL solver is fed with the sources of the conflicting clauses met during the search.

4 CACHING FOR CDCL

CDCL solvers need a falsified clause to start the conflict analysis procedure. This is also the case when an entry is recognized. In (Blomme et al., 2023), the authors proposed to simply create a clause which is falsified by all decisions on the path from the root to the leaf. However, such an approach did not provide good early experimental results. It is especially useless for conflict analysis because no literal has a reason. As such, the clause may contain literals unnecessary for the conflict.

We propose an alternative approach, which takes advantage of the notion of sources. When we have collected the original clauses O that match the entry in $F_{|I}$, we can create a clause composed of all the falsified literals in $O_{|I}$. Such a clause is by design falsified by the current assignment I. Besides, we show that this clause is entailed by the original formula, so it

Algorithm 1: CacheCDCL(ϕ).

```
Data: $\phi$ - a CNF formula
Result: Is $\phi$ SAT or UNSAT?
$\alpha \leftarrow \emptyset$;
while true do
    $conflictFound \leftarrow false$;
    $(\alpha, C) \leftarrow Propagate(\phi, \alpha)$;
    if $C \neq Undef$ then
        if $CurrentDecisionLevel() = 0$ then
            return $UNSAT$;
        $conflictFound \leftarrow true$;
        $S \leftarrow CollectSourcesConflict(C)$;
    else if $(H \leftarrow HasIsomorphism(\phi, \alpha)) \neq \emptyset$
      then
        if $CurrentDecisionLevel() = 0$ then
            return $UNSAT$;
        $conflictFound \leftarrow true$;
        $C \leftarrow CreateConflictFromIso(H, \alpha)$;
        $S \leftarrow CollectSourcesIsomorphism(H)$;
        $\alpha \leftarrow BackjumpIfNeeded(C, \alpha)$;
    if $conflictFound = true$ then
        $C_1 \leftarrow AnalyzeConflict(C)$;
        $AddToCache(S, \alpha)$;
        $\phi \leftarrow \phi \cup \{C_1\}$;
        $\alpha \leftarrow Backjump(C_1, \alpha)$;
    else
        if $needRestart$ then
            $Restart()$;
        if $needCleanDB$ then
            $CleanDB()$;
        $l \leftarrow Decide()$;
        if $l = Undef$ then
            return $SAT$;
        $\alpha \leftarrow \alpha \cup \{l\}$
```

Algorithm 2: CollectSourcesConflict(C).

```
Data: $C$ - a conflict clause
Result: $S$ - the sources collected
$S \leftarrow Sources(C)$;
for $l \in C$ do
    if $r(l) \neq \emptyset$ and $!Used[var(l)]$ then
        $Used[var(l)] \leftarrow true$;
        $S \leftarrow S \cup CollectSourcesConflict(r(l))$;
return $S$;
```

Algorithm 3: CollectSourcesIsomorphism(H).

```
Data: $H$ - the clauses recognized
Result: $S$ - the sources collected
$S \leftarrow \emptyset$;
for $C \in H$ do
    $S \leftarrow S \cup CollectSourcesConflict(C)$
return $S$;
```

Algorithm 4: CreateConflictFromIso(O, α).

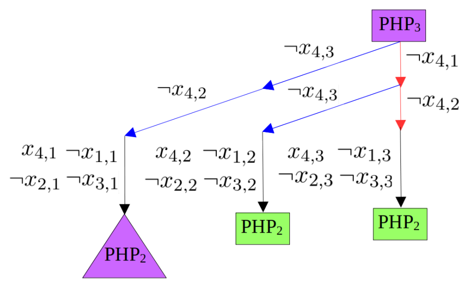
```
    Data: $O$ - the original clauses recognized, $\alpha$ - the
          current assignment
    Result: $C$ - a generated conflict clause
    $conflict \leftarrow \emptyset$;
    for $C \in O$ do
        for $l \in C$ such that $\alpha(l) = false$ do
            $conflict \leftarrow conflict \cup \{l\}$;
    return $conflict$;
```

Algorithm 5: BackjumpIfNeeded(C, α).

```
    Data: $C$ - the generated conflict clause, $\alpha$ - the
          current assignment
    Result: $\alpha$ - the updated assignment
    if $DL(C) < CurrentDecisionLevel()$ then
        $\alpha \leftarrow BackjumpUntil(DL(C), \alpha)$;
    return $\alpha$;
```

is a conflict. We know that $O_{|I} \subseteq F_{|I}$ is unsatisfiable, so $O \wedge I \models \bot$. Suppose we call $SELECT(O,I)$ the literals from I that falsify a literal in a clause of O. Then, we also have $O \wedge SELECT(O,I) \models \bot$, hence $F \models O \models \neg SELECT(O,I)$ since literals from I satisfying clauses in O do not contribute to unsatisfiability. So using $\neg SELECT(O,I)$ as conflict clause when an entry is detected is sound. Algorithm 1 is a modified CDCL procedure with the caching mechanism of (Blomme et al., 2023) enabled and our conflict clause generation procedure for caching hits. Those modifications are highlighted in blue. The conflict clause generation is given in Algorithm 4.

As an example, let us consider the problem PHP_3 with a heuristics that negatively decides the variables in reverse order. Figure 3 illustrates the behaviour of that heuristic. Here a blue arrow denotes a decision, a black arrow represents a propagation with an original clause and a red arrow represents a propagation with a learned clause. The heuristics first decides $\neg x_{4,3}$ and then $\neg x_{4,2}$. This propagates $x_{4,1}$, $\neg x_{1,1}$, $\neg x_{2,1}$ and $\neg x_{3,1}$. We end up now with a PHP_2 problem. After exploring it, we add it to the cache and the solver learns the clause $\neg x_{4,1}$, which makes us backtrack to the root of the tree. After propagating $\neg x_{4,1}$, the heuristics decides again $\neg x_{4,3}$. The clause $x_{4,1} \vee x_{4,2} \vee x_{4,3}$ propagates $x_{4,2}$ and then the mutual exclusions propagate $\neg x_{1,2}$, $\neg x_{2,2}$, and $\neg x_{3,2}$. We end up again with a PHP_2 problem that we recognize since it is in the cache. The clauses $x_{1,1} \vee x_{1,2} \vee x_{1,3}$, $x_{2,1} \vee x_{2,2} \vee x_{2,3}$ and $x_{3,1} \vee x_{3,2} \vee x_{3,3}$ and the required mutual exclusion clauses match the entry. In these clauses, we know that the literals $x_{1,2}$, $x_{2,2}$ and $x_{3,2}$ have been falsified, so we can create the falsified clause $x_{1,2} \vee x_{2,2} \vee x_{3,2}$ that will be given to the conflict analysis. By performing resolution on the clauses $\neg x_{1,2} \vee \neg x_{4,2}$, $\neg x_{2,2} \vee \neg x_{4,2}$ and $\neg x_{3,2} \vee \neg x_{4,2}$, we can learn the clause $\neg x_{4,2}$, backtrack to the decision level 0, propagate $\neg x_{4,2}$ and then continue the search.



PHP$_3$
$\neg x_{4,3}$
$\neg x_{4,1}$
$\neg x_{4,2}$
$\neg x_{4,3}$
$\neg x_{4,2}$
$x_{4,1}$ $\neg x_{1,1}$ $\neg x_{2,1}$ $\neg x_{3,1}$
$x_{4,2}$ $\neg x_{1,2}$ $\neg x_{2,2}$ $\neg x_{3,2}$
$x_{4,3}$ $\neg x_{1,3}$ $\neg x_{2,3}$ $\neg x_{3,3}$
PHP$_2$
PHP$_2$
PHP$_2$

Figure 3: Example to illustrate the integration of the cache system with the conflict analysis.

Note that if the current decision is unrelated to the hit, then the proposed conflict clause may not contain any literal from the current decision level, which breaks one classical CDCL invariant. To restore it, we compute the deepest decision level of the conflicting clause and, if it is not the current decision level, we backtrack to this level before performing the conflict analysis (cf Algorithm 5).

5 GENERALIZED ISOMORPHISMS DETECTION

We observed that the current formula $F_{|I}$ may contain (up to a renaming of variables) an entry E of the cache simplified by a subset I_1 of the current interpretation I (i.e. $\sigma(E)_{|I_1} \subseteq F_{|I}$). When this happens, since E is unsatisfiable, $\sigma(E)_{|I_1}$ is also unsatisfiable hence, $F_{|I}$ is unsatisfiable. In practice, this may happen because we look for isomorphisms only at each decision and not after each unit propagation, which would be too costly. It also happens if literals occurring in E are assigned early on the branch. To detect this case, we have to introduce a generalized isomorphisms.

As an example, let us consider the formula F containing the clauses of PHP_4 where the first clause $x_{1,1} \vee x_{1,2} \vee x_{1,3} \vee x_{1,4}$ is extended with the fresh variables a and b. Let us further assume that the cache already contains PHP_4 and that $x_{2,4}$ was assigned false early on the current branch. When both a and b are assigned false, it is clear that PHP_4 is now contained (in some way) in the current formula. However, syntactically, this is not true if we consider the simplified current formula. Indeed, this simplified formula does not contain the second clause of PHP_4 because it was reduced by $x_{2,4}$. Hence, a regular isomorphism cannot be identified. However, if a and b had been assigned before $x_{2,4}$, this identification would have been possible. Clearly, the method is highly sensitive to the order of assignments, and this should be avoided.

5.1 Formal Definition

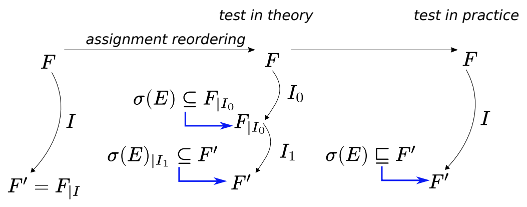


Figure 4: Principle of the generalized isomorphism test.

Let F be the initial formula, I the current interpretation, $F' = F_{|I}$ the current formula and E an entry. We want to determine if I can be partitioned into two subsets I_0 and I_1 and such that $\sigma(E)_{|I_1} \subseteq F'$. I_1 is used to simplify both $\sigma(E)$ and F, I_0 is only used to simplify F. In other words, we want to detect if by reordering the assignments, we could obtain at some point a regular isomorphism: $\sigma(E) \subseteq F_{|I_0}$. Figure 4 illustrates the relation between the different formulas, interpretations and tests. Obviously, I_0 and I_1 are unknown. It should be noticed that I_1 may satisfy a clause C of $\sigma(E)$, but we have to make sure that C is indeed present in F. Otherwise the current formula may contain only a strict subset of E, which may be satisfiable. This means that we have to keep satisfied clauses to perform our tests. Besides, a clause $C \in E$ may match a clause C' in F even if C' contains additional literals, as long as these are falsified by I_0. To formalize this, we introduce a specific inclusion relation $\sqsubseteq$ such that $\sigma(E) \sqsubseteq F'$ implies $\exists I_0$ s.t. $\sigma(E) \subseteq F_{|I_0}$.

Given a current interpretation I and two clauses C and C', $C \sqsubseteq C'$ iff $C \subseteq C'$ and $\forall l \in C' \setminus C, I(l) = false$ (l is assigned false by I). Given I the current interpretation and two formulas F_1 and F_2, $F_1 \sqsubseteq F_2$ iff every clause C of F_1 maps to a clause C' of F_2 such that $C \sqsubseteq C'$. In this work, we consider only bijective mappings to simplify the tests. Obviously, $\sigma(E) \sqsubseteq F'$ implies $\exists I_0$ s.t. $\sigma(E) \subseteq F_{|I_0}$ because $\sqsubseteq$ ignores falsified literals, which belong either to I_0, in which case we want to simply ignore them, or they belong to I_1, in which case they are simplified in both $\sigma(E)$ and F'.

Determining if such a generalized formula isomorphism exists is still in NP. Indeed, if we have an oracle that returns a consistent generalized renaming σ' that maps a literal l to a literal if $l \notin I_0$, or maps to either true or false if $l \in I_0$, then checking if $\sigma(E) \sqsubseteq F'$ (σ is σ' restricted to literals not in I_0) amounts to checking if $\sigma'(E) \subseteq F''$ where F'' is obtained by removing from F every literal falsified by I_0. This latter check is obviously polynomial.

Identifying generalized isomorphisms could be done in several ways, for example by adapting a subgraph isomorphism solver. In this work, we chose the easier option of encoding the generalized isomorphism as a subgraph isomorphism problem, even if the proposed encoding is not polynomial. Finding a polynomial encoding is the subject of future work.

5.2 Generalized Encoding

To allow the possibility to assign some literals of an entry to true or false, we have to adapt the graph encoding of (Blomme et al., 2023). Allowing some literals of a given clause to be satisfied is straightforward: we do not remove satisfied clauses before encoding the current formula. Allowing the literals of a given clause to be falsified (guessing I_0) is more involved. Indeed, a falsified literal should either be erased from the clause (if it belongs to I_0) or kept (if it belongs to I_1) in order to match an entry in the cache. As I_0 and I_1 are unknown, we have to consider both cases, for all falsified literals. So, for each clause with n falsified literals, we need to consider 2^n variants of the initial clause obtained by erasing some falsified literals (from no literal erased to each falsified literal erased) and to make sure that we only select one of them. This allows the subgraph isomorphism solver to identify which literal belongs to I_0. Note that the worst case corresponds to clauses used as reason, since all the literals but one will be falsified (there is no conflicting clause at this stage). In order to avoid matching several variants of a same clause for different clauses of an entry, we have to introduce a new type of node in our graph representation. These new nodes, that we will call *exclusion nodes*, will link all the variants of a same clause in the graph representation of the current formula. In the entry, since there is no variant, we simply add one unique exclusion node per clause. As an example, let us consider a clause $C_5 = (x_1 \lor x_2 \lor x_3 \lor x_4)$. If the literals x_3 and x_4 are falsified in the current assignment, then this new encoding will create the 4 variants shown in Figure 5.

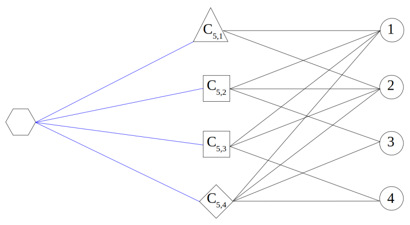


Figure 5: Graph representation of all the variants of the clause $C_5 = (x_1 \lor x_2 \lor x_3 \lor x_4)$ when x_3 and x_4 are falsified. A diamond represents a clause of size 4 and an hexagon represents an exclusion node.

6 EXPERIMENTAL RESULTS

We implemented the proposed approaches on top of MiniSat (Eén and Sörensson, 2003). We disabled database simplification and restarts. The Glasgow Subgraph Solver (GSS for short) (McCreesh et al., 2020) is called to compute subgraph isomorphisms, i.e. to query our cache. We used the same 580 UNSAT instances as in (Blomme et al., 2023).

6.1 Caching, Regular Isomorphisms

We first used the regular isomorphisms and the cache to immediately prune the search. We considered a timeout of 2 seconds for each call to GSS and of 15 minutes per benchmark. An excerpt of the results is shown in the left part of Table 1. For each instance, we give the number of conflicts, the number of entries created in the cache, the number of calls to GSS that found an isomorphism (the number between parenthesis indicates the number of different entries recognized by isomorphism), the total number of calls to GSS, the time spent by the solver (without isomorphism detection) and the cumulated time of all the calls to GSS. All times are in seconds. Our approach has solved 185 instances (113 for SAT'02 and 72 for SAT'03). We could generate a total of 63 single branch search trees (26 for SAT'02 and 37 for SAT'03) with this approach, mainly on instances from the families marg, Urquhart and xor_chain. Thus, we can often obtain a good compression on these families, with few branches in the search tree. We also observed that this approach can solve in less than 15 minutes some SAT'02 Urquhart crafted instances that MiniSat is unable to solve in more than 4 hours. The heuristics used here is not exactly the same as MiniSat because, in addition to the activity updates present in the conflict analysis, the dedicated procedure that collects the sources will also update the activity of the variables it will find.

Since the number of entries keeps growing, and those elements get larger and larger during the search, trying to recognize an entry can become very expensive. Moreover, as the subformulas can be very large as well, finding an isomorphism may take more time than the imposed limit. On large instances, some calls to GSS may be aborted and we may miss some existing isomorphisms, hence some possible compression. This occurs on problems bigger than PHP_{16}.

6.2 Caching, Generalized Isomorphisms

As a second experiment, we have integrated the generalized isomorphisms to immediately prune the search tree. This time, as this encoding is much larger than the regular one, we have imposed a time limit of 30 minutes for the solver and a time limit of 4 seconds for each call to GSS (i.e. twice as much time as for the regular isomorphism). Some results can be found in the right part of Table 1. Our approach has solved a total of 89 instances (51 for SAT'02 and 38 for SAT'03) and has generated a total of 31 single branch search trees (9 for SAT'02 and 22 for SAT'03). If we compare the two tables, and notably the instance `3col40_5_4.cnf`, we can see that the generalized isomorphisms can help to obtain smaller search trees. However, we also note that the approach is much slower due to the size of the encoding. Concerning *PHP* problems, we could not retrieve single branch search trees with these instances but the generalized isomorphism is able to detect more isomorphisms and also to develop a smaller search tree than the regular one. Figure 6 shows the search tree developed by this approach for the problem PHP_5 with both the regular and generalized isomorphisms. Red leaves denote classical conflicts while green nodes denote hits. Compared to the regular isomorphisms, we can see that the generalized isomorphisms are able to detect more hits and also to develop a smaller tree.

6.3 Comparison with Previous Work

Here we compare our two contributions with the previous work (Blomme et al., 2023). Integrating the cache within the CDCL solver increases the number of instances solved and the number of single branch search trees found compared to the post-processing approach. However, DPLL with integrated cache allows to find more single branch search trees than the CDCL approach. This is in part due to the assignment of some variables which prevented their detection. The generalized cache, even with a larger timeout, does not scale. Indeed, it creates larger graphs which limits its applicability on real benchmarks.

7 REDUCED UNSAT PROOFS

When an entry E of the cache is recognized in the current formula $F_{|I}$ and the current branch (corresponding to interpretation I) is pruned, we know that $F_{|I}$ is unsatisfiable because if contains $\sigma(E)$ where σ is the renaming identified by the subgraph isomorphism. In some cases, this gives very short search trees. However, this does not translate directly to very short proofs of unsatisfiability, because the current proof formats such as DRAT (Wetzler et al., 2014) or VeriPB (Gocht and Nordström, 2021) do not include

Table 1: Experimental results when the cache is used during the search with both regular and generalized isomorphisms.

	CDCL (integrated cache, regular isomorphisms)						CDCL (integrated cache, generalized isomorphisms)					
Instance	Conflicts	Cache	Subgraph	Calls	Time		Conflicts	Cache	Subgraph	Calls	Time	
		size	Isomorphisms		Search	GSS		size	Isomorphisms		Search	GSS
PHP_5	26	19	8 (2)	88	0.0	0.4	23	17	16 (11)	95	0.0	1.2
PHP_7	47	41	29 (8)	259	0.0	1.5	42	35	35 (21)	391	0.1	220.3
PHP_{16}	187	178	167 (32)	1020	0.7	166.1	-	-	-	-	-	-
marg2x6	20	17	18 (17)	44	0.0	0.3	20	17	18 (17)	50	0.1	3.3
marg3x3add8	32	25	20 (20)	55	0.0	0.5	31	24	21 (21)	74	0.1	14.1
marg4x4	-	-	-	-	-	-	41	39	39 (35)	186	0.1	130.5
marg6x6	86	84	84 (84)	276	0.2	15.2	-	-	-	-	-	-
Urquhart-s3-b9	21	18	17 (17)	38	0.0	0.3	21	18	17 (17)	50	0.0	3.0
Urquhart-s3-b3	29	26	27 (25)	59	0.0	0.9	29	26	27 (26)	67	0.3	18.0
Urquhart-s5-b5	95	91	91 (90)	259	0.4	55.7	-	-	-	-	-	-
x1_16	18	15	14 (14)	60	0.0	0.7	18	15	14 (14)	94	0.0	21.5
x2_32	-	-	-	-	-	-	65	54	53 (40)	568	0.1	908.4
x1_96	2177	471	106 (76)	8513	1.8	423.4	-	-	-	-	-	-
3col20_5_6	27	5	0 (0)	15	0.0	0.1	23	3	2 (2)	16	0.0	17.3
3col40_5_4	110	22	54 (3)	786	0.1	5.5	57	20	27 (4)	809	0.2	1474.8
homer06	102	95	92 (20)	462	0.5	47.7	-	-	-	-	-	-

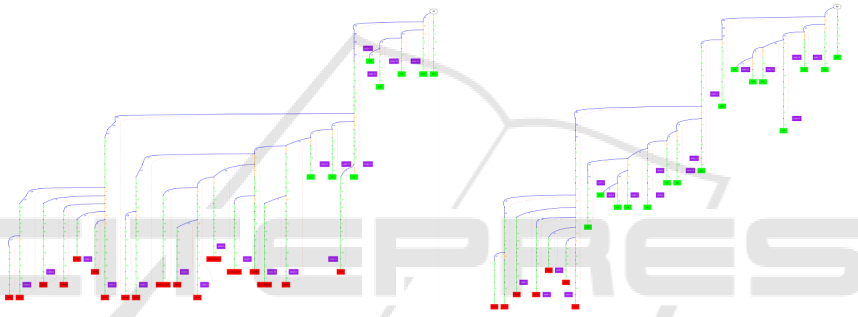


Figure 6: Search tree developed for the problem PHP_5 when the integrated cache is used with the generalized isomorphisms (on the left) and with the generalized isomorphisms (on the right) in a CDCL solver.

Table 2: Comparison with (Blomme et al., 2023). Number of instances solved (number of single branch trees).

Competition	#UNSAT	MiniSat	(Blomme et al., 2023)				This work
			DPLL (15min)		CDCL (15min)		CDCL (30min)
		(1min)	Post processing	Integrated regular	Post processing	Integrated regular	Integrated generalized
SAT'02	382	276	42 (4)	106 (42)	78 (11)	113 (26)	51 (9)
SAT'03	198	78	17 (15)	87 (53)	39 (28)	72 (37)	38 (22)

the required mechanisms to take full advantage of the recognized isomorphisms. In this section, we explain how these proof formats could be extended to obtain short proofs exploiting the recognized isomorphisms.

Basically, we have to instruct the proof checker that the proof of unsatisfiability for the current formula can be obtained by renaming literals in a previous subpart of the proof, and the checker must be able to check this. Trivially, this could be done without modifying the proof checker by storing in the cache the proof of unsatisfiability $P(E)$ next to each entry E, and adding to the generated proof $\sigma(P(E))$, that is the proof previously recorded with all literals correctly renamed. However, this basic approach will not reduce the size of the generated proof, since $P(E)$ is essentially duplicated in the proof whenever E is recognized. *Instead of copying, we use links to existing proofs, and the proof checker has to verify that these links are correct, in a reasonably efficient way.*

Let us assume that the entry E of the cache was obtained for an interpretation I_E. Since E does not contain in general all the clauses of $F_{|I_E}$, let us denote s_E a selection function which indicates which clauses of the current formula are kept. In essence, s_E directly encodes the sources of F but can be represented in a simple way by an array of Booleans or a list of clause identifiers. We have $E = s_E(F_{|I_E})$. For each entry E of the cache, we thus need to record I_E and the selection function s_E. Now, whenever an isomorphism is identified, that is, whenever $\exists E, \sigma$ such that $\sigma(E) \subseteq F_{|I}$, we can add to the proof of unsatisfiability the tuple I, I_E, s_E, σ, μ where μ maps the clauses of E to the corresponding clauses in F_I. Strictly speaking, storing μ is not necessary to check the proof, but reduces the complexity of the verification.

E is associated to a conflict C discovered in $F_{|I_E}$ and obtained by unwinding the usual conflict analysis performed by a CDCL solver. If the checker does not erase any clause, its database contains all the learnt and original clauses used by the CDLC solver to falsify C by unit propagation from I_E, and the checker has already proved that each learnt clause was actually implied by the formula. Therefore, to verify that E is unsatisfiable, the checker has to find one clause C' of its database that is actually falsified by I_E and such that the sources $S(C', I_E)$ are equal to E. This can be done easily by keepingd if the checker keeps track for every learnt clause of the clauses used in the derivation of that clause, by following the definition of the sources. Now, the checker has to verify that $\sigma(E) \subseteq F_{|I}$. Since I, I_E, s_E, σ are recorded in the proof, it is easy to compute $E = s_E(F_{|I_E})$ and check that $\sigma(E)$ is a subset of F_I. With μ, the checker can also recover the set O of clauses of F_I which are mapped to clauses of E. At this point, the checker knows that $O_{|I}$ is unsatisfiable and, as proved in Section 4, that $F \models \neg SELECT(O, I)$. Therefore, it can add $\neg SELECT(O, I)$ to its clause database. Remember that $\neg SELECT(O, I)$ is used as a conflict. The clause learnt from this conflict can be checked in the usual way.

To sum up, in order to generate and check short proofs of unsatisfiability based on our caching system, we have to extend the proof format with a new declaration that records I, I_E, s_E, σ, μ, add a new rule in the checker that uses this declaration to infer a new clause $\neg SELECT(O, I)$. This can be done in a reasonable way if the solver does not delete clauses from its database and associates to each learnt clause the clauses used to derive that learnt clause. Implementing such a checker is the subject of future work.

8 CONCLUSION

Our goal in this work was to prune as much as possible the branches of an UNSAT CDCL search tree to reduce its size for presenting it to the end user as an explanation of unsatisfiability. To do so, we extended the work of (Blomme et al., 2023) to include their caching mechanism directly in a CDCL solver and generalized it to recognize entries with assigned literals. The experimental results show that the integrated cache provides better results than the existing post-processing approach for CDCL. The generalized caching procedure improves the results on crafted benchmarks but does not scale on real benchmarks. We finally discussed the relationship between our work and the computation of small certificates of unsatisfiability. Unfortunately, the current proof formats do not support yet the features needed to benefit from our short trees to produce short certificates, but can be extended to support these reduced proofs.

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REFERENCES

Biere, A., Heule, M., van Maaren, H., and Walsh, T., editors (2021). *Handbook of Satisfiability - Second Edition*.

Blomme, A., Le Berre, D., Parrain, A., and Roussel, O. (2023). Compressing unsat search trees with caching. In *ICAART 2023*, pages 358–365. INSTICC.

Cook, S. A. (1971). The complexity of theorem-proving procedures. In *3rd Annual ACM*, pages 151–158.

Davis, M., Logemann, G., and Loveland, D. W. (1962). A machine program for theorem-proving. *Commun. ACM*, 5(7):394–397.

Davis, M. and Putnam, H. (1960). A computing procedure for quantification theory. *J. ACM*, 7(3):201–215.

Eén, N. and Sörensson, N. (2003). An extensible sat-solver. In *SAT 2003*, pages 502–518.

Gocht, S. and Nordström, J. (2021). Certifying parity reasoning efficiently using pseudo-boolean proofs. In *AAAI 2021*, pages 3768–3777. AAAI Press.

Haken, A. (1985). The intractability of resolution. *Theor. Comput. Sci.*, 39:297–308.

Ignatiev, A., Previti, A., Liffiton, M. H., and Marques-Silva, J. (2015). Smallest MUS extraction with minimal hitting set dualization. In *CP 2015*, pages 173–182.

Marques-Silva, J., Lynce, I., and Malik, S. (2021). Conflict-driven clause learning SAT solvers. In *Handbook of Satisfiability - Second Edition*, pages 133–182.

McCreesh, C., Prosser, P., and Trimble, J. (2020). The glasgow subgraph solver: Using constraint programming to tackle hard subgraph isomorphism problem variants. In *ICGT 2020*, pages 316–324.

Moskewicz, M. W., Madigan, C. F., Zhao, Y., Zhang, L., and Malik, S. (2001). Chaff: Engineering an efficient SAT solver. In *Proc. of DAC 2001*, pages 530–535.

Silva, J. P. M. and Sakallah, K. A. (1999). GRASP: A search algorithm for propositional satisfiability. *IEEE Trans. Computers*, 48(5):506–521.

Wetzler, N., Heule, M., and Jr., W. A. H. (2014). Drat-trim: Efficient checking and trimming using expressive clausal proofs. In *SAT 2014*, pages 422–429.